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WAVE PROPAGATION IN EVEN DIMENSIONAL SPACES

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Abstract. Asymptotic expansions of solutions of the wave equations in even dimensional spaces are obtained with the initial data of non-compact support. A relationship is proved between the vanishing order at the origin of the Fourier transform of the data and the decay rate of the corresponding solutions in semi-infinite cylinders or along rays inside the forward light cone.

1. Introduction.

In this paper we describe the formation of the wave propagation in $1+n$ dimensional space-time in the case where the number n of space dimensions is even and the support of the data need not be compact. More precisely, we study the asymptotic properties of solutions of the wave equation

$$\text{(W)} \qquad \begin{cases} \Box u = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ u(0,x) = \phi(x), \ \partial_t u(0,x) = \psi(x), \ x \in \mathbb{R}^n, \end{cases}$$

where $\Box = \partial_t^2 - \Delta$ is the d'Alembertian in $\mathbb{R} \times \mathbb{R}^n$ and $\phi, \psi \in \mathscr{S}$, the space of C^∞ functions rapidly decreasing at infinity. In an even dimensional world Huygens' principle does not hold and every stationary observer would be haunted by the lasting disturbance even when the initial disturbance had been localized in a compact subset. Nevertheless, the well known local decay estimates show that the amplitude of disturbance decays like $O(t^{1-n})$ on compact subsets of $\mathbb{R}^n$.

One of the purpose in this paper is to present a relationship between the rate of decay of solution u on subsets inside the light cone and the vanishing order at the origin of the Fourier transform $(\hat{\phi}, \hat{\psi})$ of the initial data (ϕ, ψ). For example, we show that the condition of local decay estimates of the form $\sup_{|x|\le\delta} |u(t,x)| = O(t^{-n-k})$ is equivalent to the condition that the derivatives up to k-th (resp. $(k-1)$-th) order of $\hat{\phi}$ (resp. $\hat{\psi}$) vanish at the origin. In the special case where $\phi, \psi \in C_0^\infty$ and u decays faster than any inverse power of t on a compact subset of $\mathbb{R}^n$, our result implies that all derivatives of $(\hat{\phi}, \hat{\psi})$ vanish at the origin, which in turn implies that u vanishes identically since $\hat{\phi}$ and $\hat{\psi}$ are the restrictions to $\mathbb{R}^n$ of holomorphic functions on $\mathbb{C}^n$. This is precisely Littman-Lui's result which provides the affirmative answer to Hörmander's problem [7]. The problem of this kind is related to the problem of Rellich type, the Sommerfeld radiation problem, and to the saturation properties for hyperbolic partial differential operators. See [1,2,4,6,8,9,10].

The basic result of this paper is an asymptotic expansion of solutions of (W) inside the light cone. The relationship mentioned above is obtained by observing the individual terms of suitable finite sum and the remainder estimate in the expansion. In order to estimate the remainder term we use the weighted Sobolev spaces $H^{m,s}, m, s \in \mathbb{R}$, defined by

$$H^{m,s} = \{f \in \mathscr{S}'; ||f||_{m,s} = ||\langle x\rangle^s(1-\Delta)^{m/2}f||_2 < \infty\},$$

where $\mathscr{S}'$ is the space of temperate distributions, $\langle x\rangle = (1+|x|^2)^{1/2}$, and $||\cdot||_p$ denotes the norm in $L^p(\mathbb{R}^n)$. We now state

Theorem 1. *Let $n \geq 2$ be even. Let $\phi, \psi \in \mathscr{S}$ and let u be the solution of (W). Then:*
(1) For any $N \geq n-1$

$$\begin{aligned}(1.1) \qquad &\operatorname*{Sup}_{(t,x)\in\Gamma^+}(t-|x|)^N|u(t,x) - \sum_{j=n-1}^{N}(F_j\psi)(t,x) - \sum_{j=n}^{N}(G_j\phi)(t,x)| \\ &\leq C_N(||\psi||_{n/2,\mathrm{Max}(N,N+2-n/2)} + ||\phi||_{n/2+1,N}),\end{aligned}$$

where $\Gamma^+ = \{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n; t > |x|\}$,

$$\begin{aligned}(F_j\psi)(t,x) = &\sum_{\ell=0}^{[(j-n+1)/2]}(-i)^{j-1}\frac{2^{j-2\ell-n/2}\Gamma(j-\ell-(n-1)/2)}{\pi^{1/2}\ell!(j-n-2\ell+1)!}(t^2-|x|^2)^{(n-1)/2-j+\ell} \\ &\times (x\cdot\nabla)^{j-n-2\ell+1}\Delta^\ell\hat{\psi}(0), \\ (G_j\phi)(t,x) = &\sum_{\ell=0}^{[(j-n)/2]}(-i)^j\frac{2^{j-2\ell-n/2}\Gamma(j-\ell-(n-1)/2)}{\pi^{1/2}\ell!(j-n-2\ell)!}t(t^2-|x|^2)^{(n-1)/2-j+\ell} \\ &\times (x\cdot\nabla)^{j-n-2\ell}\Delta^\ell\hat{\phi}(0).\end{aligned}$$

Here $[s]$ denotes the largest integer less than or equal to s, $\Gamma(\cdot)$ is the gamma function, and we follow the convention $\sum_{j=n}^{n-1}(\cdots) = 0$.
(2) For any $N \geq 1$,

$$\begin{aligned}(1.2) \qquad &\operatorname*{Sup}_{(t,x)\in(\mathbb{R}_+\times\mathbb{R}^n)\setminus\Gamma^+}(1+|x|-t)^N|u(t,x)| \\ &\leq C(\sum_{|\alpha|\leq n/2-1}||\langle x\rangle^{N+n}\partial^\alpha\psi||_\infty + \sum_{|\alpha|\leq n/2}||\langle x\rangle^{N+n}\partial^\alpha\phi||_\infty),\end{aligned}$$

where C is independent of N.

