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A CHARACTERIZATION OF COMPLETE INTEGRABILITY FOR PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER

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A CHARACTERIZATION OF COMPLETE INTEGRABILITY FOR PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER

Shyuichi IZUMIYA

Abstract. We give a characterization of the notion of complete integrability for overdetermined systems of first order partial differential equations of real valued functions.

0. Introduction

The notion of complete integrability plays an important role in the classical theory of first order differential equations (cf. Carathéodory [2] , Courant-Hilbert [3], Forsyth [4] [5]). In [7] we have given a characterization of complete integrability for single partial differential equations of first order. However, we have never seen a characterization of complete integrability for overdetermined systems. Our purpose in this paper is to generalize the result in [7] for overdetermined systems of first order differential equations.

In §1, we shall state the main result and the proof of the theorem will be given in §2. The method of the proof is a generalization of those in [7]. Some typical examples will be given in §3.

All maps considered here are differentiable of class C^∞, unless stated otherwise.

1. Main results

In this section we shall formulate our results. A system of first order differential equation is most naturally interpreted as being a closed subset of $J^1(\mathbb{R}^n, \mathbb{R})$. Unless the contrary is specifically stated, we use the following definition. *A system of first order differential equation* (or briefly, *an equation*) is a submanifold germ $(E, z_0) \subset (J^1(\mathbb{R}^n, \mathbb{R}), z_0)$ in the 1-jet space of functions of n-variables. Let θ be the canonical contact form on $J^1(\mathbb{R}^n, \mathbb{R})$ which is given by $\theta = dy - \sum_{i=1}^n p_i dx_i$, where (x, y, p) are canonical coordinates of $J^1(\mathbb{R}^n, \mathbb{R})$. We define *a geometric solution of* (E, z_0) to be an immersion $i : (L, q_0) \to (J^1(\mathbb{R}^n, \mathbb{R}), z_0)$ of an n-dimensional manifold such that $i^*\theta = 0$ and $i(L) \subset (E, z_0)$ (i.e. a Legendrian submanifold which is contained in (E, z_0)). We say that $z \in (E, z_0)$ is *a contact singular point* if $\theta(T_z E) = 0$. We denote the set of contact singular points by $\Sigma_c(E)$. We say that an equation (E, z_0) is *involutory at* $z \in (E, z_0)$ if there is a Legendrian submanifold L tangent to (E, z_0) at z. We also say that an equation (E, z_0) is *involutory* if it is involutory at any point of (E, z_0). Since single equations are automatically involutory, the notion of involutory is essential for overdetermined systems of first order partial differential equations (i.e. $\operatorname{codim} E \geq 2$) (cf. [9],[10]). An equation (E, z_0) is said to be *completely integrable* if there exists a foliation by geometric solutions on (E, z_0). In this case such a foliation is called *a complete solution* of (E, z_0).

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We can state the main theorem as follows.

Theorem 1.1. *For an equation $(E, z_0) \subset J^1(\mathbb{R}^n, \mathbb{R})$, the following are equivalent.*
(1) (E, z_0) is completely integrable.
(2) (E, z_0) is involutory and $\Sigma_c(E) = \emptyset$ or $\Sigma_c(E)$ is an n-dimensional submanifold.

The theorem gives a characterization of complete integrability of equations. By the definition, if $\Sigma_c(E)$ is an n-dimensional submanifold, it is automatically a geometric solution of (E, z_0). In fact it is *the singular solution in the strict sense* (cf. [7],[8],[11]). We can prove the similar assertions as in these papers by exactly the same way, then we will not argue about singular solutions here.

We remark that we can easily check the above conditions if the equation is the zero point set of a submersion.

