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Series ♯176. December 1992

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THE INITIAL VALUE PROBLEMS FOR QUASI-LINEAR WAVE EQUATIONS IN TWO SPACE DIMENSIONS WITH SMALL DATA

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Abstract. The present paper studies the lifespan of solutions to quasi-linear wave equations in two space dimensions. We shall show a lower bound for the lifespan. We shall also show that if the non-linear term satisfies "null-condition", the equations have global solutions. Our basic idea is to solve ordinary differential equations which are constructed from wave equations.

1. Introduction and Statement of Results. We study the lifespan of solutions of quasi-linear wave equations in two space dimensions, with small initial data, as following type;

$$\Box u(x,t) = a_{\alpha\beta}(u')\partial_\alpha\partial_\beta u(x,t), \quad (x,t)\in\mathbb{R}^2\times[0,\infty), \tag{1.1}$$

$$u(x,0)=\varepsilon f(x), \quad \partial_0 u(x,0)=\varepsilon g(x), \quad x\in\mathbb{R}^2. \tag{1.2}$$

Here $a_{\alpha\beta}(u') = a_{\beta\alpha}(u')$ and we denote $\partial_0 = \partial/\partial t$, $\partial_i = \partial/\partial x_i$ $(i=1,2)$ and $\Box = \partial_0^2-\partial_1^2-\partial_2^2$. The gradient of u is denoted by $u'=(\partial_0 u,\partial_1 u,\partial_2 u)$. We use the summation convention with subscripts $\alpha,\beta\cdots$ ranging over 0, 1, 2 and $i,j,\cdots$ over 1,2. Moreover we assume that

$$f,g\in C_0^\infty(\mathbb{R}^2) \quad \text{and} \quad f(x)=g(x)=0, \quad \text{for} \quad |x|\geqslant M \tag{1.3}$$

$$a_{\alpha\beta}(p)\in C^\infty(\mathbb{R}^3) \tag{1.4a}$$

$$a_{\alpha\beta}(p) = O(|p|^2) \tag{1.4b}$$

$$|a_{\alpha\beta}(p)| < 1/2 \tag{1.4c}$$

for $|p| < \delta$, where δ is a small positive number.

The supremum of all τ for which $C^\infty(\mathbb{R}^2 \times [0,\tau))$-solution of the Cauchy problem (1.1), (1.2) exists is called the "lifespan" T_ε. When $T_\varepsilon = \infty$, we say the Cauchy problem (1.1), (1.2) has a *global solution.*

M. Kovalyov has proved in [12] that

$$\liminf_{\varepsilon \to 0} \varepsilon^2 \log(1 + T_\varepsilon) \geqslant C, \tag{1.5}$$

where the constant C depends on f, g and $a_{\alpha\beta}$. The first aim is to determine the constant explicitly by Friedlander radiation field. Let $U(x,t)$ be a solution of linear wave equation;

$$\Box U(x,t) = 0, \quad (x,t) \in \mathbb{R}^2 \times [0,\infty), \tag{1.6}$$

$$U(x,0) = f(x), \quad \partial_0 U(x,0) = g(x), \quad x \in \mathbb{R}^2. \tag{1.7}$$

Then we can define the Friedlander radiation field $\mathcal{F}(\omega,\rho)$ by

$$\mathcal{F}(\omega,\rho) = \lim_{r\to\infty} r^{1/2} U(x,t), \quad x = r\omega,\ \omega \in S^1,\ \rho = r - t. \tag{1.8}$$

$\mathcal{F}$ is explicitly expressed by

$$\mathcal{F}(\omega,\rho) = \frac{1}{2\sqrt{2}\pi} \int_\rho^\infty (s-\rho)^{-1/2} \{R_g(\omega,s) - \partial_s R_f(\omega,s)\} ds, \tag{1.9}$$

where R_h is Radon transform of $h \in C_0^\infty(\mathbb{R}^2)$, *i.e.*,

$$R_h(\omega,s) = \int_{\omega\cdot y = s} h(y) dS_y.$$

Note that $\mathcal{F}$ satisfies

$$\mathcal{F}(\omega,\rho) = 0 \quad \text{for} \quad \rho \geqslant M, \tag{1.10}$$

$$|\partial_\rho^\ell \mathcal{F}(\omega,\rho)| \leqslant C(1 + |\rho|)^{-1/2-\ell}. \tag{1.11}$$

For the above facts, see Hörmander [3].

We write the non-linear term in (1.1) as

$$a_{\alpha\beta}(u')\partial_\alpha\partial_\beta u = Z_{\alpha\beta\gamma\delta}(\partial_\gamma u)(\partial_\delta u)(\partial_\alpha\partial_\beta u) + O(|u'|^3|u''|), \tag{1.12}$$

where

$$Z_{\alpha\beta\gamma\delta} = \left.\frac{\partial^2 a_{\alpha\beta}(u')}{\partial(\partial_\gamma u)\partial(\partial_\delta u)}\right|_{u'=0}. \tag{1.13}$$

Thus we define an important quantity

$$H = \max_{\rho\in\mathbb{R},\omega\in S^1}\{-C(-1,\omega)\partial_\rho\mathcal{F}(\omega,\rho)\partial_\rho^2\mathcal{F}(\omega,\rho)\}, \tag{1.14}$$

where

$$C(X) = Z_{\alpha\beta\gamma\delta}X_\alpha X_\beta X_\gamma X_\delta, \quad X = (X_0, X_1, X_2), \tag{1.15}$$

and $C(-1,\omega)$ is defined by setting $X_0 = -1$, $(X_1, X_2) = \omega \in S^1$. By (1.10) and (1.11), we find that H is well-defined and non-negative.

Theorem 1. $\liminf_{\varepsilon\to 0} \varepsilon^2 \log(1+T_\varepsilon) \geqslant \frac{1}{H}$.

Proof of this theorem is basically owed to the method in F. John [6]. When $a_{\alpha\beta}(u') = O(|u'|)$ in three space dimensions, he proved that

$$\liminf_{\varepsilon\to 0} \varepsilon \log(1+T_\varepsilon) \geqslant \frac{1}{H^*},$$

where

$$H^* = \max_{\rho\in\mathbb{R},\omega\in S^2}\Big\{-\frac{1}{2}C(-1,\omega)\partial_\rho^2\mathcal{F}(\omega,\rho)\Big\}.$$

We next study the interesting case $H = 0$. The condition is equivalent to the condition (i) f and g vanish identically or (ii) $C(-1,\omega) = 0$ for any $\omega \in S^1$ (see Appendix). Under the condition (i), the Cauchy problem (1.1), (1.2) has a trivial global solution $u \equiv 0$. Under the condition (ii) which is called Klainerman's null-condition, we find from (1.15) that $C(X)$

is devided by $X_0^2 - X_1^2 - X_2^2$. Hence $a_{\alpha\beta}(u')\partial_\alpha\partial_\beta u$ is represented as a linear combination of the followings:

$$C_{\alpha\beta}(\partial_\alpha\partial_\beta u)\{(\partial_0 u)^2 - (\partial_1 u)^2 - (\partial_2 u)^2\} + O(|u'|^3|u''|), \tag{1.16a}$$

$$C_{\alpha\beta}(\partial_\alpha u)\partial_\beta\{(\partial_0 u)^2 - (\partial_1 u)^2 - (\partial_2 u)^2\} + O(|u'|^3|u''|), \tag{1.16b}$$

$$C_{\alpha\beta}(\partial_\alpha u)(\partial_\beta u)\Box u + O(|u'|^3|u''|), \tag{1.16c}$$

$$C_\alpha(\partial_\alpha u)\{(\partial_\beta u)(\partial_\gamma\partial_\delta u) - (\partial_\gamma u)(\partial_\beta\partial_\delta u)\} + O(|u'|^3|u''|), \quad \beta, \gamma, \delta = 0, 1, 2, \tag{1.16d}$$

where $C_{\alpha\beta}$ and C_α are constants.

Theorem 2. If $H = 0$, there exists an $\varepsilon_0 > 0$ such that $T_\varepsilon = \infty$ for any $0 < \varepsilon < \varepsilon_0$.