Remark 1. The first few terms in the expansion in (1.1) are given by

$$(F_{n-1}\psi)(t,x) = -i^n 2^{n/2-1}\pi^{-1/2}\Gamma((n-1)/2)(t^2-|x|^2)^{-(n-1)/2}\hat{\psi}(0),$$

$$(F_n\psi)(t,x) = -i^{n-1}2^{n/2}\pi^{-1/2}\Gamma((n+1)/2)(t^2-|x|^2)^{-(n+1)/2}x\cdot\nabla\hat{\psi}(0),$$

$$(F_{n+1}\psi)(t,x) = i^n 2^{n/2-1}\pi^{-1/2}\Gamma((n+1)/2)((t^2-|x|^2)^{-(n+1)/2}\triangle\hat{\psi}(0),$$
$$+(n+1)(t^2-|x|^2)^{-(n+3)/2}(x\cdot\nabla)^2\hat{\psi}(0)),$$

$$(F_{n+2}\psi)(t,x) = i^{n-1}2^{n/2}\pi^{-1/2}\Gamma((n+3)/2)((t^2-|x|^2)^{-(n+3)/2}x\cdot\nabla\triangle\hat{\psi}(0),$$
$$+(n/3+1)(t^2-|x|^2)^{-(n+5)/2}(x\cdot\nabla)^3\hat{\psi}(0)),$$

$$(G_n\phi)(t,x) = i^n 2^{n/2}\pi^{-1/2}\Gamma((n+1)/2)t(t^2-|x|^2)^{-(n+1)/2}\hat{\phi}(0),$$

$$(G_{n+1}\phi)(t,x) = i^{n-1}2^{n/2+1}\pi^{-1/2}\Gamma((n+3)/2)t(t^2-|x|^2)^{-(n+3)/2}x\cdot\nabla\hat{\phi}(0),$$

$$(G_{n+2}\phi)(t,x) = -i^n 2^{n/2}\pi^{-1/2}\Gamma((n+3)/2)t((t^2-|x|^2)^{-(n+3)/2}\triangle\hat{\phi}(0),$$
$$+(n+3)(t^2-|x|^2)^{-(n+5)/2}(x\cdot\nabla)^2\hat{\phi}(0)),$$

$$(G_{n+3}\phi)(t,x) = -i^{n-1}2^{n/2+1}\pi^{-1/2}\Gamma((n+5)/2)t((t^2-|x|^2)^{-(n+5)/2}x\cdot\nabla\triangle\hat{\phi}(0),$$
$$+((n+5)/3)(t^2-|x|^2)^{-(n+7)/2}(x\cdot\nabla)^3\hat{\phi}(0)).$$

Remark 2. When $N = n-1$, we have from (1.1), the explicit form of F_{n-1} given above, and the well-known L^∞-estimate $\operatorname{Sup}_{t\geq 0}||u(t)||_\infty < \infty$ that

$$\operatorname*{Sup}_{(t,x)\in\Gamma+}(1+t-|x|)^{n-1}|u(t,x)| < \infty.$$

This is a sharp form of the well known local decay estimate (see [12, Theorem XI.19] for instance).

Remark 3. There are related results on the asymptotic wave profiles on the basis of the translation representations of Lax-Phillips type. See [5, 11, 13] for details.

Remark 4. Estimate (1.2) shows that u has a rapid decay outside the light cone if the data have a rapid decay at infinity. Compare with [12, Theorem XI.19]. See also Remark 6 in Section 2.

Theorem 2. *Let $n \geq 2$ be even. Let ϕ, $\psi \in \mathscr{S}$ and let u be the solution of (W). Let $\Gamma^+(\varepsilon) = \{(t,x)\in\mathbb{R}_+\times\mathbb{R}^n; |x|\leq(1-\varepsilon)t\}$ for $0<\varepsilon<1$. Then:*

(1) The following conditions are equivalent.

(i) $\operatorname*{Sup}_{(t,x)\in\Gamma+}(1+t-|x|)^n|u(t,x)| < \infty.$

(ii) *For any* ε *with* $0<\varepsilon<1$,

$$\operatorname*{Sup}_{(t,x)\in\Gamma^+(\varepsilon)}(1+t+|x|)^n|u(t,x)|<\infty.$$

(iii) *For any* $\nu\in\mathbb{R}^n$ *with* $|\nu|<1$,

$$\operatorname*{Sup}_{t\geq 0}(1+t)^n|u(t,t\nu)|<\infty.$$

(iv) *There is* $\nu_0\in\mathbb{R}^n$ *such that* $|\nu_0|<1$ *and*

$$\liminf_{t\to\infty} t^{n-1}|u(t,t\nu_0)|=0.$$

(v) *There is* $x_0\in\mathbb{R}^n$ *such that*

$$\liminf_{t\to\infty} t^{n-1}|u(t,x_0)|=0.$$

(vi) $\hat{\psi}(0)=0$.