Proposition 1.2. *Let $F = (F_1, \ldots, F_d) : (J^1(\mathbb{R}^n, \mathbb{R}), z_0) \to (\mathbb{R}^d, 0)$, be a submersion germ. Then*
1) $(F^{-1}(0), z_0)$ is involutory if and only if $[F_i, F_j]_z = 0$ $(i, j = 1, \ldots, d)$ for any $z \in F^{-1}(0)$, where

$$\begin{aligned}[F, G] =& F \cdot \frac{\partial G}{\partial z} - G \cdot \frac{\partial F}{\partial z} + \sum_{i=1}^{n} \left(\frac{\partial F}{\partial x_i} \cdot \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \cdot \frac{\partial G}{\partial x_i}\right) \\ &+ \sum_{i=1}^{n} p_i \cdot \left(\frac{\partial F}{\partial z} \cdot \frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial z} \cdot \frac{\partial F}{\partial p_i}\right).\end{aligned}$$

2) $(F^{-1}(0), z_0)$ is contact singular at z if and only if

$$rank \begin{pmatrix} \frac{\partial F_j}{\partial x_i} + p_i \frac{\partial F_j}{\partial y} \\ \frac{\partial F_j}{\partial p_i} \end{pmatrix} < d$$

at z.

Proof. For the proof of 1), see Proposition 1.6.3 in [9].

The proof of 2) is given by the fact that $F^{-1}(0)$ is contact singular at z if and only if there exists $(\lambda_1, \ldots, \lambda_d) \neq (0, \ldots, 0)$ in $\mathbb{R}^d$ such that

$$\theta \wedge (\lambda_1 dF_1 + \cdots + \lambda_d dF_d) = 0 \text{ at } z.$$

The basic ideas of the proofs of Theorem 1.1 are as follows : If (E, z_0) is involutory and $\Sigma_c(E)$ is an n-dimensional submanifold, we can assert that the equation can a single equation of the form $y = h(x, p)$, where h is a smooth function germ on the cotangent bundle $T^*\mathbb{R}^{n-d+1}$. Then we can apply the method for the single equation case in [7]. By the theorem of Kostant-Sternberg [6], a neighbourhood of L_h in $T^*\mathbb{R}^{n-d+1}$ has the same structure (i.e. corresponds to the contact structure on $J^1(\mathbb{R}^{n-d+1}, \mathbb{R})$) as a neighbourhood of the zero section of T^*L_h, where L_h is the corresponding Lagrangian submanifold to $\Sigma_c(E)$. Then we can adopt the local foliation around L_h which corresponds to the fibres of

T^*L_h as the required complete integral. Of course, if (E, z_0) is involutory and $\Sigma_c(E) = \emptyset$, then (E, z_0) is completely integral by the classical existence theorem (cf. [9]).

For the proof of the assertion that (1) implies (2) in Theorem 1.1, we use a little bit of the theory of Arnol'd-Zakalyukin, so that any Legendrian submanifold germ is contact equivalent to a non-singular one (i.e. ignoring the Legendrian fibration). This means that any complete integral germ is contact equivalent to a classical one. Then we prove that $\Sigma_c(F) = \emptyset$ or $\Sigma_c(E)$ is an n-dimensional submanifold if $F = 0$ has a classical complete solution. Since $\Sigma_c(E)$ is a contact invariant set, the statement follows.

2. Proof of Theorem 1.1

In this section we shall give a proof of Theorem 1.1. We need to prepare somewhat for the proof that (2) implies (1).

Let (E, z_0) be an equation such that z_0 is a contact singular point. The following proposition describes the local normal form of (E, z_0) up to contact diffeomorphism.