S. Klainerman [11], L. Hörmander [3], D. Christodoulou [1] and F. John [6] proved independently that the null-condition implies global existence for small data in three space dimensions. When the non-linear term is cubic form of u' in two space dimensions, P. Godin [2] proved the same results by making use of L^1-L^∞ estimates studied in L. Hörmander [4] and S. Klainerman [9]. Theorem 2 is obtained along the same lines as in P. Godin [2].

We will prove Theorem 1 in section 2 and Theorem 2 in section 3.

2. Proof of Theorem 1. First we introduce the generalized Sobolev space. Denote by $\Gamma_1, \Gamma_2, \cdots, \Gamma_7$, the vector fields

$$L_0 = t\partial_0 + x_1\partial_1 + x_2\partial_2, \ L_i = x_i\partial_0 + t\partial_i \quad (i = 1, 2),$$
$$\Omega = x_1\partial_2 - x_2\partial_1, \quad \partial_0, \partial_1, \partial_2,$$

respectively. These operators satisfy commutation relations

$$\begin{aligned} &[\Gamma_p, \Box] = \Gamma_p\Box - \Box\Gamma_p = 2\delta_{1p}\Box, \ p = 1, 2, \cdots, 7, \\ &[\Gamma, \Gamma] = \overline{\Sigma}\Gamma, \ [\Gamma, \partial] = \overline{\Sigma}\partial. \end{aligned} \tag{2.1}$$

$\overline{\Sigma}$ stands for finite linear combination with constant coefficients. For $\sigma \in \mathbb{Z}_+^7$ ($\mathbb{Z}_+$ is the set of non-negative integers), we put $\Gamma^\sigma = \Gamma_1^{\sigma_1}\Gamma_2^{\sigma_2}\cdots\Gamma_7^{\sigma_7}$. We define the norms

$$||v(t)||_k = \sum_{|\sigma|\leqslant k} ||\Gamma^\sigma v(\cdot,t)||_{L^2_x(\mathbb{R}^2)}, \tag{2.2}$$

$$|v(t)|_k = \sum_{|\sigma|\leqslant k} ||\Gamma^\sigma v(\cdot,t)||_{L^\infty_x(\mathbb{R}^2)}. \tag{2.3}$$

For convenience, when $k = 0$, we omit sub-index. Following propositions are very important in proving our theorems.

Proposition 2.1. (Klainerman's inequality [10]) For smooth function $v(x,t)$ ($x \in \mathbb{R}^n, n \geqslant 2$),

$$|v(x,t)| \leqslant C_n(1+|x|+t)^{-(n-1)/2}(1+\big|t-|x|\big|)^{-1/2}||v(t)||_{[\frac{n}{2}]+1}, \tag{2.4}$$

where [s] stands for the largest integer not exceeding s.

Proposition 2.2. (generalized energy estimate) If the solution u of (1.1), (1.2) exists in $C^\infty(\mathbb{R}^2 \times [0,T))$ and satisfies

$$\sup_{0\leqslant s\leqslant t} |u'(s)|_{[\frac{k+1}{2}]} < 1, \quad \sup_{0\leqslant s\leqslant t} |u'(s)| < \delta \quad \text{for} \quad 0 < t < T, \tag{2.5}$$

then

$$||u'(t)||_k \leqslant C_k||u'(0)||_k \exp(C_k \int_0^t |u'(s)|^2_{[\frac{k+1}{2}]} ds), \tag{2.6}$$

where δ is the one in (1.4) and $k \in \mathbb{N}$.

We prove Proposition 2.2. Multiplying Lv by $\partial_0 v$ and integrating with respect to x over $\mathbb{R}^2$, we arrive at the "energy identity" for a scalar v:

$$\frac{d}{dt}\int_{\mathbb{R}^2}\{(\partial_\alpha v)(\partial_\alpha v) - a_{00}(\partial_0 v)^2 + a_{ij}(\partial_i v)(\partial_j v)\}dx = \int_{\mathbb{R}^2} J(t,x)dx,$$

where $L = \Box - a_{\alpha\beta}(u')\partial_\alpha\partial_\beta$ and

$$\begin{aligned} J = {} & 2(\partial_0 v)(Lv) - (\partial_0 a_{00} + 2\partial_i a_{i0})(\partial_0 v)^2 - 2(\partial_j a_{ij})(\partial_0 v)(\partial_i v) \\ & + (\partial_0 a_{ij})(\partial_i v)(\partial_j v). \end{aligned} \tag{2.7}$$

Using assumption (1.4c), we get

$$||v'(t)||^2 \leqslant 3||v'(0)||^2 + 2\int_0^t ds \int_{\mathbb{R}^2} |J(s,x)|dx, \tag{2.8}$$

which implies

$$||u'(t)||^2 \leqslant 3||u'(0)||^2 + C\int_0^t |u'(s)|^2_{[\frac{k+1}{2}]}||u'(s)||_k^2 ds. \tag{2.9}$$

Using (1.1) and (2.1), we verify that for $|\sigma| \leqslant k$

$$L\Gamma^\sigma u = \Sigma\phi(u')(\Gamma^{\xi_1}\partial_{\alpha_1}u)(\Gamma^{\xi_2}\partial_{\alpha_2}u)\cdots(\Gamma^{\xi_q}\partial_{\alpha_q}u), \tag{2.10}$$

where $\phi \in C^\infty$ in u' is formed from the $a_{\alpha\beta}(u')$, and q and the multi-indices ξ_i satisfy

$$3 \leqslant q \leqslant k+2, \quad |\xi_1| + |\xi_2| + \cdots + |\xi_q| \leqslant k+1. \tag{2.11}$$

By (2.9) we can assume that

$$|\xi_p| \leqslant [\frac{k+1}{2}] \quad \text{for} \quad p \geqslant 2. \tag{2.12}$$

Therefore we find from (2.7), (2.10), (2.11), (2.12) and (2.5) that

$$\int_{\mathbb{R}^2} |J(s,x)|dx = O(|u'(s)|^2_{[\frac{k+1}{2}]}||u'(s)||_k^2).$$

Applying (2.8) to $v = \Gamma^\sigma u$ and combining with (2.9), we get

$$||u'(t)||_k^2 \leqslant C_k(||u'(0)||_k^2 + \int_0^t |u'(s)|^2_{[\frac{k+1}{2}]}||u'(s)||_k^2 ds).$$

Therefore Gronwall's inequality yields (2.6).

In order to prove Theorem 1, it is sufficient to show following lemma.

Lemma 2.1. For any $k \geqslant 9$ ($k \in \mathbb{N}$), $B > H$ and $m > 0$, there exist $J_k(B) > 0$ and $\varepsilon_k(m,B) > 0$ such that if

$$\tau < \min\{T_\varepsilon, -1 + \exp(\frac{1}{B\varepsilon^2})\} \tag{2.13}$$

and

$$|u'(t)|_{[\frac{k+1}{2}]} < \frac{m\varepsilon}{(1+t)^{1/2}} \quad \text{for} \quad 0 \leqslant t < \tau, \tag{2.14}$$

then for any $0 < \varepsilon < \varepsilon_k(m, B)$

$$|u'(t)|_{[\frac{k+1}{2}]} < \frac{J_k(B)\varepsilon}{(1+t)^{1/2}} \quad \text{for} \quad 0 \leqslant t < \tau, \tag{2.15}$$

where H is given in (1.14).