(2) *For any integer* $k\geq 1$, *the following conditions are equivalent.*

(i) $\displaystyle\operatorname*{Sup}_{(t,x)\in\Gamma^+}(1+t-|x|)^{n+k}|u(t,x)|<\infty.$

(ii) *For any* ε *with* $0<\varepsilon<1$,

$$\operatorname*{Sup}_{(t,x)\in\Gamma^+(\varepsilon)}(1+t+|x|)^{n+k}|u(t,x)|<\infty.$$

(iii) *For any* $\nu\in\mathbb{R}^n$ *with* $|\nu|<1$,

$$\operatorname*{Sup}_{t\geq 0}(1+t)^{n+k}|u(t,t\nu)|<\infty.$$

(iv) *For any* $\nu\in\mathbb{R}^n$ *with* $|\nu|<1$,

$$\liminf_{t\to\infty} t^{n+k-1}|u(t,t\nu)|=0.$$

(v) *For any* $\delta>0$ *and any* $x\in\mathbb{R}^n$ *with* $|x|<\delta$,

$$\liminf_{t\to\infty} t^{n+k-1}|u(t,x)|=0.$$

(vi) *For any multi-indices* α *and* β *such that* $|\alpha|\leq k$ *and* $|\beta|\leq k-1$,

$$\partial^\alpha\hat{\psi}(0)=\partial^\beta\hat{\phi}(0)=0.$$

Corollary (Littman-Lui [7]). *Let $n \geq 2$ be even. Let ϕ, $\psi \in C_0^\infty(\mathbb{R}^n)$ and let u be the solution of (W). Suppose one of the conditions (i)-(v) of part (2) of Theorem 2 holds for all $k \geq 1$. Then u vanishes identically.*

Remark 5. When $\hat{\psi}$, $\hat{\phi} \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$, the estimate in (ii) has been obtained in [12, Theorem XI.18] by stationary phase methods.

We prove Theorems 1 and 2 in Sections 2 and 3, respectively. We conclude this introduction by giving notations freely used in this paper. $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, $\nabla = (\partial_1, \cdots, \partial_n)$, $\partial_j = \partial/\partial x_j$, $\partial_t = \partial/\partial t$. For a muli-index $\alpha = (\alpha_1, \cdots, \alpha_n)$,
$|\alpha| = \sum_{j=1}^n \alpha_j$, $\alpha! = \prod_{j=1}^n \alpha_j!$, $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$, $\beta \leq \alpha$, $\partial^\alpha = \prod_{j=1}^n \partial_j^{\alpha_j}$, $x^\alpha = \prod_{j=1}^n x_j^{\alpha_j}$.
For $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$, $|z| = (\sum_{j=1}^n z_j \overline{z}_j)^{1/2}$. $\mathscr{F}$ denotes the Fourier transformation $(\mathscr{F}\psi)(\xi) = (2\pi)^{-n/2} \int \exp(-ix \cdot \xi)\psi(x)dx$. Different positive constants might be denoted by the same letter C.

2. Proof of Theorem 2.

Proof of Part (1). For $\phi, \psi \in \mathscr{S}$ the solution u of (W) is written as $u = v + w$, where

$$v(t,x) = (2\pi)^{-n/2} \int e^{ix\cdot\xi} |\xi|^{-1} \sin(t|\xi|)\hat{\psi}(\xi)d\xi, \tag{2.1}$$

$$w(t,x) = (2\pi)^{-n/2} \int e^{ix\cdot\xi} \cos(t|\xi|)\hat{\phi}(\xi)d\xi. \tag{2.2}$$

We first consider v. Introducing the polar coordinates, we have

(2.3)
$$\begin{aligned} v(t,x) &= \lim_{\epsilon \downarrow 0} (2\pi)^{-n/2} \int e^{ix\cdot\xi - \epsilon|\xi|} \frac{1}{2i|\xi|}(e^{it|\xi|} - e^{-it|\xi|})\hat{\psi}(\xi)d\xi \\ &= \lim_{\epsilon \downarrow 0} \frac{1}{2i(2\pi)^{n/2}} \int_0^\infty \int_{|\omega|=1} (e^{ir(x\cdot\omega + t + i\epsilon)} - e^{ir(x\cdot\omega - t + i\epsilon)})\hat{\psi}(r\omega)r^{n-2} d\omega dr, \end{aligned}$$

where $d\omega$ is the surface measure on $S^{n-1} = \{\omega \in \mathbb{R}^n; |\omega| = 1\}$. Set $\Psi_\omega(r) = r^{n-2}\hat{\psi}(r\omega)$. Integrating by parts, we have for $N \geq n-1$

$$\int_0^\infty e^{ir(x\cdot\omega \pm t + i\epsilon)} \Psi_\omega(r)dr \tag{2.4$_\pm$}$$

