



Title	Finite dimensional solution sets of extremal problems in H_1
Author(s)	Inoue, Junji; Nakazi, Takahiko
Citation	Hokkaido University Preprint Series in Mathematics, 171, 1-10
Issue Date	1992-11
DOI	10.14943/83315
Doc URL	http://hdl.handle.net/2115/68917
Type	bulletin (article)
File Information	pre171.pdf



[Instructions for use](#)

**Finite Dimensional Solution Sets of
Extremal Problems in H^1**

J. Inoue and T. Nakazi

Series #171. November 1992

HOKKAIDO UNIVERSITY
PREPRINT SERIES IN MATHEMATICS

- # 144: K. Kubota, K. Mochizuki, On small data scattering for 2-dimensional semilinear wave equations, 22 pages. 1992.
- # 145: T. Nakazi, K. Takahashi, Hyponormal Toeplitz operators and extremal problems of Hardy spaces, 30 pages. 1992.
- # 146: N. Hayashi, T. Ozawa, Remarks on nonlinear Schrödinger equations in one space dimension, 10 pages. 1992.
- # 147: M. Sato, Interface evolution with Neumann boundary condition, 16 pages. 1992.
- # 148: Y. Okabe, Langevin equations and causal analysis, 49 pages. 1992.
- # 149: Y. Giga, S. Takahashi, On global weak solutions of the nonstationary two-phase Stokes flow, 25 pages. 1992.
- # 150: G. Ishikawa, Determinacy of envelope of the osculating hyperplanes to a curve, 9 pages. 1992.
- # 151: G. Ishikawa, Developable of a curve and determinacy relative to osculation-type, 15 pages. 1992.
- # 152: H. Kubo, Global existence of solutions of semilinear wave equations with data of non compact support in odd space dimensions, 25 pages. 1992.
- # 153: Y. Watatani, Lattices of intermediate subfactors, 33 pages. 1992.
- # 154: T. Ozawa, On critical cases of Sobolev inequalities, 11 pages. 1992.
- # 155: M. Ohnuma, M. Sato, Singular degenerate parabolic equations with applications to geometric evolutions, 20 pages. 1992.
- # 156: S. Izumiya, Perestroikas of optical wave fronts and graphlike Legendrian unfoldings, 13 pages. 1992.
- # 157: A. Arai, Momentum operators with gauge potentials, local quantization of magnetic flux, and representation of canonical commutation relations, 11 pages. 1992.
- # 158: S. Izumiya, W.L. Marar, The Euler number of a topologically stable singular surface in a 3-manifold, 11 pages. 1992.
- # 159: T. Hibi, Cohen-Macaulay types of Cohen-Macaulay complexes, 26 pages. 1992.
- # 160: A. Arai, Properties of the Dirac-Weyl operator with a strongly singular gauge potential, 26 pages. 1992.
- # 161: A. Arai, Dirac operators in Boson-Fermion Fock spaces and supersymmetric quantum field theory, 30 pages. 1992.
- # 162: S. Albeverio, K. Iwata, T. Kolsrud, Random parallel transport on surfaces of finite type, and relations to homotopy, 8 pages. 1992.
- # 163: S. Albeverio, K. Iwata, T. Kolsrud, Moments of random fields over a family of elliptic curves, and modular forms, 9 pages. 1992.
- # 164: Y. Giga, M. Sato, Neumann problem for singular degenerate parabolic equations, 12 pages. 1992.
- # 165: J. Wierzbicki, Y. Watatani, Commuting squares and relative entropy for two subfactors, 18 pages. 1992.
- # 166: Y. Okabe, A new algorithm driven from the view-point of the fluctuation-dissipation theorem in the theory of KM_2O -Langevin equations, 13 pages. 1992.
- # 167: Y. Okabe, H. Mano and Y. Itoh, Random collision model for interacting populations of two species and its strong law of large numbers, 14 pages. 1992.
- # 168: A. Inoue, On the equations of stationary precesses with divergent diffusion coefficients, 25 pages. 1992.
- # 169: T. Ozawa, Remarks on quadratic nonlinear Schrödinger equations, 19 pages. 1992.
- # 170: T. Fukui, Y. Giga, Motion of a graph by nonsmooth weighted curvature, 11 pages. 1992.