Proposition 2.1. ([9,10]) *Let (E, z_0) be an involutory equation such that $z_0$is a contact singular point. Then there is a contact diffeomorphism germ $f : (J^1(\mathbb{R}^n, \mathbb{R}), z_0) \to (J^1(\mathbb{R}^n, \mathbb{R}), 0)$ such that $f(E) = \{p_1 = \cdots = p_{d-1} = y - h(x', p') = 0\}$, where $x' = (x_d, \ldots, x_n)$, $p' = (p_d, \ldots, p_n)$ and $h(x', p')$ is a function germ at 0.*

For our purpose, we may consider the equation of the form in the above proposition. We now define a map germ

$$G_h : (\mathbb{R}^{d-1} \times T^*\mathbb{R}^{n-d+1}, 0) \to (J^1(\mathbb{R}^n, \mathbb{R}), 0)$$

by

$$G_h(x_1, \ldots, x_{d-1}, x', p') = (x_1, \ldots, x_{d-1}, x', h(x', p'), p').$$

We define a 1-form on $T^*\mathbb{R}^{n-d+1}$ by $\theta_h = dh - \sum_{i=d}^{n} p_i dx_i$. Then we have the following one to one correspondence.

$$\{L \mid L \text{ is a solution of } p_1 = \cdots = p_{d-1} = y - h(x', p') = 0\}$$

$$G_h \uparrow\downarrow \Pi_*$$

$$\left\{\mathbb{R}^{d-1} \times L' \mid i : L' \subset T^*\mathbb{R}^{n-d+1} \text{ is a maximal integral submanifold of } \theta_h = 0\right\},$$

where $\Pi(x, y, p) = (x, p')$ and $\Pi_*(L) = \Pi(L)$. It follows that a solution of a equation $p_1 = \cdots = P_{d-1} = y - h(x', p') = 0$ may be regarded as a maximal isotropic submanifold of $(T^*\mathbb{R}^{n-d+1}, \theta_h)$. Since $-d\theta_h = \sum_{i=d}^{n} dp_i \wedge dx_i$ is the canonical symplectic two form, a solution of $p_1 = \cdots = p_{d-1} = y - h(x', p') = 0$ corresponds to a Lagrangian submanifold of $(T^*\mathbb{R}^{n-d+1}, \omega)$, where $\omega = -d\theta_h$. For the definition and properties of Lagrangian submanifolds, see [1]. We now quote the following very important result.

Theorem 2.2. (Kostant-Sternberg [6]) *Let (P, ω) be a symplectic manifold, L a Lagrangian submanifold and α a smooth 1-form on a neighbourhood of L in P with $\alpha \mid L = 0$ and $d\alpha = \omega$. Then there exists a tubular neighbourhood V of L in P, a neighbourhood U of zero section L in T^*L and a unique "local" vector bundle isomorphism $K : (V, L) \to (U, L)$ such that K is the identity on L and $K^*\theta_L = \alpha$. Here, θ_L is the canonical 1-form on T^*L.*

Now we can prove that (2) implies (1).

Proof; (2) $\Rightarrow$ (1). If $\Sigma_c(E)$is an n-dimensional submanifold of $J^1(\mathbb{R}^n, \mathbb{R})$, then $\pi_{T^*\mathbb{R}^{n-d+1}} \circ G_h^{-1}(\Sigma_c(E)) = L'_h$ is a Lagrangian submanifold of $T^*\mathbb{R}^{n-d+1}$, where $\pi_{T^*\mathbb{R}^{n-d+1}} : \mathbb{R}^{d-1} \times T^*\mathbb{R}^{n-d+1} \to T^*\mathbb{R}^{n-d+1}$ is the canonical projection. We may apply the Kostant-Sternberg theorem to conclude that there exist a tubular neighbourhood V of L'_h in $T^*\mathbb{R}^{n-d+1}$ and a unique (local) vector bundle isomorphism $K : V \to (T^*L'_h, \theta_{L'_h})$ such that K is the identity on L'_h and $K^*\theta_{L'_h} = -\theta_h$. Since the fibres of the cotangent bundle $T^*L'_h \to L'_h$ are maximal integral submanifolds of $\theta_{L'_h} = 0$, these fibres make a foliation whose leaves are corresponding to solutions of the original equation.

