SYSTEMS OF CLAIRAUT TYPE

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SYSTEMS OF CLAIRAUT TYPE

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Abstract. A characterization of systems of first order differential equations with (classical) complete solutions is given. Systems with affine complete solutions are also characterized.

0. Introduction

About 260 years ago Alex Claude Clairaut [2] studied the following equation which is called the Clairaut equation now : $y = x \cdot \frac{dy}{dx} + f(\frac{dy}{dx})$. It is usually taught in the first or second year course of calculus in the university and treated as one of the typical examples of non-linear equations that are easily solved. Moreover it has a quite beautiful geometric structure as follows : There exists a "general solution" that consists of lines ; $y = t \cdot x + f(t)$, where t is a parameter and the singular solution is the envelope of such a family.

In [6] we studied ordinary differential equations with beautiful geometric structure such as the Clairaut equation. In this note we shall concern systems of first order partial differential equations (briefly, equations) with (classical) complete solutions, which are the natural generalization of the Clairaut equation. Since general solutions and the singular solution of the equation can be constructed from the complete solution, this class of equations plays a principal role in classical treatises (cf. Carathéodory [1], Courant-Hilbert [3], Forsyth [4] [5]). However, we have never seen characterizations for this class of equations. Our main result (Theorem 1.1) is to give a characterization of these. In §2 we shall give a proof of the main theorem. We shall also study a class of equations with affine complete solutions in §3, which is a direct generalization of the classical Clairaut equation.

All maps considered here are differentiable of class C^∞, unless stated otherwise.

1. The main result

In this section we shall state our main result. A first order differential equation is most naturally interpreted as being a closed subset of $J^1(\mathbb{R}^n, \mathbb{R})$. Unless the contrary is specifically stated, we use the following definition. *A system of partial differential equations of*

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first order (or briefly, *an equation*) is a submersion germ $F : (J^1(\mathbb{R}^n, \mathbb{R}), z_0) \to (\mathbb{R}^d, 0)$ on the 1-jet space of functions of n-variables. Let θ be the canonical contact form on $J^1(\mathbb{R}^n, \mathbb{R})$ which is given by $\theta = dy - \sum_{i=1}^n p_i dx_i$, where (x, y, p) are canonical coordinates of $J^1(\mathbb{R}^n, \mathbb{R})$. We define *a geometric solution of* $F = 0$ to be an immersion $i : (L, q_0) \to (J^1(\mathbb{R}^n, \mathbb{R}), z_0)$ of an n-dimensional manifold such that $i^*\theta = 0$ and $i(L) \subset F^{-1}(0)$ (i.e. a Legendrian submanifold which is contained in $F^{-1}(0)$). We say that z_0 is *a π-singular point* if $F(z_0) = 0$ and rank$\,(\frac{\partial F_i}{\partial p_j}(z_0)) < n$. We denote the set of π-singular points by $\Sigma_\pi(F)$ and $\pi(\Sigma_\pi(F)) = D_F$, where $\pi(x, y, p) = (x, y)$. We call the set D_F *the discriminant set* of the equation $F = 0$.

An equation $F = 0$ is said to be *Clairaut type* if there exist smooth function germs $B_{ji}, A^\ell_{ik} : (J^1(\mathbb{R}^n, \mathbb{R}, z_0) \to \mathbb{R}$ for $i, j = 1, \ldots, n$, $k = 1, \ldots, d$ and $\ell = 1, \ldots, d$ such that

$$\frac{\partial F_\ell}{\partial x_i} + p_i \frac{\partial F_\ell}{\partial y} = \sum_{j=1}^n B_{ji} \frac{\partial F_\ell}{\partial p_j} + \sum_{k=1}^d A^\ell_{ik} F_k \quad (i = 1, \ldots, n \text{ and } \ell = 1, \ldots, d)$$

and satisfy that

$$B_{ji} = B_{ij} \tag{1}$$

$$\frac{\partial B_{jk}}{\partial x_i} + p_i \frac{\partial B_{jk}}{\partial y} + \sum_{\ell=1}^n B_{\ell i} \frac{\partial B_{jk}}{\partial p_\ell} = \frac{\partial B_{ji}}{\partial x_k} + p_k \frac{\partial B_{ji}}{\partial y} + \sum_{\ell=1}^n B_{\ell k} \frac{\partial B_{ji}}{\partial p_\ell} \tag{2}$$

at any $z \in (F^{-1}(0), z_0)$ for $i, j, k = 1, \ldots, n$.