Finite Dimensional Solution Sets of Extremal Problems in H^1

Jyunji Inoue* and Takahiko Nakazi *

Dedicated to Professor Tuyoshi Ando on his 60th birthday

For a non-zero function f in H^1 , the classical Hardy space on the unit circle, put

$$S_{|f|/f} = \{g \in H^1 : \|g\|_1 = 1, \arg f(e^{it}) = \arg g(e^{it}) \text{ a.e.t.}\},$$

then $S_{|f|/f}$ is the set of extremal functions of a well known linear extremal problem in H^1 . It is known and easy to see that if f^{-1} belongs to H^1 then the dimension of $\langle S_{|f|/f} \rangle$, the linear span of $S_{|f|/f}$, is one. A simple example shows that even if f^{-1} belongs to H^p for some p ($0 < p < 1$), the dimension of $\langle S_{|f|/f} \rangle$ may be infinite. On the other hand, a sophisticated example (will be shown in this paper) shows that even if f^{-1} locally belongs to H^1 on the unit circle except a finite set, the dimension of $\langle S_{|f|/f} \rangle$ may be infinite. In this paper it is shown that if $f \in H^1$ has the properties such that f^{-1} locally belongs to H^1 on the unit circle except a finite set and that $f^{-1} \in H^p$ for some $p > 0$, then the dimension of $\langle S_{|f|/f} \rangle$ is finite.

1 Introduction

Let U be the open unit disc with the boundary $T = \partial U$. An analytic function f on U is said to belong to N if $\sup_{0 \leq r < 1} \int_{-\pi}^{\pi} \log^+ |f(re^{it})| dt < \infty$.

*This research was partially supported by Grant-in-Aid for Scientific Research, Ministry of Education.

Each function f in N has a boundary value $f(e^{it}) = \lim_{r \rightarrow 1^-} f(re^{it})$ a.e.t. The set of all f in N satisfying

$$\lim_{r \rightarrow 1^-} \int_{-\pi}^{\pi} \log^+ |f(re^{it})| dt = \int_{-\pi}^{\pi} \log^+ |f(e^{it})| dt$$

will be denoted by N^+ . The Hardy space $H^p(U)$ ($0 < p \leq \infty$) is the subspace of all f in N^+ whose boundary function $f(e^{it})$ belongs to $L^p(T)$, the usual Lebesgue space on T . Let $1 \leq p, q \leq \infty$, $1/p + 1/q = 1$ and $\varphi \in L^q$. The bounded linear functional T_φ on H^p with kernel function φ is defined by

$$T_\varphi(f) = \int_{-\pi}^{\pi} f(e^{it}) \varphi(e^{it}) dt / 2\pi \quad (f \in H^p),$$

and the norm of T_φ is given by $\|T_\varphi\| = \sup \{ \|T_\varphi(f)\| : f \in H^p, \|f\|_p \leq 1 \}$. When T_φ is nonzero, we put $S_\varphi = \{ f \in H^p : T_\varphi(f) = \|T_\varphi\|, \|f\|_p \leq 1 \}$. The set S_φ is the intersection of the hyperplane $\{ f : T_\varphi(f) = \|T_\varphi\| \}$ with the unit ball of H^p . When $1 < p \leq \infty$, the structure of the set S_φ is simple since the set S_φ consists of exactly one point. But when $p=1$ the situation is quite different from the former case.

In 1958, deLeeuw and Rudin [1] studied the structure of S_φ , and among many fundamental results concerning extremal problems, they completely described the set S_φ when the function φ can be extended analytically to the set $\{ |z| \geq r \}$ for some $r < 1$.

Since then, the study of S_φ has been continued by several authors. Nakazi, the one of the authors of this paper, showed in [7] that if S_φ is a *weak**-compact subset of H^1 , $\langle S_\varphi \rangle$ has finite dimension, and that the set S_φ can be described completely in the same manner as deLeeuw and Rudin carried out in [1]. For continuous kernel φ , the set S_φ is *weak**-compact, and hence Nakazi's result above can be applied. It is easy to see that the above deLeeuw-Rudin's case is contained in this case.