$$
\begin{aligned}
&= \sum_{j=1}^{N} i^j (x\cdot\omega \pm t + i\varepsilon)^{-j}\,\Psi_\omega^{(j-1)}(0) \\
&\qquad + i^N (x\cdot\omega \pm t + i\varepsilon)^{-N} \int_0^\infty e^{ir(x\cdot\omega\pm t+i\varepsilon)}\Psi_\omega^{(N)}(r)dr \\
&= \sum_{j=n-1}^{N} i^j \frac{(j-1)!}{(j-n+1)!}(x\cdot\omega \pm t + i\varepsilon)^{-j}(\omega\cdot\nabla)^{j-n+1}\hat{\psi}(0) \\
&\quad + \sum_{k=0}^{n-2} i^N \binom{N}{k} \frac{(n-2)!}{(n-2-k)!}(x\cdot\omega \pm t + i\varepsilon)^{-N} \\
&\qquad \times \int_0^\infty e^{ir(x\cdot\omega\pm t+i\varepsilon)} r^{n-2-k}((\omega\cdot\nabla)^{N-k}\hat{\psi})(r\omega)dr.
\end{aligned}
$$

Substituting the finite sum of $(2.4)_\pm$ into the RHS of the last equality (2.3), we compute

(2.5)

$$
\begin{aligned}
&\frac{1}{2i(2\pi)^{n/2}} \sum_{j=n-1}^{N} \frac{i^j(j-1)!}{(j-n+1)!} \\
&\quad \times \int_{|\omega|=1} ((x\cdot\omega + t + i\varepsilon)^{-j} - (x\cdot\omega - t + i\varepsilon)^{-j})(\omega\cdot\nabla)^{j-n+1}\hat{\psi}(0)d\omega \\
&= \frac{1}{2i(2\pi)^{n}} \sum_{j=n-1}^{N} \frac{i^j(j-1)!}{(j-n+1)!} \\
&\quad \times \int(\int_{|\omega|=1} ((x\cdot\omega + t + i\varepsilon)^{-j} - (x\cdot\omega - t + i\varepsilon)^{-j})(-i\omega\cdot y)^{j-n+1}d\omega)\psi(y)dy \\
&= \frac{1}{2i(2\pi)^{n}} \sum_{j=n-1}^{N} \frac{i^j(n-2)!}{(j-n+1)!} \\
&\quad \times \int((iy\cdot\nabla)^{j-n+1}\int_{|\omega|=1} ((x\cdot\omega + t + i\varepsilon)^{1-n} - (x\cdot\omega - t + i\varepsilon)^{1-n})d\omega)\psi(y)dy \\
&= \frac{1}{2i(2\pi)^{n}} \sum_{j=n-1}^{N} \frac{i^{j-n+1}}{(j-n+1)!} \\
&\quad \times \int((iy\cdot\nabla)^{j-n+1}\int_{|\omega|=1}\int_0^\infty (e^{ir(x\cdot\omega+t+i\varepsilon)} - e^{ir(x\cdot\omega-t+i\varepsilon)})r^{n-2}drd\omega)\psi(y)dy \\
&= \sum_{j=n-1}^{N} \frac{(-1)^{j-n+1}}{(2\pi)^n(j-n+1)!}\int((y\cdot\nabla)^{j-n+1}\int e^{ix\cdot\xi-\varepsilon|\xi|}\frac{\sin(t|\xi|)}{|\xi|}d\xi)\psi(y)dy \\
&= \sum_{j=n-1}^{N} \frac{(-1)^{j-n+1}\Gamma((n-1)/2)}{2\pi^{(n+1)/2}(j-n+1)!}\int(\mathrm{Im}(y\cdot\nabla)^{j-n+1}(|x|^2-(t+i\varepsilon)^2)^{-\frac{n-1}{2}})\psi(y)dy.
\end{aligned}
$$

Noting that

(2.6)

$$(y\cdot\nabla)^{j-n+1}(|x|^2-(t+i\varepsilon)^2)^{\frac{n-1}{2}}$$
$$=\sum_{\ell=0}^{[(j-n+1)/2]}\frac{(-1)^{j-n+1-\ell}(j-n+1)!}{\ell!(j-n+1-2\ell)!}\frac{\Gamma(j-\ell-(n-1)/2)}{\Gamma((n-1)/2)}$$
$$\times(2x\cdot y)^{j-n+1-2\ell}|y|^{2\ell}(|x|^2-(t+i\varepsilon)^2)^{\frac{n-1}{2}-j+\ell},$$

(2.7)

$$\lim_{\varepsilon\downarrow 0}\mathrm{Im}(|x|^2-(t+i\varepsilon)^2)^{\frac{n-1}{2}-j+\ell}=(-1)^{j-\ell-n/2}(t^2-|x|^2)^{\frac{n-1}{2}-j+\ell}\ \text{ on }\Gamma^+,$$

