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Finite Dimensional Solution Sets of Extremal Problems in H^1

Jyunji Inoue* and Takahiko Nakazi *

Dedicated to Professor Tuyoshi Ando on his 60th birthday

For a non-zero function f in H^1, the classical Hardy space on the unit circle, put

$$S_{|f|/f} = \{g \in H^1 : \|g\|_1 = 1, \arg f(e^{it}) = \arg g(e^{it}) \ a.e.t\},$$

then $S_{|f|/f}$ is the set of extremal functions of a well known linear extremal problem in H^1. It is known and easy to see that if f^{-1} belongs to H^1 then the dimension of $< S_{|f|/f} >$, the linear span of $S_{|f|/f}$, is one. A simple example shows that even if f^{-1} belongs to H^p for some $p\,(0 < p < 1)$, the dimension of $< S_{|f|/f} >$ may be infinite. On the other hand, a sophisticated example (will be shown in this paper) shows that even if f^{-1} locally belongs to H^1 on the unit circle except a finite set, the dimension of $< S_{|f|/f} >$ may be infinite. In this paper it is shown that if $f \in H^1$ has the properties such that f^{-1} locally belongs to H^1 on the unit circle except a finite set and that $f^{-1} \in H^p$ for some $p > 0$, then the dimension of $< S_{|f|/f} >$ is finite.

1 Introduction

Let U be the open unit disc with the boundary $T = \partial U$. An analytic function f on U is said to belong to N if $\sup_{0 \leq r < 1} \int_{-\pi}^{\pi} \log^+ | f(re^{it}) | \, dt < \infty$.

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Each function f in N has a boundary value $f(e^{it}) = \lim_{r\to 1-} f(re^{it})$ a.e.t. The set of all f in N satisfying

$$\lim_{r\to 1-} \int_{-\pi}^{\pi} \log^{+} \mid f(re^{it}) \mid dt = \int_{-\pi}^{\pi} \log^{+} \mid f(e^{it}) \mid dt$$

will be denoted by N^{+}. The Hardy space $H^p(U)$ $(0 < p \leq \infty)$ is the subspace of all f in N^{+} whose boundary function $f(e^{it})$ belongs to $L^p(T)$, the usual Lebesgue space on T. Let $1 \leq p,\, q \leq \infty,\ 1/p + 1/q = 1$ and $\varphi \in L^q$. The bounded linear functional T_φ on H^p with kernel function φ is defined by

$$T_\varphi(f) = \int_{-\pi}^{\pi} f(e^{it})\,\varphi(e^{it})\,dt/2\pi \qquad (f \in H^p),$$

and the norm of T_φ is given by $\|T_\varphi\| = \sup\{\mid T_\varphi(f) \mid : f \in H^p, \|f\|_p \leq 1\}$. When T_φ is nonzero, we put $S_\varphi = \{f \in H^p : T_\varphi(f) = \|T_\varphi\|, \|f\|_p \leq 1\}$. The set S_φ is the intersection of the hyperplane $\{f : T_\varphi(f) = \|T_\varphi\|\}$ with the unit ball of H^p. When $1 < p \leq \infty$, the structure of the set S_φ is simple since the set S_φ consists of exactly one point. But when p=1 the situation is quite different from the former case.

In 1958, deLeeuw and Rudin [1] studied the structure of S_φ, and among many fundamental results concerning extremal problems, they completely described the set S_φ when the function φ can be extended analytically to the set $\{\mid z \mid \geq r\}$ for some $r < 1$.

Since then, the study of S_φ has been continued by several authors. Nakazi, the one of the authors of this paper, showed in [7] that if S_φ is a *weak**-compact subset of H^1, $< S_\varphi >$ has finite dimension, and that the set S_φ can be described completely in the same manner as deLeeuw and Rudin carried out in [1]. For continuous kernel φ, the set S_φ is *weak**−compact, and hence Nakazi's result above can be applied. It is easy to see that the above deLeeuw-Rudin's case is contained in this case.