Let $\varphi = |f|/f$ for some nonzero function $f \in H^1$. Even if f is a polynomial, φ may not be continuous but $\langle S_\varphi \rangle$ is finite dimensional. We are interested in the problem to decide the functions $f \in H^1$ such that $\langle S_\varphi \rangle$ for $\varphi = |f|/f$ has finite dimension. If f^{-1} belongs to H^1 then $S_{|f|/f} = \{f\}$, that is the dimension of $\langle S_{|f|/f} \rangle$ is one ([7]). We want to relax the hypothesis on f^{-1} . It is known that f^{-1} belongs to N_+ if and only if f is outer. Hence without loss of generality we may assume that f^{-1} belongs

to N_+ . Regarding $N_+ \cap L^p = H^p$, we want to define a local version of the Hardy space H^p .

For each $c \in T$, I_c denotes an arc in T which contains c at the center of I_c and put $L^p(I_c) = \{f : f \text{ is measurable on } T \text{ and } \int_{I_c} |f(e^{it})|^p dt < \infty\}$.

Definition. For the function $g \in N_+$ we say that g locally belongs to H^p at $c \in T$ when $g \in N_+ \cap L^p(I_c)$ for some I_c .

If g locally belongs to H^p at every point on T , then it is easy to see that g belongs to H^p . We can conjecture that if $f \in H^1$ and $f^{-1} \in N_+$ locally belongs to H^1 on the unit circle except a finite set, then the linear span of $S_{|f|/f}$ is finite dimensional. However, we have a counter example which will be given in §3. In the above conjecture, if $f^{-1} \in H^p$ for some $0 < p < 1$ instead of $f^{-1} \in N_+$ then the conjecture is valid. This is our main result which will be proved in §2.

In the proof of the main result (Theorem 1) the following Lemma 1, which may be some independent interest, plays an essential role. If $A = \emptyset$ in (b) of Lemma 1, it follows easily that F must be constant. This is the Neuwirth-Newman Theorem [8].

Lemma 1. Let F be a function in N^+ such that

- (a) F belongs to H^p for some $p > 0$,
- (b) F locally belongs to $H^{1/2}$ except a finite set A of T ,
- (c) F is outer and $F(e^{it}) \geq 0$ a.e. on T .

Then F can be extended to a rational function.

2 The main theorem and its corollaries

In this section the main theorem in this paper and its corollaries will be proved. The proof of Lemma 1 which plays an essential role in the proof will be given in §4.

Notation : In the rest of this paper , the notation $[a]$ for a real number a is used to express the largest integer which is equal or less than a .

Theorem 1. Suppose f is a nonzero function in H^1 . If f^{-1} locally belongs to H^1 on the unit circle except a finite set A of T and $f^{-1} \in H^p$ for some $p > 0$, then the linear span of $S_{|f|/f}$ has finite dimension.

Moreover, if $A = \{\alpha_j : j = 1, 2, \dots, n\}$ and if $p_j = \sup\{q : f^{-1} \text{ locally belongs to } H^q \text{ at } \alpha_j\}$ for $j = 1, 2, \dots, n$, then we have

$$\dim \langle S_{|f|/f} \rangle \leq 1 + \sum_{j=1}^n 2 \left[\frac{p_j + 1}{2p_j} \right]$$

Proof. If $S_{|f|/f} = \{f/\|f\|\}$, the theorem is trivial, and we suppose that $S_{|f|/f} \neq \{f/\|f\|\}$. Let g be an arbitrary outer function in $S_{|f|/f}$ and put $F(z) = g(z)/f(z)$ ($z \in U$). We claim that $F(z)$ has the following properties:

- (a) F locally belongs to $H^{1/2}$ except a finite set A of T ,
- (b) At $z = \alpha_j \in A$, F locally belongs to H^q for each q ($0 < q < \frac{p_j}{1+p_j}$),
- (c) F is outer and $F(e^{it}) \geq 0$ a.e. on T .