We shall prove Theorem 1 by assuming Lemma 2.1. Let $U(x,t)$ be the solution of (1.6), (1.7). We find from (1.4b) that

$$\Gamma^\sigma u|_{t=0} = \varepsilon \Gamma^\sigma U|_{t=0} + O(\varepsilon^3) \quad \text{for any} \quad \sigma \in \mathbb{Z}_+^7. \tag{2.16}$$

Therefore by (2.16)

$$||u'(0)||_k \leqslant C_k \varepsilon.$$

Moreover by $k \geqslant 9$ and (2.4)

$$|u'(0)|_{[\frac{k+1}{2}]} \leqslant |u'(0)|_{k-2} \leqslant C_k ||u'(0)||_k \leqslant C_k \varepsilon. \tag{2.17}$$

Letting $m > \max\{2J_k(B), C_k\}$, we get for sufficiently small τ

$$|u'(t)|_{[\frac{k+1}{2}]} < \frac{m\varepsilon}{(1+t)^{1/2}} \quad \text{for} \quad 0 \leqslant t < \tau. \tag{2.18}$$

If (2.18) holds for any τ, then $T_\varepsilon = \infty$. Hence there exists a τ $(0 < \tau < T_\varepsilon)$ such that (2.18) holds and

$$|u'(\tau)|_{[\frac{k+1}{2}]} = \frac{m\varepsilon}{(1+\tau)^{1/2}}. \tag{2.19}$$

Suppose that $\tau < -1 + \exp(\frac{1}{B\varepsilon^2})$, then we can apply Lemma 2.1 and obtain

$$|u'(t)|_{[\frac{k+1}{2}]} < \frac{J_k(B)\varepsilon}{(1+t)^{1/2}} < \frac{m\varepsilon}{2(1+t)^{1/2}} \quad \text{for} \quad 0 \leqslant t < \tau.$$

This contradicts (2.19). Therefore we have

$$T_\varepsilon > \tau \geqslant -1 + \exp(\frac{1}{B\varepsilon^2}) \quad \text{for} \quad 0 < \varepsilon < \varepsilon_k(m, B).$$

Since B is arbitrary except for condition "$B > H$", Theorem 1 follows.

Now we prove Lemma 2.1. First we verify that (2.15) holds for $0 \leqslant t \leqslant \varepsilon^{-1}$. By (2.17) we get

$$||u'(0)||_k < C_k \varepsilon.$$

Then for sufficiently small t

$$||u'(t)||_k < 2C_k^2 \varepsilon,$$

also by (2.4) and $k \geqslant 9$

$$|u'(t)|_{[\frac{k+1}{2}]} < \frac{2C_k^3 \varepsilon}{(1+t)^{1/2}} < \delta,$$

for sufficiently small ε. Using Proposition 2.2, we find that these inequality will continue to hold as long as

$$4C_k^7 \varepsilon^2 \log(1+t) < \log 2.$$

Therefore if we take ε such that

$$4C_k^7 \varepsilon^2 \log(1+\varepsilon^{-1}) < \log 2,$$

(2.15) holds for $0 \leqslant t < \varepsilon^{-1}$.

Moreover we have $u(x,t) = 0$ for $|x| \geqslant t + M$ (see [8], Appendix 1). Therefore we can restrict ourselves in the region

$$\varepsilon^{-1} \leqslant t < \tau, \quad |x| \leqslant t + M. \tag{2.20}$$

In order to show (2.15) in the region (2.20), we introduce "*pseudo characteristic rays*" in (r,t)-plane, which is given by solutions of ordinary differential equations;

$$\frac{dr}{dt} = \kappa(r,t), \tag{2.21}$$

where $\omega \in S^1$ is fixed and

$$\kappa(r,t) = 1 + \frac{1}{2} C(-1,\omega)(\partial_0 u)^2. \tag{2.22}$$

For each point (r,t) with $r \geqslant 0$, $\varepsilon^{-1} \leqslant t < \tau$, there exists such a curve through this point. Continuing this curve backwards, we arrive at a point (r_1, t_1) for which either $r_1 = 0, t_1 > \varepsilon^{-1}$ or $r_1 \geqslant 0, t_1 = \varepsilon^{-1}$. We call S_λ the solution of (2.21) with $t_1 - r_1 = \lambda$.

Along S_λ, we find that

$$\left|\frac{d(t-r-\lambda)}{dt}\right| = |1-\kappa| \leqslant C|u'|^2 \leqslant \frac{Cm^2\varepsilon^2}{1+t}, \tag{2.23}$$

$$|t-r-\lambda| \leqslant Cm^2\varepsilon^2\log(1+t) \leqslant \frac{Cm^2}{B}, \tag{2.24}$$

where we have used (2.21), (2.22), (2.14) and (2.13). We take $\varepsilon_k(m,B)$ such that $\varepsilon_k(m,B) < \delta m^{-1}$, then Proposition 2.2, (2.14) and (2.13) yield

$$||u'(t)||_k < C_k\varepsilon\exp(C_k\int_0^t \frac{m^2\varepsilon^2}{1+s}ds) < C_k\varepsilon\exp(\frac{C_k m^2}{B}).$$

Therefore by $k \geqslant 9$ and Proposition 2.1

$$|u'(t)|_{[\frac{k+1}{2}]+2} \leqslant |u'(t)|_{k-2} \leqslant \frac{C_k\varepsilon\exp(\frac{C_k m^2}{B})}{(1+t)^{1/2}(1+|t-r|)^{1/2}}. \tag{2.25}$$

We set

$$\lambda_0 = \frac{Cm^2}{B} + \exp(\frac{2C_k m^2}{B}). \tag{2.26}$$

Then by (2.24), (2.25) and (2.26) along S_λ with $\lambda \geqslant \lambda_0$

$$|u'(t)|_{[\frac{k+1}{2}]} \leqslant \frac{C_k\varepsilon\exp(\frac{C_k m^2}{B})}{(1+t)^{1/2}(1+\exp(\frac{2C_k m^2}{B}))^{1/2}} \leqslant \frac{C_k\varepsilon}{(1+t)^{1/2}} \quad \text{for} \quad 0 \leqslant t < \tau.$$

This implies that (2.15) is valid along S_λ with $\lambda \geqslant \lambda_0$, then it is sufficient to show (2.15) along S_λ with $-M \leqslant \lambda \leqslant \lambda_0$, $\varepsilon^{-1} \leqslant t < \tau$.

For functions $\varphi(x,t;\varepsilon), \psi(x,t;\varepsilon)$ we write $\varphi = O^*(\psi)$, if for any $k \geqslant 9, B > H$ and $m > 0$, there exist $J_k(B) > 0$ and $\varepsilon_k(m,B) > 0$ such that

$$|\varphi(x,t;\varepsilon)| < J_k(B)\psi(x,t;\varepsilon) \quad \text{for} \quad 0 < \varepsilon < \varepsilon_k(m,B),$$

as long as (2.13),(2.14) hold, along S_λ with $-M \leqslant \lambda \leqslant \lambda_0$ and $\varepsilon^{-1} \leqslant t < \tau$. Then our purpose is to prove

$$|u'(t)|_{[\frac{k+1}{2}]} = O^*(\varepsilon(1+t)^{-1/2}). \tag{2.27}$$

If we take $\varepsilon_k(m,B) < \lambda_0^{-1/p}$, then we find

$$\lambda_0 = O^*(\varepsilon^{-p}) \quad \text{for fixed} \quad p > 0. \tag{2.28}$$

For later we shall assume that $p < \frac{1}{8}$. We find from (2.28) and (2.24) that

$$t = r + O^*(\varepsilon^{-p}), \quad r^{-1} = t^{-1} + O^*(\varepsilon^{-p}t^{-2}), \quad r^{1/2} = t^{1/2} + O^*(\varepsilon^{-p}t^{-1/2}). \tag{2.29}$$

Then it follows from (2.25), (2.29) and (2.28) that

$$|u'(t)|_{[\frac{k+1}{2}]+2} = O^*(\varepsilon^{1-p}t^{-1/2}). \tag{2.30}$$

Since

$$\begin{aligned}|\Gamma^\sigma u(x,t)| &= \left| -\int_r^{t+M} \omega_i \partial_i \Gamma^\sigma u(s\omega, t) ds \right| \\ &= O\Big(\int_r^{t+M} |u'(t)|_{[\frac{k+1}{2}]+2} ds\Big) = O^*(\varepsilon^{1-2p}t^{-1/2}),\end{aligned}$$

for $|\sigma| \leqslant [\frac{k+1}{2}] + 2$, we have

$$|u(t)|_{[\frac{k+1}{2}]+2} = O^*(\varepsilon^{1-2p}t^{-1/2}). \tag{2.31}$$

The operator ∂_α can be written as

$$\partial_i = -\omega_i \partial_0 + \frac{1}{t} L_i + \frac{\omega_i}{t+r} L_0 - \frac{r\omega_i\omega_j}{t(t+r)} L_j, \quad \partial_0 = \frac{1}{t^2 - r^2}(tL_0 - x_i L_i),$$