Let $\varphi = \mid f \mid /f$ for some nonzero function $f \in H^1$. Even if f is a polynomial, φ may not be continuous but $< S_\varphi >$ is finite dimensional. We are interested in the problem to decide the functions $f \in H^1$ such that $< S_\varphi >$ for $\varphi = \mid f \mid /f$ has finite dimension. If f^{-1} belongs to H^1 then $S_{|f|/f} = \{f\}$, that is the dimension of $< S_{|f|/f} >$ is one ([7]). We want to relax the hypothesis on f^{-1}. It is known that f^{-1} belongs to N_+ if and only if f is outer. Hence without loss of generality we may assume that f^{-1} belongs to N_+. Regarding $N_+ \cap L^p = H^p$, we want to define a local version of the Hardy space H^p.

For each $c \in T$, I_c denotes an arc in T which contains c at the center of I_c and put $L^p(I_c) = \{f : f \text{ is measurable on } T \text{ and } \int_{I_c} | f(e^{it}) |^p \, dt < \infty\}$.

Definition. For the function $g \in N_+$ we say that g locally belongs to H^p at $c \in T$ when $g \in N_+ \bigcap L^p(I_c)$ for some I_c.

If g locally belongs to H^p at every point on T, then it is easy to see that g belongs to H^p . We can conjecture that if $f \in H^1$ and $f^{-1} \in N_+$ locally belongs to H^1 on the unit circle except a finite set, then the linear span of $S_{|f|/f}$ is finite dimensional. However, we have a counter example which will be given in §3. In the above conjecture, if $f^{-1} \in H^p$ for some $0 < p < 1$ instead of $f^{-1} \in N_+$ then the conjecture is valid. This is our main result which will be proved in §2 .

In the proof of the main result (Theorem 1) the following Lemma 1 , which may be some independent interest, plays an essential role. If $A = \emptyset$ in (b) of Lemma 1, it follows easily that F must be constant. This is the Neuwirth-Newman Theorem [8].

Lemma 1. *Let F be a function in N^+ such that*

(a) *F belongs to H^p for some $p > 0$,*

(b) *F locally belongs to $H^{1/2}$ except a finite set A of T,*

(c) *F is outer and $F(e^{it}) \geq 0$ a.e. on T.*

Then F can be extended to a rational function.

2 The main theorem and its corollaries

In this section the main theorem in this paper and its corollaries will be proved. The proof of Lemma 1 which plays an essential role in the proof will be given in §4.

Notation : In the rest of this paper , the notation $[a]$ for a real number a is used to express the largest integer which is equal or less than a.

Theorem 1. *Suppose f is a nonzero function in H^1 . If f^{-1} locally belongs to H^1 on the unit circle except a finite set A of T and $f^{-1} \in H^p$ for some $p > 0$, then the linear span of $S_{|f|/f}$ has finite dimension.*

Moreover, if $A = \{\alpha_j : j = 1, 2, ..., n\}$ and if $p_j = \sup\{q : f^{-1}$ locally belongs to H^q at $\alpha_j\}$ for $j = 1, 2, ..., n$, then we have

$$\dim < S_{|f|/f} >\leq 1 + \sum_{j=1}^{n} 2\, [\frac{p_j+1}{2p_j}]$$

Proof. If $S_{|f|/f} = \{f/\|f\|\}$, the theorem is trivial, and we suppose that $S_{|f|/f} \neq \{f/\|f\|\}$. Let g be an arbitrary outer function in $S_{|f|/f}$ and put $F(z) = g(z)/f(z)$ $(z \in U)$. We claim that $F(z)$ has the following properties:

(a) F locally belongs to $H^{1/2}$ except a finite set A of T,

(b) At $z = \alpha_j \in A$, F locally belongs to H^q for each q $(0 < q < \frac{p_j}{1+p_j})$,

(c) F is outer and $F(e^{it}) \geq 0$ a.e. on T.