We also say that an $n - d + 1$-parameter family of function germs

$$f : (\mathbb{R}^{n-d+1} \times \mathbb{R}^n, (t_0, x_0)) \to (\mathbb{R}, y_0)$$

is *a (classical)complete solution of* $F = 0$ if $F_k(x, f(t, x), \frac{\partial f}{\partial x}(t, x)) = 0$ for $k = 1, \ldots, d$ and rank$\,(\frac{\partial f}{\partial t_i}, \frac{\partial^2 f}{\partial t_i \partial x_j}) = n - d + 1$. Our main result is the following.

Theorem 1.1. *For an equation germ $F = 0$, the following are equivalent.*
(1) $F = 0$ is the Clairaut type equation.
(2) $F = 0$ has a (classical) complete solution.

In this case, if $\Sigma_\pi(F) \neq \emptyset$, then $\Sigma_\pi(F)$ is a geometric solution (i.e. the singular solution) of $F = 0$ and the discriminant set D_F is the envelope of the family of graphs of the complete solution.

By the classical existence theorem (see [7]), if $F = 0$ is a π-regular equation, there exists a (classical) complete solution. Then we can assert that π-regular equation is Clairaut type by the above theorem.

We now give two examples which describe the above assertion.

Examples 1.2. 1) The following equation is a generalization of the classical Clairaut equation:

$$F_i(p_1,\dots,p_n)=0 \quad (i=1,\dots,d-1),\ F_d(x,y,p)=y-\sum_{i=1}^{n} p_i x_i - f(p_1,\dots,p_n)=0,$$

where F_i, f are function germ.

Since $F=(F_1,\dots,F_d)$ is a submersion, we have rank$\left(\frac{\partial F_i}{\partial x_j}\right)=d-1$ Then the set $F^{-1}(0)$ is locally parametrized by an immersion $a(t)=(a_1(t),\dots,a_n(t))$, where $t=(t_1,\dots,t_{n-d+1})$. It follows that we can get a complete solution

$$y=\sum_{i=1}^{n} a_i(t)x_i+f(a_1(t),\dots,a_n(t)).$$

We can easily check that

$$\frac{\partial F_\ell}{\partial x_i}+p_i\frac{\partial F_\ell}{\partial y}=0$$

on $F^{-1}(0)$. This means that we can choose $B_{ij}=0$.

2) Consider the following equation : $F_1=p_1^2-y=0,\ F_2=p_2=0\ (n=2)$.

Then we have

$$\frac{\partial F_1}{\partial x_1}+p_1\frac{\partial F_1}{\partial y}=-p_1,\ \frac{\partial F_1}{\partial x_2}+p_2\frac{\partial F_1}{\partial y}=-p_2,$$
$$\frac{\partial F_2}{\partial x_1}+p_1\frac{\partial F_2}{\partial y}=0,\ \frac{\partial F_2}{\partial x_2}+p_2\frac{\partial F_2}{\partial y}=0$$

and

$$\frac{\partial F_1}{\partial p_1}=2p_1,\ \frac{\partial F_1}{\partial p_2}=0,\ \frac{\partial F_2}{\partial p_1}=0,\ \frac{\partial F_2}{\partial p_2}=1.$$