(2.8)

$$\int(x\cdot y)^{j-n+1-2\ell}|y|^{2\ell}\psi(y)dy=(-i)^{n-j-1}(2\pi)^{n/2}(x\cdot\nabla)^{j-n-2\ell+1}\Delta^\ell\hat{\psi}(0),$$

we conclude that the finite sum of (2.5) tends to $\sum_{j=n-1}^{N}(F_j\psi)(t,x)$ pointwisely on Γ^+ as $\varepsilon\downarrow 0$. By (2.3) and $(2.4)_\pm$ the remainder of the expansion is equal to

(2.9)

$$\lim_{\varepsilon\downarrow 0}\sum_{k=0}^{n-2}\binom{N}{k}\frac{i^{N-1}(n-2)!}{2(2\pi)^{n/2}(n-2-k)!}$$
$$\times\int_{|\omega|=1}\int_0^\infty((x\cdot\omega+t+i\varepsilon)^{-N}e^{it r}-(x\cdot\omega-t+i\varepsilon)^{-N}e^{-itr})e^{ir(x\cdot\omega+i\varepsilon)}$$
$$\times r^{n-2-k}((\omega\cdot\nabla)^{N-k}\hat{\psi})(r\omega)drd\omega$$
$$=\lim_{\varepsilon\downarrow 0}\sum_{k=0}^{n-2}\binom{N}{k}\frac{i^{N-1}(n-2)!}{2(2\pi)^{n/2}(n-2-k)!}$$
$$\times\int((x\cdot\frac{\xi}{|\xi|}+t+i\varepsilon)^{-N}e^{it|\xi|}-(x\cdot\frac{\xi}{|\xi|}-t+i\varepsilon)^{-N}e^{-it|\xi|})$$
$$\times e^{ix\cdot\xi-\varepsilon|\xi|}|\xi|^{-k-1}(\frac{\xi}{|\xi|}\cdot\nabla)^{N-k}\hat{\psi}(\xi)d\xi.$$

By the equality

$$(\frac{\xi}{|\xi|}\cdot\nabla)^{N-k}\hat{\psi}(\xi)=(-i)^{N-k}\sum_{|\alpha|=N-k}\frac{(N-k)!}{\alpha!}(\frac{\xi}{|\xi|})^\alpha\mathscr{F}(y^\alpha\psi)(\xi),$$

we have

$(2.10)_\pm$

$$\int|(x\cdot\frac{\xi}{|\xi|}\pm t+i\varepsilon)^{-N}e^{\pm it|\xi|}e^{ix\cdot\xi-\varepsilon|\xi|}|\xi|^{-k-1}(\frac{\xi}{|\xi|}\cdot\nabla)^{N-k}\hat{\psi}(\xi)|d\xi$$
$$\le(t-|x|)^{-N}\sum_{|\alpha|=N-k}\frac{(N-k)!}{\alpha!}\int|\xi|^{-k-1}|\mathscr{F}(y^\alpha\psi)(\xi)|d\xi$$

since $|x \cdot \frac{\xi}{|\xi|} \pm t + i\varepsilon| \geq t - |x|$ on Γ^+. We estimate the last integral in $(2.10)_\pm$. For $k \leq n/2 - 2$, we have by the Schwarz inequality

$$\begin{aligned}(2.11)\qquad \int_{|\xi|<1} |\xi|^{-k-1} |\mathscr{F}(y^\alpha \psi)(\xi)| d\xi &\leq C||\mathscr{F}(y^\alpha \psi)||_2 \\ &\leq C||\psi||_{0,N-k}.\end{aligned}$$

For $k \geq n/2 - 1$, we have by the Schwarz and Hardy inequalities (see [3])

$$\begin{aligned}(2.12)\qquad &\int_{|\xi|<1} |\xi|^{-k-1} |\mathscr{F}(y^\alpha \psi)(\xi)| d\xi \\ &\leq \Big(\int_{|\xi|<1} |\xi|^{-n+2\delta} d\xi\Big)^{1/2} \Big(\int ||\xi|^{n/2-k-1-\delta} \mathscr{F}(y^\alpha \psi)(\xi)|^2 d\xi\Big)^{1/2} \\ &\leq C_\delta ||(-\triangle)^{(k+1-n/2+\delta)/2} \mathscr{F}(y^\alpha \psi)||_2 \\ &= C_\delta || |y|^{k+1-n/2+\delta} y^\alpha \psi||_2 \leq C_\delta ||\psi||_{0,N+2-n/2},\end{aligned}$$

where $0 < \delta < 1$. Similarly, for $k \leq n/2 - 1$

$$\begin{aligned}(2.13)\qquad &\int_{|\xi|>1} |\xi|^{-k-1} |\mathscr{F}(y^\alpha \psi)(\xi)| d\xi \\ &\leq C_\delta \Big(\int_{|\xi|>1} ||\xi|^{n/2-k-1+\delta} \mathscr{F}(y^\alpha \psi)(\xi)|^2 d\xi\Big)^{1/2} \\ &\leq C_\delta ||\mathscr{F}(y^\alpha \psi)||_{0,n/2-k} \leq C_\delta ||\psi||_{n/2-k,N-k}.\end{aligned}$$

For $k \geq n/2$,

$$\begin{aligned}(2.14)\qquad &\int_{|\xi|>1} |\xi|^{-k-1} |\mathscr{F}(y^\alpha \psi)(\xi)| d\xi \\ &\leq C\Big(\int_{|\xi|>1} |\mathscr{F}(y^\alpha \psi)(\xi)|^2 d\xi\Big)^{1/2} \leq C||y^\alpha \psi||_2 \leq C||\psi||_{0,N-k}.\end{aligned}$$

Combining $(2.10)_\pm$-(2.14), the RHS of (2.9) is dominated by

$$C(t - |x|)^{-N} (||\psi||_{0,N+2-n/2} + ||\psi||_{n/2,N}).$$

This proves the expansion of v defined by (2.1), which is equivalent to (1.1) when $\phi = 0$. The expansion of w, namely, (1.1) with $\psi = 0$, is proved similarly. This completes the proof of Part(1). Q.E.D.