(a) and (b) are deduced from the conditions on f and g by using the Hölder's inequality, and (c) is obvious from the conditions on f and g.

By (a), (b) and (c) using Lemma 1 of §4, F can be extended to a rational function with poles only at points in A. We also express this extension by the same symbol $F(z)$. Note that if α_j is a pole of order n_j of F, n_j must be an even integer by (c), and hence we have $n_j = 2m_j$ with $m_j \leq [\frac{p_j+1}{2p_j}]$. Since $F(z)$ has no zeros on U, its zeros are all on T. Therefore, with some $\gamma > 0$, we can represent $F(z) = g(z)/f(z)$ in the form

$$F(z) = \frac{\gamma \prod_{k=1}^{N}(z - z_k)(\bar{z}_k z - 1)}{\prod_{j=1}^{n}(z - \alpha_j)^{m_j}(1 - \bar{\alpha}_j z)^{m_j}}, \tag{1}$$

where $N = m_1 + ... + m_n$ and $z_k \in T$ $j = 1, ..., N$. From (1), it follows that each outer function g in $S_{|f|/f}$ has the expression of the form

$$g(z) = \gamma f(z) \frac{\prod_{k=1}^{N}(z - z_k)(\bar{z}_k z - 1)}{\prod_{j=1}^{n}(z - \alpha_j)^{m_j}(1 - \bar{\alpha}_j z)^{m_j}} \qquad (z \in U),$$

with the relations $m_j \leq [\frac{p_j+1}{2p_j}]$ $j = 1, 2, ..., n$.

By using the Nakazi's results in [7], we can conclude that

$$\dim < S_{|f|/f} > \leq 1 + \sum_{j}^{n} 2\,[\frac{p_j+1}{2p_j}]$$

and this completes the proof.

In the following three corollaries of Theorem 1, we need not assume $f^{-1} \in H^p$ for some $p > 0$.

Corollary 1. *Suppose f is a schlicht function in H^1. If f^{-1} locally belongs to H^1 on T except a finite set A, then* $\dim < S_{|f|/f} > \leq \max\{3, 2 \mid A \mid +1\}$, *where $\mid A \mid$ denotes the number of elements of A.*

Proof. We divide the situation into two cases: (i) $f(z) \neq 0$ for each $z \in U$, (ii) There exists $\alpha \in U$ such that $f(\alpha) = 0$. In the case (i), f^{-1} is a schlicht function and hence belongs to H^p for all $p\,(0 < p < 1/2)$ at each point of A. Thus we have $\dim < S_{|f|/f} > \leq 2 \mid A \mid +1$ by Theorem 1. In the case (ii), we can represent f in the form $f(z) = (z-\alpha)\,g(z)$ with g^{-1} is a bounded analytic function on U, and hence $g/\|g\|_1$ is an exposed point of the unit ball of H^1. Hence we get $\dim < S_{|f|/f} > =3$ (cf. [7]). Thus the proof is complete.

Corollary 2. *Suppose f is a nonzero function in H^1. If there exists a nonzero function $h \in H^1$ such that hf is an analytic polynomial of degree n, then* $\dim < S_{|f|/f} > \leq 2n+1$.

Proof. Let $f = qg$ be a factorization with an inner function q and an outer function g, and put $s = hf$. Since s is an analytic polynomial of degree n, we can represent s in the form $s(z) = \beta \prod_{j=1}^{L}(z-\alpha_j)^{m_j}$, where $\alpha_1, ..., \alpha_L, \beta \in C$, and $m_1 + ... + m_L = n$. It is easy to see that q is a finite Blaschke product whose zeros are contained in $A \cap U$, where $A = \{\alpha_1, ..., \alpha_L\}$, and α_j may be a zero of q with order at most m_j. On the other hand, $g^{-1} (= qh/s)$ locally belongs to H^1 at each point of $T \backslash A$, and locally belongs to H^q with $0 < q < m_j^{-1}/(m_j^{-1}+1)$ at each point of $\alpha_j \in T \cap A$. Hence we can apply Theorem1 to g, and get $\dim < S_{|g|/g} > \leq 1 + \sum_{|\alpha_j|=1} 2[\frac{m_j+2}{2}]$. Thus we have