(see [6], Appendix 2 and [7]) and these representations yield

$$\partial_\alpha v = -\omega_\alpha \partial_0 v + O(t^{-1}|v|_1) = -\omega_\alpha \partial_r v + O(t^{-1}|v|_1), \tag{2.32}$$

$$\partial_\alpha v = O(\frac{1}{|t-r|}|v|_1), \tag{2.33}$$

$$\partial_\alpha \partial_\beta v = \omega_\alpha \omega_\beta \partial_0^2 v + O(t^{-1}|v'|_1) = \omega_\alpha \omega_\beta \partial_r^2 v + O(t^{-1}|v'|_1), \tag{2.34}$$

$$(\partial_0 + \partial_r)v = O(t^{-1}|v|_1), \quad (\partial_0 + \partial_r)^2 v = O(t^{-2}|v|_2), \tag{2.35}$$

where $\partial_r = \omega_i \partial_i$.

The operator L can be written in the form

$$\begin{aligned}Lv = {} & 2t^{-1/2}(\partial_0 + \kappa\partial_r)(t^{1/2}\partial_0 v) - (\partial_0 + \partial_r)^2 v + \frac{t-r}{tr}\partial_0 v \\ & - 2(\kappa - 1)(\partial_0 + \partial_r)\partial_0 v - \frac{\delta_{ij} - \omega_i\omega_j}{t^2} L_i L_j v + \frac{\omega_i}{tr} L_i v \\ & - a_{\alpha\beta}(u')\partial_\alpha\partial_\beta v + C(-1, \omega)\partial_0^2 v.\end{aligned}$$

Then by (2.29), (2.33), (2.34), (2.35) and (2.30)

$$\frac{d}{dt}(t^{1/2}\partial_0 v) = \frac{1}{2}t^{1/2}Lv + O^*(t^{-3/2}|v|_2). \tag{2.36}$$

We apply (2.36) to $v = \Gamma^\sigma u$ with $|\sigma| \leqslant [\frac{k+1}{2}]$ below. When $v = \Gamma^\sigma u$

$$t^{1/2}\partial_0\Gamma^\sigma u(t)|_{t=\varepsilon^{-1}} = O(\varepsilon^{-1/2}|u'(\varepsilon^{-1})|_{[\frac{k+1}{2}]}) = O(\varepsilon). \tag{2.37}$$

Now we show (2.27) by induction. We first show

$$|u'(t)| = O^*(\varepsilon t^{-1/2}). \tag{2.38}$$

Let $v = u$ in (2.36). Then it follows from (2.31) and $Lu = 0$ that

$$\frac{d}{dt}(t^{1/2}\partial_0 u) = O^*(\varepsilon^{1-2p}t^{-2}). \tag{2.39}$$

Integrating (2.39) from ε^{-1} to t, we find from (2.37) that

$$|t^{1/2}\partial_0 u(t)| \leqslant |s^{1/2}\partial_0 u(s)|_{s=\varepsilon^{-1}}| + O^*\left(\int_{\varepsilon^{-1}}^t \varepsilon^{1-2p}s^{-2}ds\right) = O^*(\varepsilon), \tag{2.40}$$

which implies

$$\partial_0 u(t) = O^*(\varepsilon t^{-1/2}). \tag{2.41}$$

Using (2.32), (2.31) and (2.41),

$$\begin{aligned}\partial_i u(t) &= -\omega_i\partial_0 u + O(t^{-1}|u|)\\ &= O^*(\varepsilon t^{-1/2}) + O^*(\varepsilon^{1-2p}t^{-3/2}) = O^*(\varepsilon).\end{aligned}$$

Next we shall show

$$|u'(t)|_1 = O^*(\varepsilon t^{-1/2}). \tag{2.42}$$

We begin the proof of (2.42) by showing

$$\partial_\alpha\partial_\beta u(t) = O^*(\varepsilon t^{-1/2}). \tag{2.43}$$

Letting $v = \partial_0 u$ in (2.36), it follows from $Lu = 0$, (1.15), (2.32), (2.34), (2.30) and (2.31) that

$$\begin{aligned}
L(\partial_0 u) &= \Box(\partial_0 u) - a_{\alpha\beta}(u')\partial_\alpha\partial_\beta\partial_0 u \\
&= (\partial_0 a_{\alpha\beta}(u'))\partial_\alpha\partial_\beta u \\
&= Z_{\alpha\beta\gamma\delta}\partial_0\{(\partial_\gamma u)(\partial_\delta u)\}\partial_\alpha\partial_\beta u + O(|u'|^2|u''|^2) \\
&= 2C(-1,\omega)(\partial_0^2 u)^2\partial_0 u + O(|u'|^2|u''|^2 + t^{-1}|u|_1|u'|_1^2) \\
&= 2C(-1,\omega)t^{-1}(t^{\frac{1}{2}}\partial_0^2 u)^2\partial_0 u + O^*(\varepsilon^{3-4p}t^{-2}).
\end{aligned}$$

Then we write the differential equation (2.36) as

$$\frac{d}{dt}W_1(t) = C(-1,\omega)t^{-1/2}W_1(t)^2\partial_0 u(t) + O^*(\varepsilon^{1-p}t^{-3/2}), \tag{2.44}$$

where

$$W_1(t) = t^{1/2}\partial_0^2 u(t). \tag{2.45}$$

Notice that by (2.30),

$$W_1(t) = O^*(\varepsilon^{1-p}). \tag{2.46}$$

The following facts play an important role in our proof.

$$|\varepsilon\partial_0^\ell U(x,t) - (-1)^\ell\partial_\rho^\ell\mathcal{F}(\omega,\rho)| = O(\varepsilon r^{-3/2}), \tag{2.47}$$

$$|\partial_0^\ell u(x,\varepsilon^{-1}) - \varepsilon\partial_0^\ell U(x,\varepsilon^{-1})| = O(\varepsilon^3), \tag{2.48}$$

with $\ell = 1,2$. (2.47) can be proved by using Lemma 2.1.1 in [3]. We prove (2.48) for $\ell = 1$ because another case can be proved in the similar way. By (2.16), the function $u - \varepsilon U$ satisfies

$$\Box(u - \varepsilon U) = a_{\alpha\beta}(u')\partial_\alpha\partial_\beta u, \tag{2.49}$$

$$||\partial_0 u(0) - \varepsilon\partial_0 U(0)|| = O(\varepsilon^3). \tag{2.50}$$

Applying Proposition 2.1 and classical energy estimate to (2.49), (2.50), we find that

$$\begin{aligned}
|\partial_0 u(\varepsilon^{-1}) - \varepsilon\partial_0 U(\varepsilon^{-1})| &\leqslant C(1+\varepsilon^{-1})^{-1/2}||\partial_0 u(\varepsilon^{-1}) - \varepsilon\partial_0 U(\varepsilon^{-1})|| \\
&\leqslant C\varepsilon^{1/2}(||\partial_0 u(0) - \varepsilon\partial_0 U(0)|| + \int_0^{\varepsilon^{-1}} ||a_{\alpha\beta}(u'(s))\partial_\alpha\partial_\beta u(s)||ds).
\end{aligned}$$

Since $0 \leqslant s \leqslant \varepsilon^{-1}$, we find that

$$||a_{\alpha\beta}(u'(s))\partial_\alpha\partial_\beta u(s)|| \leqslant C\left(\int_{\mathbb{R}^2} |u'(s)|^6_{[\frac{k+1}{2}]} ds\right)^{1/2} = O(\varepsilon^3 s^{-1/2}).$$

Therefore we have

$$|\partial_0 u(\varepsilon^{-1}) - \varepsilon\partial_0 U(\varepsilon^{-1})| = O(\varepsilon^{7/2} + \varepsilon^{1/2}\int_0^{\varepsilon^{-1}} \varepsilon^3 s^{-1/2} ds) = O(\varepsilon^3).$$

This implies (2.44) with $\ell = 1$.