It follows that

$$\begin{aligned}
\frac{\partial F_1}{\partial x_1}+p_1\frac{\partial F_1}{\partial y}&=-\frac{1}{2}\cdot\frac{\partial F_1}{\partial p_1}+0\cdot\frac{\partial F_1}{\partial p_2}+0\cdot F_1+0\cdot F_2,\\
\frac{\partial F_1}{\partial x_2}+p_2\frac{\partial F_1}{\partial y}&=0\cdot\frac{\partial F_1}{\partial p_1}+0\cdot\frac{\partial F_1}{\partial p_2}+0\cdot F_1-1\cdot F_2,\\
\frac{\partial F_2}{\partial x_1}+p_1\frac{\partial F_2}{\partial y}&=-\frac{1}{2}\cdot\frac{\partial F_2}{\partial p_1}+0\cdot\frac{\partial F_2}{\partial p_1}+0\cdot F_1+0\cdot F_2,\\
\frac{\partial F_2}{\partial x_2}+p_2\frac{\partial F_2}{\partial y}&=0\cdot\frac{\partial F_2}{\partial p_1}+0\cdot\frac{\partial F_2}{\partial p_2}+0\cdot F_1+0\cdot F_2.
\end{aligned}$$

The complete solution is given by $y=\frac{1}{4}(x_1+t)^2$.

In classical textbooks (see [1],[3],[4],[5]), the notion of singular solution has been appeared accompany with the notion of complete solutions. In these, the singular solution has been defined to be the envelope of the family of graphs of the complete solution. Theorem 1.1 gives a characterization of this class of equations as the class of Clairaut type equations.

2. Proof of Theorem 1.1

In this section we shall give a proof of Theorem 1.1. For our purpose, we need some preparations on elementary properties of Legendrian singularities. For a Legendrian immersion germ $i : (L, q_0) \to J^1(\mathbb{R}^n, \mathbb{R})$, $q_0 \in L$ is said to be *a Legendrian singular point* if $\pi \circ i$ is not an immersion at q_0. Then we have the following lemma.

Lemma 2.1. *For an equation $F : (J^1(\mathbb{R}^n, \mathbb{R}), z_0) \to (\mathbb{R}^d, 0)$, the following are equivalent.*

(1) *$F = 0$ has a (classical) complete solution.*

(2) *There exists a foliation on $F^{-1}(0)$ by geometric solutions of $F = 0$ with leaves are Legendrian non singular.*

Proof. Suppose that $f : (\mathbb{R}^{n-d+1} \times \mathbb{R}^n, (t_0, x_0)) \to (\mathbb{R}, y_0)$ is a (classical) complete solution of $F = 0$. Then we define a map germ $j^1_* f : (\mathbb{R}^{n-d+1} \times \mathbb{R}^n, (t_0, x_0)) \to (J^1(\mathbb{R}^n, \mathbb{R}), z_0)$ by $j^1_*(t, x) = (x, f(t, x), \frac{\partial f}{\partial x}(t, x))$. We can easily show that $j^1_* f$ is an immersion if and only if $\mathrm{rank}(\frac{\partial f}{\partial t_i}, \frac{\partial^2 f}{\partial t_i \partial x_j}) = n - d + 1$. It follows that $j^1_* f$ gives a local parametrization of $F^{-1}(0)$ and the family $\{\mathrm{Image} j^1_* f_t\}_{t \in (\mathbb{R}^{n-d+1}, t_0)}$ gives a desired foliation, where $f_t(x) = f(t, x)$.

For the converse, we remark that q_0 is a Legendrian non singular point of Legendrian immersion $i : (L, q_0) \to J^1(\mathbb{R}^n, \mathbb{R})$ if and only if $\tilde{\pi} \circ i$ is a local diffeomorphism at q_0, where $\tilde{\pi}(x, y, p) = x$.

Suppose that there exists a foliation which satisfies the condition (2). Then we have an $n - d + 1$-parameter family of smooth sections $s : (\mathbb{R}^{n-d+1} \times \mathbb{R}^n, (t_0, x_0)) \to (J^1(\mathbb{R}^n, \mathbb{R}), z_0)$ of $\tilde{\pi}$ (i.e. $\tilde{\pi} \circ s(t, x) = x$) such that s is an immersion, $s(\mathbb{R}^{n-d+1} \times \mathbb{R}^n) = F^{-1}(0)$ and $s_t^* \theta = 0$ for any $t \in (\mathbb{R}^{n-d+1}, t_0)$, where $s_t(x) = s(t, x)$. It follows that there exists a family of function germs $f : (\mathbb{R}^{n-d+1} \times \mathbb{R}^n, (t_0, x_0)) \to (\mathbb{R}^n, y_0)$ such that $j^1_* f(t, x) = s(t, x)$. Since s is an immersion, then f is a (classical) complete solution of $F = 0$.