In order to prove the converse direction. We need some preliminaries. We say that an $n-d+1$-parameter family of function germs $f : (\mathbb{R}^{n-d+1} \times \mathbb{R}^n, (t_0, x_0)) \to (\mathbb{R}, y_0)$ is *a classical complete solution* of (E, z_0) if

$$\{(x, f(t,x), \frac{\partial f}{\partial x}(t,x)) | (t,x) \in (\mathbb{R}^{n-d+1} \times \mathbb{R}^n, (t_0, x_0))\} \subset (E, z_0)$$

and $\operatorname{rank}(\frac{\partial f}{\partial t_i}, \frac{\partial^2 f}{\partial t_i \partial x_j}) = n - d + 1$. Then we define a map germ

$$j_*^1 f : (\mathbb{R}^{n-d+1} \times \mathbb{R}^n, (t_0, x_0)) \to (J^1(\mathbb{R}^n, \mathbb{R}), (x_0, y_0, p_0))$$

by $j_*^1 f(t,x) = (x, f(t,x), \frac{\partial^2 f}{\partial t_i \partial x_j}(t,x))$. Since $f(t,x)$ is a classical complete solution, $j_*^1 f$ is a local parametrization of (E, z_0). It follows that the family $\{\text{Image}\, j_*^1 f_t\}_{t \in (\mathbb{R}^n, t_0)}$ gives a local foliation on (E, z_0). For a Legendrian immersion germ $i : (L, q_0) \to J^1(\mathbb{R}^n, \mathbb{R})$, $q_0 \in L$ is said to be *a Legendrian singular point* if $\pi \circ i$ is not an immersion at q_0. We remark that q_0 is a Legendrian non-singular point if and only if $\tilde{\pi} \circ i$ is a local diffeomorphism at q_0, where $\tilde{\pi}(x, y, p) = x$. Then we have the following lemma.

Lemma 2.3. 1) *If (E, z_0) has a classical complete solution, then $\Sigma_c(E) = \emptyset$ or $\Sigma_c(F)$ is an n-dimensional submanifold.*
2) *If all leaves of a complete solution of (E, z_0) are Legendrian non-singular, then it is a classical complete solution.*

Proof. 1) By the above argument, $j_*^1 f$ is a local parametrization of (E, z_0). Since $j_*^1 f^*\theta = \sum_{i=1}^{n-d+1} \frac{\partial f}{\partial t_i}(t,x) dt_i$, then $j_*^1 f(t,x) \in \Sigma_c(E)$ if and only if $\frac{\partial f}{\partial t_i}(t,x) = 0$ for $i = 1, \ldots, n - d + 1$. Since $\operatorname{rank}(0, \frac{\partial f^2}{\partial t_i \partial x_j}) = \operatorname{rank}(\frac{\partial f}{\partial t_i}, \frac{\partial f^2}{\partial t_i \partial x_j}) = n - d + 1$ at $j_*^1 f(t,x) \in \Sigma_c(E)$, we have $\operatorname{rank}(\frac{\partial f^2}{\partial t_i \partial x_j}, \frac{\partial f^2}{\partial x_i \partial x_j}) = n - d + 1$. It follows that $\Sigma_c(E)$ is an n-dimensional submanifold.

2) Suppose that there exists a complete solution whose leaves are Legendrian non-singular. Then we have an n-parameter family of smooth sections

$$s : (\mathbb{R}^{n-d+1} \times \mathbb{R}^n, (t_0, x_0)) \to (J^1(\mathbb{R}^n, \mathbb{R}), z_0)$$

of $\tilde{\pi}$ (i.e. $\tilde{\pi} \circ s(t,x) = x$) such that s is an immersion, $s(\mathbb{R}^n \times \mathbb{R}^n) = (E, z_0)$ and $s_t^*\theta = 0$ for any $t \in (\mathbb{R}^{n-d+1}, t_0)$, where $s_t(x) = s(t,x)$. It follows that there exists a family of function germs $f : (\mathbb{R}^{n-d+1} \times \mathbb{R}^n, (t_0, x_0)) \to (\mathbb{R}^n, y_0)$ such that $j_*^1 f(t,x) = s(t,x)$. Since s is an immersion, f is a (classical) complete solution of (E, z_0).