It follows from (2.40), (2.47) and (2.48) that

$$\begin{aligned}
t^{1/2}\partial_0 u(t) &= (\varepsilon^{-1})^{1/2}\partial_0 u(\varepsilon^{-1}) + O^*(\varepsilon^{3/2}) \\
&= (\varepsilon^{-1})^{1/2}\varepsilon\partial_0 U(\varepsilon^{-1}) + O^*(\varepsilon^{3/2}) \\
&= (\varepsilon^{-1} - \lambda)^{1/2}\varepsilon\partial_0 U(\varepsilon^{-1}) + O^*(\varepsilon^{3/2}) + O(\{(\varepsilon^{-1})^{1/2} - (\varepsilon^{-1} - \lambda)^{1/2}\}\varepsilon) \\
&= -\varepsilon\partial_\rho\mathcal{F}(\omega, -\lambda) + O^*(\varepsilon^{3/2}).
\end{aligned}$$

On the other hand, by (2.47) and (2.48)

$$\begin{aligned}
W_1(\varepsilon^{-1}) &= (\varepsilon^{-1})^{1/2}\partial_0^2 u(\varepsilon^{-1}) \\
&= (\varepsilon^{-1})^{1/2}\varepsilon\partial_0^2 U(\varepsilon^{-1}) + O(\varepsilon^{5/2}) \\
&= (\varepsilon^{-1} - \lambda)^{1/2}\varepsilon\partial_0^2 U(\varepsilon^{-1}) + O(\varepsilon^{5/2}) + O(\{(\varepsilon^{-1})^{1/2} - (\varepsilon^{-1} - \lambda)^{1/2}\}\varepsilon) \\
&= \varepsilon\partial_\rho^2\mathcal{F}(\omega, -\lambda) + O(\varepsilon^{3/2}).
\end{aligned} \tag{2.52}$$

Then by (2.46), (2.51) and (2.52) we rewrite (2.44)

$$\frac{d}{dt}W_1(t) = -\varepsilon C(-1, \omega)\partial_\rho\mathcal{F}(\omega, -\lambda)t^{-1}W_1(t)^2 + O^*(\varepsilon^{1-p}t^{-3/2} + \varepsilon^{7/2-2p}t^{-1}), \tag{2.53}$$

$$W_1(\varepsilon^{-1}) = \varepsilon\partial_\rho^2\mathcal{F}(\omega, -\lambda) + O(\varepsilon^{3/2}). \tag{2.54}$$

If the solution $W_1(t)$ of (2.53), (2.54) satisfies

$$W_1(t) = O^*(\varepsilon), \tag{2.55}$$

then we obtain (2.43). Indeed, using (2.45), we have

$$\partial_0^2 u(t) = O^*(\varepsilon t^{-1/2}), \tag{2.56}$$

then (2.34), (2.30) and (2.56) yield (2.43).

We shall show (2.55). Multiplying sgnW_1 to both sides of (2.53),

$$\frac{d}{dt}|W_1(t)| \leqslant -\varepsilon C(-1,\omega)\partial_\rho\mathcal{F}(\omega,-\lambda)t^{-1}|W_1(t)|^2 + |L(t)|, \tag{2.57}$$

where

$$L(t) = O^*(\varepsilon^{1-p}t^{-3/2} + \varepsilon^{7/2-2p}t^{-1}).$$

Note that

$$\int_{\varepsilon^{-1}}^{t} |L(s)|ds = O^*(\varepsilon^{5/4}). \tag{2.58}$$

Replacing if necessary W_1 by $-W_1$, we may assume

$$W_1(\varepsilon^{-1}) \geqslant 0.$$

We set

$$\beta(t) = W_1(\varepsilon^{-1}) + J_k(B)\varepsilon^{5/4} - \varepsilon C(-1,\omega)\partial_\rho\mathcal{F}(\omega,-\lambda)\int_{\varepsilon^{-1}}^{t} |W_1(s)|^2 s^{-1}ds,$$

then we find that

$$0 \leqslant |W_1(t)| \leqslant \beta(t).$$

If $C(-1,\omega)\partial_\rho\mathcal{F}(\omega,-\lambda) \geqslant 0$, by (2.57)

$$\frac{d}{dt}|W_1(t)| \leqslant |L(t)|.$$

Integrating this inequality from ε^{-1} to t, we obtain by (2.37)

$$|W_1(t)| \leqslant W_1(\varepsilon^{-1}) + O^*(\varepsilon^{5/4}) = O^*(\varepsilon).$$

Therefore we assume $C(-1,\omega)\partial_\rho\mathcal{F}(\omega,-\lambda) < 0$. In this case,

$$\begin{aligned}\frac{d}{dt}\beta(t) &= -\varepsilon C(-1,\omega)\partial_\rho\mathcal{F}(\omega,-\lambda)t^{-1}|W(t)|^2 \\ &\leqslant -\varepsilon C(-1,\omega)\partial_\rho\mathcal{F}(\omega,-\lambda)t^{-1}\beta(t)^2.\end{aligned} \tag{2.59}$$

Now we consider a Cauchy problem;

$$\frac{d}{dt}Z(t) = -\varepsilon C(-1,\omega)\partial_\rho \mathcal{F}(\omega,-\lambda)t^{-1}Z(t)^2, \tag{2.60}$$

$$Z(\varepsilon^{-1}) = \beta(\varepsilon^{-1}) = W_1(\varepsilon^{-1}) + J_k(B)\varepsilon^{5/4}. \tag{2.61}$$

Then by (2.59) and (2.60)

$$\begin{aligned}
&\frac{d}{dt}\{(Z(t)-\beta(t))\exp(\varepsilon C(-1,\omega)\partial_\rho\mathcal{F}(\omega,-\lambda)\int_{\varepsilon^{-1}}^{t}(z(s)+\beta(s))s^{-1}ds)\}\\
=&\{\frac{d}{dt}Z(t)-\frac{d}{dt}\beta(t)+\varepsilon C(-1,\omega)\partial_\rho\mathcal{F}(\omega,-\lambda)t^{-1}(Z(t)^2-\beta(t)^2)\}\\
&\times\exp(\varepsilon C(-1,\omega)\partial_\rho\mathcal{F}(\omega,-\lambda)\int_{\varepsilon^{-1}}^{t}(Z(s)+\beta(s))s^{-1}ds)\geqslant 0.
\end{aligned}$$

Since $Z(\varepsilon^{-1}) = \beta(\varepsilon^{-1})$, we have

$$\beta(t) \leqslant Z(t).$$

Solving (2.60), (2.61) explicitly, we have by (2.13) and (2.45)

$$\begin{aligned}
Z(t) &= \frac{Z(\varepsilon^{-1})}{1-\varepsilon Z(\varepsilon^{-1})(-C(-1,\omega))\partial_\rho\mathcal{F}(\omega,-\lambda)\log t}\\
&= \frac{O^*(\varepsilon)}{1-\varepsilon(\varepsilon\partial_\rho^2\mathcal{F}(\omega,-\lambda)+O^*(\varepsilon^{5/4}))(-C(-1,\omega))\partial_\rho\mathcal{F}(\omega,-\lambda)\log t}\\
&= \frac{O^*(\varepsilon)}{1-\frac{1}{B}\{(-C(-1,\omega))\partial_\rho\mathcal{F}(\omega,-\lambda)\partial_\rho^2\mathcal{F}(\omega,-\lambda)+O^*(\varepsilon^{1/4})\}}\\
&\leqslant \frac{O^*(\varepsilon)}{\frac{1}{B}(B-H+O^*(\varepsilon^{1/4}))} = O^*(\varepsilon).
\end{aligned}$$

Hence we have

$$0 \leqslant |W_1(t)| \leqslant \beta(t) \leqslant Z(t) = O^*(\varepsilon),$$

then (2.55) holds.