Now we can give the proof that (1) implies (2).

Proof of Theorem 1.1, (1) $\Rightarrow$ (2). By the assumption, there exist function germs

$$B_{ij}, A^\ell_{ik} : (J^1(\mathbb{R}^n, \mathbb{R}), z_0) \to \mathbb{R}$$

such that

$$\frac{\partial F_\ell}{\partial x_i} + p_i \frac{\partial F_\ell}{\partial y} = \sum_{j=1}^{n} B_{ji} \frac{\partial F_\ell}{\partial p_j} + \sum_{k=1}^{d} A^\ell_{ik} F_k \quad (i = 1, \ldots, n \text{ and } \ell = 1, \ldots, d),$$

$B_{ji} = B_{ij}$ and $\frac{\partial B_{jk}}{\partial x_i} + p_i \frac{\partial B_{jk}}{\partial y} + \sum_\ell^n B_{\ell i} \frac{\partial B_{jk}}{\partial p_\ell} = \frac{\partial B_{ji}}{\partial x_k} + p_k \frac{\partial B_{ji}}{\partial y} + \sum_\ell^n B_{\ell k} \frac{\partial B_{ji}}{\partial p_\ell}$ at any $z \in (F^{-1}(0), z_0)$ for $i, j, k = 1, \ldots, n$.

We consider linearly independent vector fields

$$V_i = \frac{\partial}{\partial x_i} + p_i \frac{\partial}{\partial y} - \sum_{j=1}^{n} B_{ji} \frac{\partial}{\partial p_j} \qquad (i = 1, \dots, n)$$

on $(J^1(\mathbb{R}^n, \mathbb{R}), z_0)$. Let $c(t)$ be an integral curve of V_i such that $c(0) \in F^{-1}(0)$. Then we can show that $\frac{dF(c(t))}{dt}|_{t=0} = \frac{\partial F_\ell}{\partial x_i} + p_i \frac{\partial F_\ell}{\partial y} - \sum_{j=1}^n B_{ji} \frac{\partial F_\ell}{\partial p_j} = 0$ It follows that $V_i(z) \in T_z F^{-1}(0)$ for any $z \in F^{-1}(0)$. Since V_i are linearly independent, then we can define an n-dimensional distribution E on $F^{-1}(0)$ which is generated by vectors $V_i(z)$ on each point $z \in F^{-1}(0)$. By the direct calculation, we have

$$[V_i, V_k] = \sum_{j=1}^{n} \left(\frac{\partial B_{ji}}{\partial x_k} - \frac{\partial B_{jk}}{\partial x_i} + p_k \frac{\partial B_{ji}}{\partial y} - p_i \frac{\partial B_{jk}}{\partial y} + \sum_{\ell=1}^{n} B_{\ell k} \frac{\partial B_{ji}}{\partial p_\ell} - \sum_{\ell=1}^{n} B_{\ell i} \frac{\partial B_{jk}}{\partial p_\ell} \right) \frac{\partial}{\partial p_j}$$

for any $i, k = 1, \dots, n$. By the assumption, $[V_i, V_k](z) \in E_z$ for any $z \in F^{-1}(0)$. Then the distribution E is integrable and there exists an n-dimensional foliation on $F^{-1}(0)$ by the Frobenius theorem. Since $\theta(V_i) = 0$, then leaves of this foliation are Legendrian submanifold. By the definition of V_i, we have $d\tilde{\pi}(V_i) = \frac{\partial}{\partial x_i}$. It follows that leaves are Legendrian non singular. Then this foliation gives a (classical) complete solution by Lemma 2.1.

The converse direction is rather a direct.