$$\dim < S_{|f|/f} > \leq \sum_{\alpha_j \in U} 2m_j + 1 + \sum_{|\alpha_j|=1} 2\,[\frac{m_j+2}{2}] \leq 1 + 2n.$$

This completes the proof.

Corollary 3. *Suppose* $f \in H^1$ *is continuous and* $\Re[f(z)] \geq 0$ *on* $\bar{U}$. *If* f *has at most a finite number of zeros on* T, *the dimension of* $< S_{|f^n|/f^n} >$ *is finite for any positive integer* n.

Proof. Since $\Re[f(z)] \geq 0$ on U, we have $\Re[f^{-1}(z)] \geq 0$ on U, and hence f^{-1} belongs to H^p for each $p < 1$. Therefore, f^{-n} belongs to H^p for some $p > 0$, and f^{-n} locally belongs to H^1 except a finite number of points on T. Thus the result follows from Theorem 1.

If q is an inner function and $f = 1 - q$ then $\Re f$ is nonnegative a.e. on T and hence $f/\|f\|_1$ is an exposed point of the unit ball of H^1, that is $\dim < S_{|f|/f} > = 1$. If q is a finite Blaschke product then by Corollary 3 $\dim < S_{|f^2|/f^2} >< \infty$. However, it is easy to see that if q is not a finite Blaschke product then $\dim S_{|f^2|/f^2} = \infty$.

If $g \in H^1$ is continuous and $| g(z) | \leq 1$ on $\bar{U}$, and $f = 1 - g$, then $\Re f$ is nonnegative. If the peak set of g is a finite set on the unit circle, then by Corollary 3 $\dim S_{|f^n|/f^n} < \infty$ for any positive integer $n > 0$. A well known Rudin-Carleson theorem [3] tells us how to construct such a g.

Corollary 4. *If* $q(z)$ *is an inner function such that* i) $q(z)$ *is not a finite Blaschke product* ii) *The singular support of* $q(z)$ *is a finite set* A. *Then for each number* $\alpha \in T$, $B = \{z \in T \backslash A : q(z) = \alpha\}$ *is an infinite set.*

Proof. If B were a finite set, $f = (\alpha - q)^2$ would satisfy the conditions of Theorem 1 and hence $\dim < S_{|f|/f} >$ must be finite. But it is well known and easy to prove that if $b(z)$ is an inner function which is not a finite Blaschke product, we have $\dim < S_{|(\alpha-b)^2|/(\alpha-b)^2} >= \infty$. Thus we get a contradiction, which completes the proof.

3 Examples

In this section we give counter examples which we mentioned in the abstract and §1. That is, to get the conclusion we cannot omit in Theorem 1 either of (a) and (b) below:

(a) $f(z)$ locally belongs to H^1 except finite points of T,

(b) $f^{-1} \in H^p$ for some $p > 0$.

Example 1. Let $0 < p < 1$ and let b be an infinite Blaschke product on the unit disc such that its zeros converge to 1. Choose p' such that $0 < p < p' < 1$ and put $f = (1-b)^{1/p'}$. We claim that f satisfies (b) of the conditions above, but $\dim < S_{|(1-b)^2|/(1-b)^2} >= \infty$.

Indeed, since $(f^{-1})^p = (1-b)^{-p/p'}$ with $p/p' < 1$, we have $f^{-1} \in H^p$, and since $f = (1-b)^2(1-b)^{-2+(1/p')}$ with $(1-b)^{-2+(1/p')} \in H^1$ we have $\dim < S_{|f|/f} >= \infty$. Note that f doesn't satisfy (a) of the conditions above by Corollary 4.