Now we prove (2.42). Let $v = \Gamma u$ in (2.36) (Γ is an arbitrary one) and set

$$W(t) = t^{1/2}\partial_0\Gamma u(t). \tag{2.62}$$

It follows from (2.1), (2.32), (2.38), (2.43) and (2.62) that

$$\begin{aligned}
L(\Gamma u) &= \Box\Gamma u - a_{\alpha\beta}(u')\partial_\alpha\partial_\beta\Gamma u \\
&= \Box\Gamma u - \Gamma\Box u + \Gamma(a_{\alpha\beta}(u')\partial_\alpha\partial_\beta u) - a_{\alpha\beta}(u')\partial_\alpha\partial_\beta\Gamma u \\
&= C\Box u + (\Gamma a_{\alpha\beta}(u'))\partial_\alpha\partial_\beta u + a_{\alpha\beta}(u')(\Gamma\partial_\alpha\partial_\beta u - \partial_\alpha\partial_\beta\Gamma u) \\
&= C a_{\alpha\beta}(u')\partial_\alpha\partial_\beta u + (\Gamma a_{\alpha\beta}(u'))\partial_\alpha\partial_\beta u + a_{\alpha\beta}(u')\partial_\gamma\partial_\delta u \\
&= O(|u'|^2|u''| + |\Gamma u'||u'||u''|) \\
&= O(|u'|^2|u''| + |\partial_\alpha\Gamma u||u'||u''|) \\
&= O(|u'|^2|u''| + |\partial_0\Gamma u||u'||u''| + t^{-1}|u|_2|u'||u''|) \\
&= O^*(\varepsilon^3 t^{-3/2} + \varepsilon^2 t^{-3/2}|W| + \varepsilon^{3-2p}t^{-5/2}) \\
&= O^*(\varepsilon^3 t^{-3/2} + \varepsilon^2 t^{-3/2}|W|).
\end{aligned}$$

Therefore by (2.36) and (2.31)

$$\frac{d}{dt}W(t) = O^*(\varepsilon^3 t^{-1} + \varepsilon^2 t^{-1}|W(t)| + \varepsilon^{1-2p}t^{-2}).$$

Integrating this equation from ε^{-1} to t and using (2.13) and (2.37), we have

$$\begin{aligned}
|W(t)| &\leqslant |W(\varepsilon^{-1})| + O^*\left(\int_{\varepsilon^{-1}}^{t}(\varepsilon^3 s^{-1} + \varepsilon^2|W(s)|s^{-1} + \varepsilon^{1-p}s^{-2})ds\right) \\
&= O^*(\varepsilon) + O^*\left(\varepsilon^2\int_{\varepsilon^{-1}}^{t}|W(s)|s^{-1}ds\right).
\end{aligned}$$

Hence Gronwall's inequality and (2.13) lead to

$$|W(t)| = O^*(\varepsilon\exp(\varepsilon^2\log t)) = O^*(\varepsilon e^{1/B}) = O^*(\varepsilon),$$

i.e.,

$$\partial_0\Gamma u(t) = O^*(\varepsilon t^{-1/2}). \tag{2.63}$$

We also find from (2.1), (2.32), (2.31), (2.38) and (2.63) that (2.42) holds.

Finally we prove (2.27). It is sufficient to show that

$$|u'(t)|_l = O^*(\varepsilon t^{-1/2}), \tag{2.64}$$

under the assumption

$$|u'(t)|_{\ell-1} = O^*(\varepsilon t^{-1/2}), \tag{2.65}$$

where $1 \leqslant \ell \leqslant [\frac{k+1}{2}]$. Let $v = \Gamma^\ell u$ (for $\ell \in \mathbb{Z}_+, \Gamma^\ell$ stands for $\sum_{|\sigma|=\ell} \Gamma^\sigma$) in (2.36) and set

$$W(t) = t^{1/2}\partial_0 \Gamma^\ell u(t). \tag{2.66}$$

Using (2.1), (2.32), (2.38), (2.42) and (2.65), we have

$$\begin{aligned}
L(\Gamma^\ell u) &= \Box \Gamma^\ell u - a_{\alpha\beta}(u')\partial_\alpha\partial_\beta \Gamma^\ell u \\
&= \Gamma^\ell \Box u + \sum_{\mu<\ell} C_\mu \Gamma^\mu \Box u - a_{\alpha\beta}(u')\partial_\alpha\partial_\beta \Gamma^\ell u \\
&= \Gamma^\ell (a_{\alpha\beta}(u')\partial_\alpha\partial_\beta u) + \sum_{\mu<\ell} C_\mu \Gamma^\mu (a_{\alpha\beta}(u')\partial_\alpha\partial_\beta u) - a_{\alpha\beta}(u')\partial_\alpha\partial_\beta \Gamma^\ell u \\
&= \sum_{\nu<\ell} \binom{\ell}{\nu} (\Gamma^{\ell-\nu} a_{\alpha\beta}(u'))(\Gamma^\nu \partial_\alpha\partial_\beta u) + a_{\alpha\beta}(u')\Gamma^\ell \partial_\alpha\partial_\beta u \\
&\quad + \sum_{\mu<\ell} C_\mu \Gamma^\mu (a_{\alpha\beta}(u')\partial_\alpha\partial_\beta u) - a_{\alpha\beta}(u')\partial_\alpha\partial_\beta \Gamma^\ell u \\
&= O(|u'|_\eta |u'|_\zeta |u'|_\xi) \quad (\eta,\zeta,\xi \leqslant \ell,\ \eta+\zeta+\xi \leqslant \ell+1) \\
&= O(|u'|^3_{\ell-1} + |u'|_0 |u'|_1 |u'|_\ell) \\
&= O^*(\varepsilon^3 t^{-3/2} + \varepsilon^2 t^{-1}|u'|_\ell) \\
&= O^*(\varepsilon^3 t^{-3/2} + \varepsilon^2 t^{-3/2} V(t)),
\end{aligned}$$

where

$$V(t) = t^{1/2}|u'(t)|_\ell. \tag{2.67}$$

Therefore by (2.36) and (2.31)

$$\frac{d}{dt}W(t) = O^*(\varepsilon^3 t^{-1} + \varepsilon^2 t^{-1} V(t) + \varepsilon^{1-2p} t^{-2}).$$

Integrating this equation from ε^{-1} to t and using (2.13) and (2.37), we have

$$\begin{aligned}
|W(t)| &\leqslant |W(\varepsilon^{-1})| + O^*\left(\int_{\varepsilon^{-1}}^t (\varepsilon^3 s^{-1} + \varepsilon^2 V(s)s^{-1} + \varepsilon^{1-2p}s^{-2})ds\right) \\
&= O^*(\varepsilon) + O^*\left(\varepsilon^2 \int_{\varepsilon^{-1}}^t V(s)s^{-1}ds\right).
\end{aligned}$$

Then by (2.66), (2.1), (2.32), (2.37), (2.65) and (2.31)

$$V(t) = O^*(\varepsilon) + O^*(\varepsilon^2 \int_{\varepsilon^{-1}}^t V(s)s^{-1}ds).$$

Gronwall's inequality and (2.13) yield

$$V(t) = O^*(\varepsilon \exp(\varepsilon^2 \log t)) = O^*(\varepsilon e^{1/B}) = O^*(\varepsilon). \tag{2.68}$$

We find from (2.67) and (2.68) that (2.64) holds and this complete the proof of Theorem 1.

3. Proof of Theorem 2. For a function $h \in C^\infty(\mathbb{R}^2 \times [0,T))$, we denote by $E_T(h)$ the solution of the Cauchy problem;

$$\Box E_T(h)(x,t) = h(x,t), \quad (x,t) \in \mathbb{R}^2 \times [0,T),$$
$$E_T(h)(x,0) = 0, \quad \partial_0 E_T(h)(x,0) = 0, \quad x \in \mathbb{R}^2.$$

Using an L^1-L^∞ estimate in Corollary 6.2 in [4] (also see [2], Lemma 4.1), we can prove;

Proposition 3.1 Suppose that $b \in \mathbb{R}$, and $h_1, h_2 \in C^\infty(\mathbb{R}^2 \times [0,T))$ have a compact support in x for fixed t, then there exists $C > 0$ such that

$$(1+t)|E_T(h_1 h_2)(x,t)|^2 \leqslant C\Big(\sum_{|\theta|\leqslant 1}\int_0^t \frac{\|\Gamma^\theta h_1(s)\|^2}{(1+s)^b}ds\Big)\Big(\sum_{|\theta|\leqslant 1}\int_0^t \frac{\|\Gamma^\theta h_2(s)\|^2}{(1+s)^{1-b}}ds\Big), \tag{3.1}$$

for $0 \leqslant t < T$. We also need following proposition which is proved in [13].