Proof of Theorem 1.1, (2) $\Rightarrow$ (1). Let $y = f(t, x)$ be a complete solution of $F = 0$. If we calculate the partial derivative of $F_\ell(x, f(t, x), \frac{\partial f}{\partial x}(t, x)) = 0$ with respect to x_i, then we have $\frac{\partial F_\ell}{\partial x_i} + f_{x_i} \frac{\partial F_\ell}{\partial y} + \sum_{j=1}^n \frac{\partial^2 f}{\partial x_j \partial x_i} \frac{\partial F_\ell}{\partial p_j} = 0$ at $(x, f(t, x), \frac{\partial f}{\partial x}(t, x)) \in F^{-1}(0)$.

Since the map germ $f_*^1 f$ is an immersion germ, then there exist function germs $B_{ji} : (J^1(\mathbb{R}^n, \mathbb{R}), z_0) \to \mathbb{R}$ such that $B_{ji} \circ j_*^1 f = \frac{\partial^2 f}{\partial x_j \partial x_i}$ for $i, j = 1, \dots n$. For any $z \in F^{-1}(0)$, there exists $(t, x) \in (\mathbb{R}^{n-d+1} \times \mathbb{R}^n, (t_0, x_0))$ such that $(x, f(t, x), \frac{\partial f}{\partial x}(t, x)) = z$. Then we have $\frac{\partial F_\ell}{\partial x_i} + p_i \frac{\partial F_\ell}{\partial y} = \sum_{j=1}^n B_{ji} \frac{\partial F_\ell}{\partial p_j}$ on $F^{-1}(0)$. This means that there exists a function germ $A_{ik}^\ell : (J^1(\mathbb{R}^n, \mathbb{R}), z_0) \to \mathbb{R}$ such that $\frac{\partial F_\ell}{\partial x_i} + p_i \frac{\partial F_\ell}{\partial y} = \sum_{j=1}^n B_{ji} F_{p_j} + \sum_{k=1}^d A_{ik}^\ell F_k$ for $i, j = 1, \dots, n$ and $\ell = 1, \dots, d$.

On the other hand, if we calculate the partial derivative of the equality $\frac{\partial^2 f}{\partial x_j \partial x_i}(t, x) = B_{ji}(x, f(t, x), \frac{\partial f}{\partial x}(t, x))$ with respect ro x_k, then we have $\frac{\partial^3 f}{\partial x_j \partial x_i \partial x_k} = \frac{\partial B_{ji}}{\partial x_k} + \frac{\partial B_{ji}}{\partial y} \frac{\partial f}{\partial x_k} + \sum_{\ell=1}^n \frac{\partial B_{ji}}{\partial p_\ell} \frac{\partial f}{\partial x_\ell \partial x_k}$. Since $\frac{\partial f}{\partial x_k}(t, x) = p_k$, $\frac{\partial^2 f}{\partial x_\ell \partial x_k} = B_{\ell k}$ and f is smooth, then $F = 0$ is Clairaut type. This completes the proof that (2) implies (1).

Proof of the second part of Theorem 1.1. By the first part of the theorem, we may assume that there exists a (classical) complete solution $y = f(t, x)$ of $F = 0$ and $\Sigma_\pi(F) \neq \emptyset$. By the definition, $j_*^1(t, x) \in \Sigma_\pi(F)$ if and only if

$$\operatorname{rank} \begin{pmatrix} E & \frac{\partial f}{\partial x} \\ 0 & \frac{\partial f}{\partial t} \end{pmatrix} = n \quad \text{at } (t, x).$$

It is equivalent to the fact that $\frac{\partial f}{\partial t_i}(t,x)=0$. Then the Jacobian matrix of this equation is given by $J(\frac{\partial f}{\partial t_1},\dots,\frac{\partial f}{\partial t_{n-d+1}})=(\frac{\partial f}{\partial t_i\partial x_j},\frac{\partial f}{\partial t_i\partial t_k})$. Since

$$\operatorname{rank}(\frac{\partial f}{\partial t_1},\frac{\partial f}{\partial t_i\partial x_j})=\operatorname{rank}(0,\frac{\partial f}{\partial t_i\partial x_j})=n-d+1$$

at the point (t,x) with $j^1_*f(t,x)\in\Sigma_\pi(F)$, then we have $\operatorname{rank}J(\frac{\partial f}{\partial t_1},\dots,\frac{\partial f}{\partial t_{n-d+1}})=n-d+1$. It follows that $\Sigma_\pi(F)=j^1_*f(\{\frac{\partial f}{\partial t_i}=0|i=1,\dots,n-d+1\})$ is an n-dimensional submanifold.