Example 2. (cf. [5]) Let $f(z)$ be a function defined by

$$F(z) = \prod_{k=1}^{\infty} \frac{(z-\alpha_k)(1-\bar{\alpha}_k z)}{(z-1)(1-z)} \qquad (z \in U) \tag{2}$$

where $\alpha_k = e^{-i/k^2}$, $k = 1, 2, 3, \ldots$. It is easy to see that the right hand side infinite product in (2) converges uniformly on each compact set of $C\backslash\{1\}$, and $F(e^{it}) \geq 0$ on $T\backslash\{1\}$. Moreover $F(z)$ is an outer function. To see this, we consider $\log F(z)$ on $\bar{U}\backslash\{1\}$ such that $\Im[\log F(-1)] = 0$. Since $\Im[\log F(e^{it})]$ is a monotone decreasing step function on $(0, 2\pi)$, with jumps 2π at $t = 1/k^2\,(k = 1, 2, \ldots)$, and hence $\Im[\log F]$ is a real harmonic function belonging to the Zygmund's class. Therefore the harmonic conjugates of $\Im[\log F]$ are contained in h^1 (cf. [6]), which implies that $F(z)$ is outer.

Choose $\varepsilon_k > 0$ so that

$$\left.\begin{array}{ll} \frac{1}{k^2} > \frac{1}{k^2} - \varepsilon_k > \frac{1}{(k+1)^2} + \varepsilon_{k+1}, & \\ |F(e^{it})| \leq 1 & t \in (\frac{1}{k^2} - \varepsilon_k, \frac{1}{k^2} + \varepsilon_k) \end{array}\right. \qquad k = 1, 2, \ldots$$

and put $\Omega = \bigcup_{k=1}^{\infty} (\frac{1}{k^2} - \varepsilon_k, \frac{1}{k^2} + \varepsilon_k)$. If we define a function g on $T\backslash\{1\}$ by

$$g(e^{it}) = \begin{cases} \min\{\frac{1}{|F(e^{it})|}, 1\}: & t \in [0, 2\pi)\backslash\Omega \\ \frac{1}{\varepsilon_k k^4}: & t \in (\frac{1}{k^2} - \varepsilon_k, \frac{1}{k^2} + \varepsilon_k), \qquad k = 1, 2, \ldots \end{cases}$$

then $\log g(e^{it})$ belongs to $L^1(T)$. Using this $g(e^{it})$ we define an outer function $f(z) \in H^1$ by

$$f(z) = \exp \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log g(e^{it})\, dt/2\pi. \qquad (z \in U)$$

Then it is easy to see that $f(z)$ locally belongs to H^1 except one point $z = 1$. But since $f(z)F(z) \in H^1$ and $\arg f(e^{it}) = \arg f(e^{it})F(e^{it})$ a.e. on $[0, 2\pi)$ we can see that

$$\dim < S_{|f|/f} >= \dim < S_{|fF|/fF} >= \infty.$$

4 Proof of Lemma 1

In this section, we prove Lemma 1 used in the proof of the main theorem in §2. In the following, C^+ and C_+ denote the half planes given by $C^+ = \{z : \Im z > 0\}$ and $C_+ = \{z : \Re z > 0\}$. $H^p(C^+)$ (resp. $H^p(C_+)$) is the Hardy space on C^+ (resp. C_+) in the usual sense (cf. [3, p.51]). $\psi(z)$ denotes the linear fractional transformation defined by $\psi(z) = (z-i)/(z+i)$, which maps C^+ conformally onto U.