Proof of Part(2). We use the representation formula of solutions by means of the spherical mean:

$$u(t,x) = (2\pi)^{-n/2}(t^{-1}\partial_t)^{n/2-1}t^{n-1}M_\psi(t,x) \tag{2.15}$$
$$+ (2\pi)^{-n/2}\partial_t(t^{-1}\partial_t)^{n/2-1}t^{n-1}M_\phi(t,x),$$

where

$$M_f(t,x) = \int_{|y|<1}(1-|y|^2)^{-1/2}f(x-ty)dy. \tag{2.16}$$

By (2.15), we have

$$u(t,x) = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}\Big(\sum_{j=0}^{n/2-1} a_j t^{j+1}\partial_t^j M_\psi(t,x) + \sum_{j=0}^{n/2} b_j t^j \partial_t^j M_\phi(t,x)\Big) \tag{2.17}$$

where $a_0 = b_0 = 1$;

$$a_j = \frac{2^{j+1}}{(j-1)!}\sum_{k=j}^{n/2-1}\binom{n/2-1}{k}\frac{(k+1)!(2k-j-1)!}{(2k+2)!(k-j)!},$$
$$b_j = a_{j-1} + (j+1)a_j, \quad \text{for} \quad 1 \le j \le n/2-1;$$
$$b_{n/2} = a_{n/2-1}.$$

In view of (2.17), the required estimate (1.2) follows from

$$\operatorname*{Sup}_{(t,x)\in(\mathbb{R}_+\times\mathbb{R}^n)\setminus\Gamma+}(1+|x|-t)^N t^j |M_f(t,x)| \tag{2.18}$$
$$\le C\|\langle x\rangle^{N+2j}f\|_\infty, \quad 0 \le j \le n/2, \ f \in \mathscr{S}.$$

Since $(1+|x|-t)^N \le (1+|x-ty|)^N$ for $|x| \ge t$, $|y| \le 1$, we have

$$(1+|x|-t)^N|f(x-ty)| \le (1+|x-ty|^2)^{-j}\|\langle x\rangle^{N+2j}f\|_\infty,$$

so that

(2.19)
$$(1+|x|-t)^N t^j|M_f(t,x)|$$
$$\le \|\langle x\rangle^{N+2j}f\|_\infty t^j\int_0^1 r^{n-1}(1-r^2)^{-1/2}\Big(\int_{|\omega|=1}(1+|x-tr\omega|^2)^{-j}d\omega\Big)dr.$$

We consider the last surface integral for $1 \le j \le n/2$. For simplicity we set $a = 1+|x|^2+t^2r^2$ and $b = -2tr$. Then $1+|x-tr\omega|^2 = a+bx\cdot\omega$ and $|bx\cdot\omega| \le b|x| = 2tr|x| \le t^2r^2+|x|^2 = a-1$. By expanding the integrand, we obtain

(2.20)

$$\begin{aligned}
&\int_{|\omega|=1} (1+|x-tr\omega|^2)^{-j}\,d\omega \\
&\quad = a^{-j}\int_{|\omega|=1}(1+a^{-1}bx\cdot\omega)^{-j}\,d\omega \\
&\quad = \sum_{\ell=0}^{\infty}\frac{(j+2\ell-1)!}{(j-1)!(2\ell)!}a^{-j-2\ell}b^{2\ell}\int_{|\omega|=1}(x\cdot\omega)^{2\ell}\,d\omega \\
&\quad = 2\pi^{(n-1)/2}\sum_{\ell=0}^{\infty}\frac{(j+2\ell-1)!}{(j-1)!(2\ell)!}\frac{\Gamma(\ell+1/2)}{\Gamma(\ell+n/2)}a^{-j}(a^{-2}b^2|x|^2)^{\ell} \\
&\quad = 2\pi^{(n-1)/2}\sum_{\ell=0}^{\infty}\frac{2^{j-1}\Gamma(\ell+j/2)\Gamma(\ell+(j+1)/2)}{(j-1)!\Gamma(\ell+1)\Gamma(\ell+n/2)}a^{-j}(a^{-2}b^2|x|^2)^{\ell} \\
&\quad \le \frac{2^j\pi^{(n-1)/2}}{(j-1)!}\sum_{\ell=0}^{\infty}\frac{\Gamma(\ell+j/2)}{\Gamma(\ell+1)}a^{-j}(a^{-2}b^2|x|^2)^{\ell} \\
&\quad = \frac{2^j\pi^{(n-1)/2}\Gamma(j/2)}{(j-1)!}(a^2-b^2|x|^2)^{-j/2}.
\end{aligned}$$