On the other hand, $(j^1_*f)^*\theta=0$ if and only if $\frac{\partial f}{\partial t_i}(t,x)=0$. This means that $\Sigma_\pi(F)$ is a Legendrian submanifold. Furthermore, we consider the family of graphs of complete solution which is given by the equation $f(t,x)-y=0$. Then we ca show that the set

$$\{(x,f(t,x))|\text{There exists } t\in(\mathbb{R}^n,t_0) \text{ such that } \frac{\partial f}{\partial t_i}(t,x)=0\ (i=1,\dots n-d+1)\}$$

is the envelope of this family by the usual method of the elementary calculus. This set is equal to the discriminant set D_F by the previous arguments. This completes the proof of Theorem 1.1.

3. The Clairaut system

In this section we shall study equations with affine complete solutions.

Theorem 3.1. *For an equation $F:(J^1(\mathbb{R}^n,\mathbb{R}),z_0)\to(\mathbb{R}^d,0)$ with $\Sigma_\pi(F)\neq\emptyset$, the following are equivalent.*
(1) *There exist smooth function germs $A^\ell_{ik}:(J^1(\mathbb{R}^n,\mathbb{R}),z_0)\to\mathbb{R}$ such that*

$$\frac{\partial F_\ell}{\partial x_i}+p_i\frac{\partial F_\ell}{\partial y}=\sum_{k=1}^d A^\ell_{ik}F_k \quad \text{for } i=1,\dots,n,\ \ell=1,\dots,d.$$

(2) *There exists a (classical) complete solution of $F=0$ such that all members are affine planes.*
(3) *There exists a submersion germ $G:(\mathbb{R}^n,p_0)\to(\mathbb{R}^d,0)$ and a function germ $f:(\mathbb{R}^n,p_0)\to\mathbb{R}$ such that*

$$F^{-1}(0)=\{(x,y,p)|G(p_1,\dots,p_n)=0 \text{ and } y=\sum_{i=1}^n x_ip_i-f(p_1,\dots,p_n)\}.$$

Proof. Suppose that the equation $F=0$ satisfies the condition (1). By the proof of Theorem 1.1, the vector fields $V_i=\frac{\partial}{\partial x_i}+p_i\frac{\partial}{\partial y}$ generate a completely integrable distribution E. By the definition of V_i, maximal integral submanifolds of E are affine Legendrian subspaces in $J^1(\mathbb{R}^n,\mathbb{R})$, so that the assertion (2) follows.

Suppose that a family of hyperplanes $y = \sum_{i=1}^{n} a_i(t)x_i + b(t)$ is a complete solution of $F = 0$, where $t \in (\mathbb{R}^{n-d+1}, t_0)$. Since $\Sigma_\pi(F) \neq \emptyset$, we can calculate that rank $(\frac{\partial a_i(t)}{\partial t_j}(t_0) = n-d+1$, so that the germ $a : (\mathbb{R}^{n-d+1}, t_0) \to (\mathbb{R}^n, p_0)$ defined by $a(t) = (a_1(t), \ldots, a_n(t))$ is an immersion germ. It follows that there exists a submersion germ $G : (\mathbb{R}^n, p_0) \to (\mathbb{R}^{d-1}, 0)$ such that $(G^{-1}(0), p_0) = (\mathrm{Image}\, a, p_0)$. We can also find a function germ $f : (\mathbb{R}^n, p_0) \to \mathbb{R}$ such that $f \circ a(t) = b(t)$. Then we have the following inclusion:

$$F^{-1}(0) \supset \{(x, y, p) | G(p_1, \ldots, p_n) = 0 \text{ and } y = \sum_{i=1}^{n} x_i p_i - f(p_1, \ldots, p_n)\}.$$

However, both manifolds are codimension d, then these are equal as germs. This completes the proof that (2) implies (3). The remained assertion can be proved by the direct calculation just like as that of in the proof of Theorem 1.1.

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