Lemma 1. *Let F be a function in N^+ such that*
(a) *F belongs to H^p for some $p > 0$,*
(b) *F locally belongs to $H^{1/2}$ except a finite subset A of T,*
(c) *F* is outer and *$F(e^{it}) \geq 0$ a.e. on T.*
Then F can be extended to a rational function.

Proof. By conditions on F, $F(z)$ can be extended beyond every boundary point in $T \setminus A$ to a holomorphic function on $C \cup \{\infty\} \setminus A$, which we also express by $F(z)$. Therefore if we can prove that each point $\alpha \in A$ is at most a pole of $F(z)$, it follows that F is a rational function.

In the rest of the proof of this lemma , $M_i (i = 1, 2, ..., 5)$ stands for an appropriate positive constant. Let α be a point of A. To prove that α is a pole of F, we can assume without loss of generality that $\alpha = 1$. If we put $G(z) = F(z)^p(1-z)^2$, then trivially $G(z)/(1-z)^2 = F(z)^p \in H^1(U)$, and we have $G(\psi(z)) = F(\psi(z))^p(1-\psi(z))^2 \in H^1(C^+)$ (cf. [4] p.130). From this we get

$$| F(\psi(z)) |^p | 1 - \psi(z) |^2 \leq M_1 \qquad (\Im z \geq 1)$$

and hence

$$| F(\psi(z)) | \leq \frac{M_1^{1/p}}{| 1 - \psi(z) |^{2/p}} \leq M_2 | z |^{2/p} \qquad (\Im z \geq 1) \qquad (3)$$

By the conditions (b) and (c), it follows that $\bar{F}(\psi(z)) = F(\psi(\bar{z}))$ on $C \backslash A$, and hence we get

$$| F(\psi(z)) | \leq M_2 | z |^{2/p} \qquad (\Im z \leq -1) \tag{4}$$

Next, since $F(\psi(z+a))^p(1-\psi(z+a))^2 \in H^1(C^+)$ for each $a > 0$, we have by Fejer-Riesz inequality (cf. [2], p.46) modified to hold for a function in $H^p(C^+)$

$$\int_0^\infty | F(\psi(it+a)) |^p | 1-\psi(it+a) |^2 \frac{1}{(1+t)^2} dt$$

$$\leq \frac{1}{2} \int_{-\infty}^\infty | F(\psi(x+a)) |^p | 1-\psi(x+a) |^2 \frac{1}{1+x^2} dx$$

$$\leq \sup_{a>0} \int_{-\infty}^\infty | F(\psi(x+a)) |^p | 1-\psi(x+a) |^2 dx \leq \|G \circ \psi\|_1 < \infty \tag{5}$$

Since

$$| 1-\psi(it+a) | = | 1-(it+a-i)/(it+a+i) |$$

$$= | 2i/(it+a+i) | \geq | 1/(it+a+1) | \qquad (t > 0, a > 0),$$

we get

$$M_3 \geq \int_0^\infty | F(\psi(it+a)) |^p | 1-\psi(it+a)) |^2 \frac{1}{(t+1)^2} dt$$

$$\geq \frac{1}{2} \int_0^\infty | F(\psi(it+a)) |^p | \frac{1}{it+a+1} |^4 dt \qquad (a > 0) \tag{6}$$

Therefore if we choose $a > 0$ large enough to assure $\Re\alpha < a \quad (\alpha \in A)$, we get that $F(\psi(z+a))/(z+a+1)^{4/p} |_{C_+}$ belongs to $H^p(C_+)$. From this we get

$$| F(\psi(z)) | \leq M_4 | z |^{4/p} \qquad (\Re z \geq a) \tag{7}$$

In the same way, we have

$$| F(\psi(z)) | \leq M_5 | z |^{4/p} \qquad (\Re z \leq -b) \tag{8}$$

for a large positive number b. From (3), (4), (7) and (8), we can conclude that ∞ is at most a pole of $F(\psi)$, that is $\alpha = 1$ is at most a pole of F. This completes the proof of Lemma 1.

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