Proof of Theorem 1.1; (1) $\Rightarrow$ (2). Suppose that there exists a complete solution of (E, z_0). Let (L, z_0) be a germ of a leaf of the complete solution at the point z_0. By the Arnol'd-Zakalyukin theory ([1], Corollary 20.2), there exist a partition (I, J) of the set $\{1, \ldots, n\}$ and a function germ $S(x_I, p_J)$ such that

$$L = \{(x_I, -\frac{\partial S}{\partial p_J}, S(x_I, p_J) - < \frac{\partial S}{\partial p_J}, p_J >, \frac{\partial S}{\partial x_I}, p_J)\},$$

where $< x_J, p_J >$ is the canonical inner product. We now define a contact diffeomorphism germ by

$$C_{(I,J)}(x, y, p) = (x_I, p_J, y - < x_J, p_J >, p_I, -x_J).$$

Then we have $C_{(I,J)}(L) = \{(x_I, p_J, S(x_I, p_J), \frac{\partial S}{\partial x_I}, \frac{\partial S}{\partial p_J})$. It follows that $C_{(I,J)}(L)$ is Legendrian non-singular at $C_{(I,J)}(z_0)$. By Lemma 2.3, we have a classical complete solution of $(C_{(I,J)}(E), C_{(I,J)}(z_0))$. Then $\Sigma_c(E) = C_{(I,J)}^{-1}(\Sigma_c(C_{(I,J)}(E))$ is also an n-dimensional submanifold. This complete the proof.

3. Examples

In this section we shall give typical examples of completely integrable systems.

Examples 3.1. (The Clairaut system) Consider the following system of equations :

$$E = \{p_1 = \cdots = p_{d-1} = y - \sum_{i=d}^{n} x_i p_i + f(p_d, \ldots, p_n) = 0\},$$

where f is a smooth function. The classical complete solution is given by $f(t,x) = \sum_{i=d}^{n} x_i t_{i-d+1} + f(t_1, \ldots, t_{n-d+1})$. The contact singular set is given by

$$\{\frac{\partial f}{\partial p_d} - p_d = \cdots = \frac{\partial f}{\partial p_n} - p_n = p_1 = \cdots = p_{d-1} = y - \sum_{i=d}^{n} x_i p_i + f(p_d, \ldots, p_n) = 0\}.$$

Example 3.2. Consider the equation :

$$E = \{p_1 = \cdots = p_{d-1} = y - f(x_1, \ldots, x_n) = 0\}.$$

The complete solution is given by

$$\{(u_1, \ldots, u_{d-1}, t, f(t), 0, u_d, \ldots, u_n) | (t, u) \in \mathbb{R}^{n-d+1} \times \mathbb{R}^n\},$$

where $t = (t_1, \ldots, t_{n-d+1})$ is the parameter.

The contact singular set is given by

$$\{p_1 = \cdots p_{d-1} = y - f(x_d, \ldots, x_n) = \frac{\partial f}{\partial p_d} - p_d = \cdots = \frac{\partial f}{\partial p_n} - p_n = 0\}.$$

Example 3.3. Consider the following equation :

$$E = \{p_1 = y - 2p_2^3 = 0\} \quad (n = 2).$$

We have a complete solution $s : (\mathbb{R} \times \mathbb{R}^2, 0) \longrightarrow J^1(\mathbb{R}^2, \mathbb{R})$ defined by $s(t,u,v) = (3u^2 + t, v, 2u^3, 0, u)$, where t is the parameter. In this case the contact singular set is $\Sigma_c(E) = \{p_1 = p_2 = y = 0\}$.

Of course, we can easily check that all of the above examples are involutory.

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