A direct calculation shows

$$\begin{aligned}
a^2-b^2|x|^2 &= 1+(|x|^2-t^2r^2)^2+2|x|^2+2t^2r^2 \\
&\ge 1+2|x|^2 \ge 1+2t^2 \ge (1/2)(1+t)^2.
\end{aligned} \tag{2.21}$$

By (2.20) and (2.21),

$$\int_{|\omega|=1}(1+|x-tr\omega|^2)^{-j}\,d\omega \le \frac{2^{3j/2}\pi^{(n-1)/2}\Gamma(j/2)}{(j-1)!}(1+t)^{-j}. \tag{2.22}$$

Combining (2.19) and (2.22), we obtain (2.18), as required. Q.E.D.

Remark 6. *In the same way as above, we have the estimate*

$$\begin{aligned}
&\operatorname*{Sup}_{(t,x)\in(\mathbb{R}_+\times\mathbb{R}^n)\setminus\Gamma^+} M(|x|-t)|u(t,x)| \\
&\quad \le C\Big(\sum_{|\alpha|\le n/2-1}\|\langle x\rangle^n M(|x|)\partial^\alpha\psi\|_\infty + \sum_{|\alpha|\le n/2}\|\langle x\rangle^n M(|x|)\partial^\alpha\phi\|_\infty\Big)
\end{aligned}$$

whenever the RHS is finite, where M is a positive increasing function on $\mathbb{R}_+$, e.g. $M(s) = \exp(\lambda s^\mu)$ with λ, $\mu \ge 0$.

3. Proof of Theorem 2.

Proof of Part (1). Clearly, (i)⇒ (ii)⇒ (iii)⇒ (iv) and (ii)⇒(v).

(iv)⇒(vi): Assume (iv). There is a sequence $\{t_j\}$ such that $0 < t_1 < \cdots < t_j \uparrow \infty$ as $j \to \infty$ and

$$\lim_{j\to\infty} t_j^{n-1}|u(t_j, t_j\nu_0)| = 0. \tag{3.1}$$

By (1.1) with $N = n$,

$$\begin{aligned}(1-|\nu_0|)^n t_j^n |u(t_j, t_j\nu_0) - F_{n-1}\psi(t_j, t_j\nu_0)| \\ \leq (1-|\nu_0|)^n t_j^n |F_n\psi(t_j, t_j\nu_0) + G_n\phi(t_j, t_j\nu_0)| + C,\end{aligned} \tag{3.2}$$

where C is independent of t_j. By the definition of F_{n-1}, F_n, G_n,

$$\lim_{j\to\infty} t_j^{n-1} F_{n-1}\psi(t_j, t_j\nu_0) = -i^n 2^{n/2-1}\pi^{-1/2}\Gamma((n-1)/2)(1-|\nu_0|^2)^{-(n-1)/2}\hat{\psi}(0), \tag{3.3}$$

$$\lim_{j\to\infty} t_j^{n-1} F_n\psi(t_j, t_j\nu_0) = 0, \tag{3.4}$$

$$\lim_{j\to\infty} t_j^{n-1} G_n\phi(t_j, t_j\nu_0) = 0. \tag{3.5}$$

Dividing both sides of (3.2) by t_j, letting $j \to \infty$, and using (3.1), (3.3)-(3.5), we obtain $\hat{\psi}(0) = 0$.

(v)⇒ (vi): The proof proceeds in the same way as above.

(vi)⇒ (i): Since $\operatorname{Sup}_{t\geq 0}\|u(t)\|_\infty < \infty$, it suffices to prove

$$\operatorname*{Sup}_{(t,x)\in\Gamma+} (t-|x|)^n |u(t,x)| < \infty. \tag{3.6}$$

By assumption, $F_{n-1}\psi \equiv 0$ and therefore (3.6) follows from (1.1) with $N = n$ and the definition of F_n and G_n. Q.E.D.

Proof of Part (2). There is a sequence $\{t_j\}$ such that $0 < t_1 < \cdots < t_j \uparrow \infty$ as $j \to \infty$ and

$$\lim_{j\to\infty} t_j^{n+k-1}|u(t_j, t_j\nu)| = 0. \tag{3.7}$$

By Part (1), $\hat{\psi}(0) = 0$. By (1.1) with $N = n+1$,

$$(1-|\nu|)^{n+1} t_j^{n+1} |u(t_j, t_j\nu) - F_n\psi(t_j, t_j\nu) - G_n\phi(t_j, t_j\nu)| \leq (1-|\nu|)^{n+1} t_j^{n+1} |F_{n+1}\psi(t_j, t_j\nu) + G_{n+1}\phi(t_j, t_j\nu)| + C, \tag{3.8}$$

where C is independent of t_j, By definition,

$$\lim_{j\to\infty} t_j^n F_n\psi(t_j, t_j\nu) = -i^{n-1} 2^{n/2} \pi^{-1/2} \Gamma((n+1)/2)(1-|\nu|^2)^{-(n+1)/2} \nu \cdot \nabla\hat{\psi}(0), \tag{3.9}$$