(a) and (b) are deduced from the conditions on f and g by using the Hölder's inequality, and (c) is obvious from the conditions on f and g .

By (a), (b) and (c) using Lemma 1 of §4, F can be extended to a rational function with poles only at points in A . We also express this extension by the same symbol $F(z)$. Note that if α_j is a pole of order n_j of F , n_j must be an even integer by (c), and hence we have $n_j = 2m_j$ with $m_j \leq [\frac{p_j+1}{2p_j}]$. Since $F(z)$ has no zeros on U , its zeros are all on T . Therefore, with some $\gamma > 0$, we can represent $F(z) = g(z)/f(z)$ in the form

$$F(z) = \frac{\gamma \prod_{k=1}^N (z - z_k)(\bar{z}_k z - 1)}{\prod_{j=1}^n (z - \alpha_j)^{m_j} (1 - \bar{\alpha}_j z)^{m_j}}, \quad (1)$$

where $N = m_1 + \dots + m_n$ and $z_k \in T$ $j = 1, \dots, N$. From (1), it follows that each outer function g in $S_{|f|/f}$ has the expression of the form

$$g(z) = \gamma f(z) \frac{\prod_{k=1}^N (z - z_k)(\bar{z}_k z - 1)}{\prod_{j=1}^n (z - \alpha_j)^{m_j} (1 - \bar{\alpha}_j z)^{m_j}} \quad (z \in U),$$

with the relations $m_j \leq [\frac{p_j+1}{2p_j}]$ $j = 1, 2, \dots, n$.

By using the Nakazi's results in [7], we can conclude that

$$\dim \langle S_{|f|/f} \rangle \leq 1 + \sum_j^n 2 \left[\frac{p_j + 1}{2p_j} \right]$$

and this completes the proof.

In the following three corollaries of Theorem 1, we need not assume $f^{-1} \in H^p$ for some $p > 0$.

Corollary 1. Suppose f is a schlicht function in H^1 . If f^{-1} locally belongs to H^1 on T except a finite set A , then $\dim \langle S_{|f|/f} \rangle \leq \max \{ 3, 2|A| + 1 \}$, where $|A|$ denotes the number of elements of A .

Proof. We divide the situation into two cases: (i) $f(z) \neq 0$ for each $z \in U$, (ii) There exists $\alpha \in U$ such that $f(\alpha) = 0$. In the case (i), f^{-1} is a schlicht function and hence belongs to H^p for all p ($0 < p < 1/2$) at each point of A . Thus we have $\dim \langle S_{|f|/f} \rangle \leq 2|A| + 1$ by Theorem 1. In the case (ii), we can represent f in the form $f(z) = (z - \alpha)g(z)$ with g^{-1} is a bounded analytic function on U , and hence $g/\|g\|_1$ is an exposed point of the unit ball of H^1 . Hence we get $\dim \langle S_{|f|/f} \rangle = 3$ (cf. [7]). Thus the proof is complete.

Corollary 2. Suppose f is a nonzero function in H^1 . If there exists a nonzero function $h \in H^1$ such that hf is an analytic polynomial of degree n , then $\dim \langle S_{|f|/f} \rangle \leq 2n + 1$.

Proof. Let $f = qg$ be a factorization with an inner function q and an outer function g , and put $s = hf$. Since s is an analytic polynomial of degree n , we can represent s in the form $s(z) = \beta \prod_{j=1}^L (z - \alpha_j)^{m_j}$, where $\alpha_1, \dots, \alpha_L, \beta \in \mathbb{C}$, and $m_1 + \dots + m_L = n$. It is easy to see that q is a finite Blaschke product whose zeros are contained in $A \cap U$, where $A = \{\alpha_1, \dots, \alpha_L\}$, and α_j may be a zero of q with order at most m_j . On the other hand, $g^{-1} (= qh/s)$ locally belongs to H^1 at each point of $T \setminus A$, and locally belongs to H^q with $0 < q < m_j^{-1}/(m_j^{-1} + 1)$ at each point of $\alpha_j \in T \cap A$. Hence we can apply Theorem 1 to g , and get $\dim \langle S_{|g|/g} \rangle \leq 1 + \sum_{|\alpha_j|=1} 2 \left[\frac{m_j + 2}{2} \right]$. Thus we have

$$\dim \langle S_{|f|/f} \rangle \leq \sum_{\alpha_j \in U} 2m_j + 1 + \sum_{|\alpha_j|=1} 2 \left[\frac{m_j + 2}{2} \right] \leq 1 + 2n.$$

This completes the proof.