Proposition 3.2 If functions u, v are smooth and

$$u(x,t) = v(x,t) = 0 \quad \text{for} \quad |x| \geqslant t + M,$$

then we have

$$\|u(t)v'(t)\| \leqslant C_M \sum_{|\theta|=1} |\Gamma^\theta v(t)| \, \|u'(t)\|. \tag{3.2}$$

Let k be fixed and $k \geqslant 9$. For functions $\varphi(x,t;\varepsilon), \psi(x,t;\varepsilon)$, we denote $\varphi = O^*(\psi)$ if for any $m > 0$, there exist $J > 0$ and $\varepsilon(m) > 0$ such that for $0 < \varepsilon < \varepsilon(m)$

$$|\varphi(x,t;\varepsilon)| < J\psi(x,t;\varepsilon) \quad \text{for} \quad 0 \leqslant t < \tau,$$

as long as $\tau < T_\varepsilon$ and

$$|u(t)|_{[\frac{k+1}{2}]+1} < \frac{m\varepsilon}{(1+t)^{1/2}} \quad \text{for} \quad 0 \leqslant t < \tau. \tag{3.3}$$

By the similar argument in Section 2, it is sufficient to prove;

Lemma 3.1. $\quad |u(t)|_{[\frac{k+1}{2}]} = O^*(\varepsilon(1+t)^{-1/2}).$

We conclude this section by proving Lemma 3.1. Denote by F the right-hand side of (1.1). By (2.1) we have

$$\Box\Gamma^\sigma u = \sum_{\lambda \leqslant \sigma} C_\lambda \Gamma^\lambda F \quad (C_\sigma = 1).$$

Then we can write

$$\Gamma^\sigma u = W^\sigma + \sum_{\lambda \leqslant \sigma} C_\lambda E_\tau(\Gamma^\lambda F),$$

where W^σ is a solution of linear wave equation;

$$\Box W^\sigma(x,t) = 0, \quad (x,t) \in \mathbb{R}^2 \times [0,\tau),$$
$$W^\sigma(x,0) = \Gamma^\sigma u(x,0), \quad \partial_0 W^\sigma(x,0) = \partial_0 \Gamma^\sigma u(x,0), \quad x \in \mathbb{R}^2.$$

Since, as well known,

$$|W^\sigma(x,t)| \leqslant \frac{C_\sigma \varepsilon}{(1+t)^{1/2}} \quad \text{for} \quad 0 \leqslant t < \tau,$$

it is sufficient to show

$$E_\tau(\Gamma^\lambda F) = O^*(\varepsilon(1+t)^{-1/2}) \quad \text{for} \quad |\lambda| \leqslant [\frac{k+1}{2}] + 1. \tag{3.4}$$

As stated in introduction, we may assume that F has a form in (1.16a-d). We verify (3.4) for each case.

Case (1.16a). We have to show

$$E_\tau(\Gamma^\lambda(\partial_\alpha\partial_\beta u Q(u'))) = O^*(\varepsilon(1+t)^{-1/2}), \tag{3.5}$$

$$E_\tau(\Gamma^\lambda(|u'|^3|u''|)) = O^*(\varepsilon(1+t)^{-1/2}), \tag{3.6}$$

where $Q(u') = (\partial_0 u)^2 - (\partial_1 u)^2 - (\partial_2 u)^2$. It follows from Proposition 2.2 and (3.3) that

$$\begin{aligned} ||u'(t)||_k &\leqslant C\varepsilon \exp(C\int_0^t |u'(s)|^2_{[\frac{k+1}{2}]} ds) \\ &\leqslant C\varepsilon \exp(Cm^2\varepsilon^2 \log(1+t)) \\ &\leqslant C\varepsilon(1+t)^p \qquad \text{for} \quad 0 \leqslant t < \tau, \end{aligned}$$

if $Cm^2\varepsilon^2 < p$. For later we shall assume that $p < \dfrac{1}{4}$. Hence we have

$$||u'(t)||_k = O^*(\varepsilon(1+t)^p). \tag{3.7}$$

Now we shall prove (3.5). Using Proposition 3.1 with $b = -1$, we get

$$(1+t)^{1/2}|E_\tau(\Gamma^\lambda(\partial_\alpha\partial_\beta u Q(u')))| \leqslant C \sum_{\mu+\nu=\lambda} I_\mu(t) J_\nu(t), \tag{3.8}$$

where

$$I_\mu(t) = \Big(\sum_{|\theta|\leqslant 1} \int_0^t ||\Gamma^\theta\Gamma^\mu \partial_\alpha\partial_\beta u(s)||^2 (1+s)^{-2} ds\Big)^{1/2},$$

$$J_\nu(t) = \Big(\sum_{|\theta|\leqslant 1} \int_0^t ||\Gamma^\theta\Gamma^\nu Q(u'(s))||^2 (1+s) ds\Big)^{1/2}.$$

Since $k \geqslant 9, |\theta| + |\mu| + 1 \leqslant [\dfrac{k+1}{2}] + 3 \leqslant k$. Therefore, by (3.7) we get

$$I_\mu(t) = O^*\left(\left(\int_0^t \varepsilon^2(1+s)^{2p-2} ds\right)^{1/2}\right) = O^*(\varepsilon). \tag{3.9}$$

On the other hand, since

$$Q(u') = t^{-1}(L_0 u \partial_0 u - L_i u \partial_i u),$$

$\Gamma^\theta\Gamma^\nu Q(u')$ is represented as the sum of

$$\Gamma^\rho(t^{-1})\Gamma^\eta(L_\alpha u)\Gamma^\xi(\partial_\alpha u), \quad \rho+\eta+\xi=\theta+\nu, \ \alpha=0,1,2.$$

We can verify in the support of u

$$\Gamma^\rho(t^{-1}) \leqslant Ct^{-1}(1+t^{-|\rho|}). \tag{3.10}$$

Moreover we find that

$$||\Gamma^\eta(L_\alpha u(t))\Gamma^\xi(\partial_\alpha u(t))|| \leqslant C|u(t)|_{[\frac{\ell+1}{2}]}||u'(t)||_\ell, \tag{3.11}$$

where $\ell = [\frac{k+1}{2}]+3$. Indeed, if we set $\zeta=\eta+\chi$ ($|\chi|=1$, $\Gamma^\chi = L_\alpha$), we obtain $|\zeta+\xi| \leqslant [\frac{k+1}{2}]+3=\ell$. If $|\xi| \geqslant [\frac{\ell+1}{2}]$, then $|\zeta| \leqslant [\frac{\ell+1}{2}]$ and (3.11) holds. If $|\xi| < [\frac{\ell+1}{2}]$, *i.e.*, $|\xi| \leqslant [\frac{\ell+1}{2}]-1$, we have by using Proposition 3.2

$$\begin{aligned} ||\Gamma^\zeta u(t)\Gamma^\xi\partial_\alpha u(t)|| &\leqslant C\sum_{|\iota|=1}|\Gamma^{\xi+\iota}u(t)|\,||\Gamma^\zeta u'(t)|| \\ &\leqslant C|u(t)|_{[\frac{\ell+1}{2}]}||u'(t)||_\ell, \end{aligned}$$

which implies (3.11). Since $k\geqslant 9$, we get $\ell<k$ and $[\frac{\ell+1}{2}] \leqslant [\frac{k+1}{2}]$. Then, by (3.3) and (3.7)

$$\begin{aligned} ||\Gamma^\zeta u(t)\Gamma^\xi\partial_\alpha u(t)|| &\leqslant C|u(t)|_{[\frac{k+1}{2}]}||u'(t)||_k \\ &= O^*(\varepsilon(1+t)^{p-1/2}). \end{aligned} \tag{3.12}$$