$$\lim_{j\to\infty} t_j^n G_n\phi(t_j, t_j\nu) = i^n 2^{n/2} \pi^{-1/2} \Gamma((n+1)/2)(1-|\nu|^2)^{-(n+1)/2} \hat{\phi}(0), \tag{3.10}$$

$$\lim_{j\to\infty} t_j^n F_{n+1}\psi(t_j, t_j\nu) = 0, \tag{3.11}$$

$$\lim_{j\to\infty} t_j^n G_{n+1}\phi(t_j, t_j\nu) = 0. \tag{3.12}$$

Divinding both sides by t_j, letting $j \to \infty$, and using (3.7), (3.9)-(3.12), we obtain

$$i\nu \cdot \nabla\hat{\psi}(0) + \hat{\phi}(0) = 0. \tag{3.13}$$

Letting $\nu = 0$ in (3.13), we have $\hat{\phi}(0) = 0$. Then (3.13) proves $\nabla\hat{\psi}(0) = 0$ since ν is arbitrary in the open unit ball.

We assume that $\partial^\alpha\hat{\psi}(0) = \partial^\beta\hat{\phi}(0) = 0$ for any α, β with $|\alpha| \leq k-1$, $|\beta| \leq k-2$ and we prove that $\partial^\alpha\hat{\psi}(0) = \partial^\beta\hat{\phi}(0) = 0$ for any α, β with $|\alpha| = k$, $|\beta| = k-1$. By assumption, $F_\ell\psi = G_\ell\phi = 0$ for all $\ell \leq n+k-2$. By (1.1) with $N = n+k$,

$$(1-|\nu|)^{n+k} t_j^{n+k} |u(t_j, t_j\nu) - F_{n+k-1}\psi(t_j, t_j\nu) - G_{n+k-1}\phi(t_j, t_j\nu)| \leq (1-|\nu|)^{n+k} t_j^{n+k} |F_{n+k}\psi(t_j, t_j\nu) + G_{n+k}\phi(t_j, t_j\nu)| + C, \tag{3.14}$$

where C is independent of t_j. By definition,

$$\lim_{j\to\infty} t_j^{n+k-1} F_{n+k-1}\psi(t_j, t_j\nu) = \sum_{\ell=0}^{[k/2]} (-1)^{n+k} \frac{2^{n/2+k-2\ell-1}\Gamma((n-1)/2+k-\ell)}{\pi^{1/2}\ell!(k-2\ell)!}(1-|\nu|^2)^{\ell-k-(n-1)/2}(\nu\cdot\nabla)^{k-2\ell}\Delta^\ell\hat{\psi}(0), \tag{3.15}$$

(3.16)
$$\lim_{j\to\infty} t_j^{n+k-1} G_{n+k-1}\phi(t_j, t_j\nu)$$
$$= \sum_{\ell=0}^{[(k-1)/2]} (-1)^{n+k-1} \frac{2^{n/2+k-2\ell-1}\Gamma((n-1)/2+k-\ell)}{\pi^{1/2}\ell!(k-2\ell-1)!}(1-|\nu|^2)^{\ell-k-(n-1)/2} \times (\nu\cdot\nabla)^{k-2\ell-1}\Delta^{\ell}\hat{\phi}(0),$$

(3.17)
$$\lim_{j\to\infty} t_j^{n+k-1} F_{n+k}\psi(t_j, t_j\nu) = 0,$$

(3.18)
$$\lim_{j\to\infty} t_j^{n+k-1} G_{n+k}\phi(t_j, t_j\nu) = 0.$$

As before, we conclude from (3.7), (3.14)-(3.18) that the sum of the RHS of (3.15) and (3.16) is zero. By letting $\nu = 0$, we see that $\Delta^{[k/2]}\hat{\psi}(0) = 0$ if k is even and that $\Delta^{[(k-1)/2]}\hat{\phi}(0) = 0$ if k is odd.

Counting the degree of ν in the summand of (3.15) and (3.16) according to those two cases of k and considering the corresponding coefficient, we conclude that $\Delta^{\ell}\hat{\psi}(0) = 0$ for $1 \le \ell \le [k/2]$ and $\Delta^{\ell}\hat{\phi}(0) = 0$ for $1 \le \ell \le [(k-1)/2]$ since there is order of degree of ν in the terms of the RHS of (3.15) and (3.16) when we count the degree from zero. At the last step, we have

$$(\nu\cdot\nabla)^k\hat{\psi}(0) = 0,$$

from which we obtain $\partial^\alpha\hat{\psi}(0) = 0$ for $|\alpha| = k$. This is in turn implies that

$$(\nu\cdot\nabla)^{k-1}\hat{\phi}(0) = 0,$$

from which we obtain $\partial^\beta\hat{\phi}(0) = 0$ for $|\beta| = k-1$. This proves (v).
(v)$\Rightarrow$ (vi): The proof proceeds similarly as above.
(vi)$\Rightarrow$ (i): The proof is similar to the corresponding one in Part (1). Note that

$$\operatorname*{Sup}_{(t,x)\in\Gamma^+} (t-|x|)^{n+k}(|F_{n+k}\psi(t,x)| + |G_{n+k}\psi(t,x)|) < \infty.$$

Q.E.D.

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