Corollary 3. Suppose $f \in H^1$ is continuous and $\Re[f(z)] \geq 0$ on \bar{U} . If f has at most a finite number of zeros on T , the dimension of $\langle S_{|f^n|/f^n} \rangle$ is finite for any positive integer n .

Proof. Since $\Re[f(z)] \geq 0$ on U , we have $\Re[f^{-1}(z)] \geq 0$ on U , and hence f^{-1} belongs to H^p for each $p < 1$. Therefore, f^{-n} belongs to H^p for some $p > 0$, and f^{-n} locally belongs to H^1 except a finite number of points on T . Thus the result follows from Theorem 1.

If q is an inner function and $f = 1 - q$ then $\Re f$ is nonnegative a.e. on T and hence $f/\|f\|_1$ is an exposed point of the unit ball of H^1 , that is $\dim \langle S_{|f|/f} \rangle = 1$. If q is a finite Blaschke product then by Corollary 3 $\dim \langle S_{|f^2|/f^2} \rangle < \infty$. However, it is easy to see that if q is not a finite Blaschke product then $\dim S_{|f^2|/f^2} = \infty$.

If $g \in H^1$ is continuous and $|g(z)| \leq 1$ on \bar{U} , and $f = 1 - g$, then $\Re f$ is nonnegative. If the peak set of g is a finite set on the unit circle, then by Corollary 3 $\dim S_{|f^n|/f^n} < \infty$ for any positive integer $n > 0$. A well known Rudin-Carleson theorem [3] tells us how to construct such a g .

Corollary 4. If $q(z)$ is an inner function such that i) $q(z)$ is not a finite Blaschke product ii) The singular support of $q(z)$ is a finite set A . Then for each number $\alpha \in T$, $B = \{z \in T \setminus A : q(z) = \alpha\}$ is an infinite set.

Proof. If B were a finite set, $f = (\alpha - q)^2$ would satisfy the conditions of Theorem 1 and hence $\dim \langle S_{|f|/f} \rangle$ must be finite. But it is well known and easy to prove that if $b(z)$ is an inner function which is not a finite Blaschke product, we have $\dim \langle S_{|(\alpha-b)^2|/(\alpha-b)^2} \rangle = \infty$. Thus we get a contradiction, which completes the proof.

3 Examples

In this section we give counter examples which we mentioned in the abstract and §1. That is, to get the conclusion we cannot omit in Theorem 1 either of (a) and (b) below:

- (a) $f(z)$ locally belongs to H^1 except finite points of T ,
- (b) $f^{-1} \in H^p$ for some $p > 0$.

Example 1. Let $0 < p < 1$ and let b be an infinite Blaschke product on the unit disc such that its zeros converge to 1. Choose p' such that $0 < p < p' < 1$ and put $f = (1-b)^{1/p'}$. We claim that f satisfies (b) of the conditions above, but $\dim < S_{|(1-b)^{2/p}/(1-b)^2} > = \infty$.

Indeed, since $(f^{-1})^p = (1-b)^{-p/p'}$ with $p/p' < 1$, we have $f^{-1} \in H^p$, and since $f = (1-b)^2(1-b)^{-2+(1/p')}$ with $(1-b)^{-2+(1/p')} \in H^1$ we have $\dim < S_{|f|/f} > = \infty$. Note that f doesn't satisfy (a) of the conditions above by Corollary 4.

Example 2. (cf. [5]) Let $f(z)$ be a function defined by

$$F(z) = \prod_{k=1}^{\infty} \frac{(z - \alpha_k)(1 - \bar{\alpha}_k z)}{(z-1)(1-z)} \quad (z \in U) \quad (2)$$

where $\alpha_k = e^{-i/k^2}$, $k = 1, 2, 3, \dots$. It is easy to see that the right hand side infinite product in (2) converges uniformly on each compact set of $C \setminus \{1\}$, and $F(e^{it}) \geq 0$ on $T \setminus \{1\}$. Moreover $F(z)$ is an outer function. To see this, we consider $\log F(z)$ on $\bar{U} \setminus \{1\}$ such that $\Im[\log F(-1)] = 0$. Since $\Im[\log F(e^{it})]$ is a monotone decreasing step function on $(0, 2\pi)$, with jumps 2π at $t = 1/k^2$ ($k = 1, 2, \dots$), and hence $\Re[\log F]$ is a real harmonic function belonging to the Zygmund's class. Therefore the harmonic conjugates of $\Re[\log F]$ are contained in h^1 (cf. [6]), which implies that $F(z)$ is outer.

Choose $\varepsilon_k > 0$ so that

$$\frac{1}{k^2} > \frac{1}{k^2} - \varepsilon_k > \frac{1}{(k+1)^2} + \varepsilon_{k+1}, \quad k = 1, 2, \dots$$

$$|F(e^{it})| \leq 1 \quad t \in \left(\frac{1}{k^2} - \varepsilon_k, \frac{1}{k^2} + \varepsilon_k\right)$$

and put $\Omega = \bigcup_{k=1}^{\infty} \left(\frac{1}{k^2} - \varepsilon_k, \frac{1}{k^2} + \varepsilon_k\right)$. If we define a function g on $T \setminus \{1\}$ by

$$g(e^{it}) = \begin{cases} \min \left\{ \frac{1}{|F(e^{it})|}, 1 \right\} : & t \in [0, 2\pi) \setminus \Omega \\ \frac{1}{\varepsilon_k k^4} : & t \in \left(\frac{1}{k^2} - \varepsilon_k, \frac{1}{k^2} + \varepsilon_k\right), \quad k = 1, 2, \dots \end{cases}$$

then $\log g(e^{it})$ belongs to $L^1(T)$. Using this $g(e^{it})$ we define an outer function $f(z) \in H^1$ by

$$f(z) = \exp \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log g(e^{it}) dt / 2\pi. \quad (z \in U)$$

Then it is easy to see that $f(z)$ locally belongs to H^1 except one point $z = 1$. But since $f(z)F(z) \in H^1$ and $\arg f(e^{it}) = \arg f(e^{it})F(e^{it})$ a.e. on $[0, 2\pi)$ we can see that

$$\dim \langle S_{|f|/f} \rangle = \dim \langle S_{|fF|/fF} \rangle = \infty.$$

4 Proof of Lemma 1

In this section, we prove Lemma 1 used in the proof of the main theorem in §2. In the following, C^+ and C_+ denote the half planes given by $C^+ = \{z : \Im z > 0\}$ and $C_+ = \{z : \Re z > 0\}$. $H^p(C^+)$ (resp. $H^p(C_+)$) is the Hardy space on C^+ (resp. C_+) in the usual sense (cf. [3, p.51]). $\psi(z)$ denotes the linear fractional transformation defined by $\psi(z) = (z-i)/(z+i)$, which maps C^+ conformally onto U .

Lemma 1. Let F be a function in N^+ such that

- (a) F belongs to H^p for some $p > 0$,
- (b) F locally belongs to $H^{1/2}$ except a finite subset A of T ,
- (c) F is outer and $F(e^{it}) \geq 0$ a.e. on T .

Then F can be extended to a rational function.

Proof. By conditions on F , $F(z)$ can be extended beyond every boundary point in $T \setminus A$ to a holomorphic function on $C \cup \{\infty\} \setminus A$, which we also express by $F(z)$. Therefore if we can prove that each point $\alpha \in A$ is at most a pole of $F(z)$, it follows that F is a rational function.

In the rest of the proof of this lemma, $M_i (i = 1, 2, \dots, 5)$ stands for an appropriate positive constant. Let α be a point of A . To prove that α is a pole of F , we can assume without loss of generality that $\alpha = 1$. If we put $G(z) = F(z)^p(1-z)^2$, then trivially $G(z)/(1-z)^2 = F(z)^p \in H^1(U)$, and we have $G(\psi(z)) = F(\psi(z))^p(1-\psi(z))^2 \in H^1(C^+)$ (cf. [4] p.130). From this we get

$$|F(\psi(z))|^p |1-\psi(z)|^2 \leq M_1 \quad (\Im z \geq 1)$$

and hence

$$|F(\psi(z))| \leq \frac{M_1^{1/p}}{|1-\psi(z)|^{2/p}} \leq M_2 |z|^{2/p} \quad (\Im z \geq 1) \quad (3)$$

By the conditions (b) and (c), it follows that $\bar{F}(\psi(z)) = F(\psi(\bar{z}))$ on $C \setminus A$, and hence we get

$$|F(\psi(z))| \leq M_2 |z|^{2/p} \quad (\Re z \leq -1) \quad (4)$$

Next, since $F(\psi(z+a))^p (1-\psi(z+a))^2 \in H^1(C^+)$ for each $a > 0$, we have by Fejer-Riesz inequality (cf. [2], p.46) modified to hold for a function in $H^p(C^+)$.

$$\begin{aligned} & \int_0^\infty |F(\psi(it+a))^p| |1-\psi(it+a)|^2 \frac{1}{(1+t)^2} dt \\ & \leq \frac{1}{2} \int_{-\infty}^\infty |F(\psi(x+a))^p| |1-\psi(x+a)|^2 \frac{1}{1+x^2} dx \\ & \leq \sup_{a>0} \int_{-\infty}^\infty |F(\psi(x+a))^p| |1-\psi(x+a)|^2 dx \leq \|G \circ \psi\|_1 < \infty \end{aligned} \quad (5)$$

Since

$$\begin{aligned} & |1-\psi(it+a)| = |1-(it+a-i)/(it+a+i)| \\ & = |2i/(it+a+i)| \geq |1/(it+a+1)| \quad (t > 0, a > 0), \end{aligned}$$

we get

$$\begin{aligned} M_3 & \geq \int_0^\infty |F(\psi(it+a))^p| |1-\psi(it+a)|^2 \frac{1}{(t+1)^2} dt \\ & \geq \frac{1}{2} \int_0^\infty |F(\psi(it+a))^p| \frac{1}{|it+a+1|^4} dt \quad (a > 0) \end{aligned} \quad (6)$$

Therefore if we choose $a > 0$ large enough to assure $\Re \alpha < a$ ($\alpha \in A$), we get that $F(\psi(z+a))/(z+a+1)^{4/p} |_{C_+}$ belongs to $H^p(C_+)$. From this we get

$$|F(\psi(z))| \leq M_4 |z|^{4/p} \quad (\Re z \geq a) \quad (7)$$

In the same way, we have

$$|F(\psi(z))| \leq M_5 |z|^{4/p} \quad (\Re z \leq -b) \quad (8)$$

for a large positive number b . From (3), (4), (7) and (8), we can conclude that ∞ is at most a pole of $F(\psi)$, that is $\alpha = 1$ is at most a pole of F . This completes the proof of Lemma 1.

REFERENCES

1. deLeeuw, K. and Rudin, W., Extreme points and extremum problems in H^1 , Pacific J. Math. 8 (1958), 467-485.
2. Duren, P. L., Theory of H^p spaces, Academic Press Inc., New York and London 1970.
3. Garnett, J. B., Bounded analytic functions, Academic press. Inc. 1981.
4. Hoffman, K., Banach spaces of analytic functions, Prentice-Hall Inc., Englewood Cliffs, New Jersey 1962.
5. Inoue, J., An example of a non-exposed extreme function on the unit ball of H^1 , preprint.
6. Koosis, P., introduction to H^p spaces, Cambridge University Press, 1980
7. Nakazi, T., Exposed points and extremal problems in H^1 , J. Funct. Anal. 53 (1983), 224-230.
8. Neuwirth, J. and Newman, D. J., Positive $H^{1/2}$ functions are constants, Proc. Amer. Math. Soc. 18 (1967) 958

Jyunji Inoue
Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo 060
Japan

Takahiko Nakazi
Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo 060
Japan