Combining (3.10) and (3.12), we get

$$||\Gamma^\theta\Gamma^\nu Q(u'(t))|| = O^*(\varepsilon t^{-1}(1+t^{-|\theta+\nu|})(1+t)^{p-1/2}). \tag{3.13}$$

On the other hand, as shown in section 2,

$$||\Gamma^\theta\Gamma^\nu Q(u'(t))|| \leqslant C\varepsilon \quad \text{for} \quad 0\leqslant t\leqslant 1. \tag{3.14}$$

Hence by (3.13) and (3.14)

$$\begin{aligned} J_\nu(t) &= \left(\sum_{|\theta|\leqslant 1}\int_0^1 ||\Gamma^\theta\Gamma^\nu Q(u'(s))||^2(1+s)ds + \sum_{|\theta|\leqslant 1}\int_1^t ||\Gamma^\theta\Gamma^\nu Q(u'(s))||^2(1+s)ds\right)^{1/2} \\ &\leqslant \left(C\varepsilon + O^*\left(\int_1^t \varepsilon^2(1+s)^{2p-2}ds\right)\right)^{1/2} = O^*(\varepsilon). \end{aligned} \tag{3.15}$$

Therefore (3.5) follows from (3.9) and (3.15).

Next we shall prove (3.6). Using Proposition 3.1 with $b = \frac{1}{2}$, we get

$$(1+t)^{1/2}|E_\tau(\Gamma^\lambda(|u'|^3|u''|))| \leqslant C \sum_{\mu+\nu=\lambda} I_\mu(t)J_\nu(t), \tag{3.16}$$

where

$$I_\mu(t) = \Big(\sum_{|\theta|\leqslant 1} \int_0^t ||\Gamma^\theta\Gamma^\mu(|u'(s)|^2)||^2(1+s)^{-1/2}ds\Big)^{1/2},$$

$$J_\nu(t) = \Big(\sum_{|\theta|\leqslant 1} \int_0^t ||\Gamma^\theta\Gamma^\mu(|u'(s)|\,|u''(s)|)||^2(1+s)^{-1/2}ds\Big)^{1/2}.$$

Since $|\theta+\mu| \leqslant [\frac{k+1}{2}]+2 \leqslant k$, we can verify

$$||\Gamma^\theta\Gamma^\mu(|u'(s)|^2)|| \leqslant C|u(s)|_{[\frac{k+1}{2}]+1}||u'(s)||_k.$$

Using (3.3) and (3.7), we get

$$||\Gamma^\theta\Gamma^\mu(|u'(s)|^2)|| \leqslant Cm\varepsilon^2(1+s)^{p-1/2} = O^*(\varepsilon(1+s)^{p-1/2}).$$

Then we have

$$I_\mu(t) = O^*\Big(\Big(\int_0^t \varepsilon^2(1+s)^{2p-3/2}ds\Big)^{1/2}\Big) = O^*(\varepsilon). \tag{3.17}$$

Similarly we also find that

$$J_\nu(t) = O^*(\varepsilon). \tag{3.18}$$

Therefore (3.16), (3.17) and (3.18) imply (3.6).

Case (1.16b). We have to show

$$E_\tau(\Gamma^\lambda(\partial_\alpha u\partial_\beta Q(u'))) = O^*(\varepsilon(1+t)^{-1/2}).$$

This can be obtained similarly to (3.5), by using (3.10), (3.11) and Proposition 3.1, 3.2 but $\ell = [\frac{k+1}{2}]+4$ in (3.11).

Case (1.16c). We have to show

$$E_\tau(\Gamma^\lambda(\partial_\alpha u\partial_\beta u\Box u)) = O^*(\varepsilon(1+t)^{-1/2}).$$

Using (1.1), we get

$$\begin{aligned}\Gamma^\lambda(\partial_\alpha u\partial_\beta u\Box u) &= \Gamma^\lambda(\partial_\alpha u\partial_\beta u(\partial_\gamma u\partial_\delta u\Box u + O(|u'|^3|u''|))) \\ &= O(\Gamma^\lambda(|u'|^4|u''|)).\end{aligned}$$

Therefore this case can be reduced to (3.6).

Case (1.16d). We have to show

$$E_\tau(\Gamma^\lambda(\partial_\alpha u(\partial_\beta u\partial_\gamma\partial_\delta u - \partial_\gamma u\partial_\beta\partial_\delta u))) = O^*(\varepsilon(1+t)^{-1/2}).$$

Using (2.32) and (2.34), we have

$$\begin{aligned}\partial_\alpha u(\partial_\beta u\partial_\gamma\partial_\delta u - \partial_\gamma u\partial_\beta\partial_\delta u) &= O(t^{-1}(|u'|^2|u'|_1 + |u'||u''||u|_1)) \\ &= O(t^{-1}|u'||u|_1|u'|_1).\end{aligned}$$

Therefore this case can be verified similarly to (3.5) as $Q = t^{-1}|u'||u|_1$, then the proof of Theorem 2 is complete.

Appendix. If $H = 0$, we find from the definition of H (1.14) that

$$C(-1,\omega)\partial_\rho\mathcal{F}(\omega,\rho)\partial_\rho^2\mathcal{F}(\omega,\rho) = \frac{1}{2}C(-1,\omega)\partial_\rho((\partial_\rho\mathcal{F}(\omega,\rho))^2) \geqslant 0,$$

for any $\omega \in S^1$ and $\rho \in \mathbb{R}$. For fixed ω, if $C(-1,\omega) \neq 0$, then $\partial_\rho((\partial_\rho\mathcal{F}(\omega,\rho))^2)$ is of constant sign in ρ. Using (1.10) and (1.11), we find that $\partial_\rho\mathcal{F}(\omega,\rho) = 0$ for any $\rho \in \mathbb{R}$, *i.e.*, $\mathcal{F}(\omega,\rho) \equiv$ const in ρ. Therefore (1.10) implies $\mathcal{F}(\omega,\rho) = 0$ for any $\rho \in \mathbb{R}$.

We set

$$\Omega = S^1\backslash\{\omega \in S^1 | C(-1,\omega) = 0 \text{ and } \mathcal{F}(\omega,\rho) = 0 \quad \text{for any } \rho \in \mathbb{R}\}.$$

We claim that Ω is either the set of ω such that $C(-1,\omega) = 0$, or the set of ω such that $\mathcal{F}(\omega,\rho) = 0$ for any $\rho \in \mathbb{R}$. Set

$$F = \{\omega \in \Omega |\ C(-1,\omega) \neq 0\}.$$

Clearly F is open in Ω. On the other hand, by the above argument we have

$$F = \{\omega \in \Omega |\ \mathcal{F}(\omega, \rho) = 0 \text{ for any } \rho \in \mathbb{R}\}.$$

Then we find that F is also closed in Ω. Therefore F is equal to either ϕ or Ω. When $F = \phi, \Omega$ is the set of ω such that $C(-1, \omega) = 0$. When $F = \Omega$, Ω is the set of ω such that $\mathcal{F}(\omega, \rho) = 0$ for any $\rho \in \mathbb{R}$.

Since

$$S^1 = \Omega \cup \{\omega \in S^1 \big| C(-1, \omega) = 0 \text{ and } \mathcal{F}(\omega, \rho) = 0 \text{ for any } \rho \in \mathbb{R}\},$$

either "$C(-1, \omega) = 0$ for any $\omega \in S^1$" or "$\mathcal{F}(\omega, \rho) = 0$ for any $\omega \in S^1$ and $\rho \in \mathbb{R}$", when $H = 0$. Moreover if $\mathcal{F}(\omega, \rho) = 0$ for any $\omega \in S^1$ and $\rho \in \mathbb{R}$, then, by (1.9),

$$\mathcal{F}(\omega, \rho) + \mathcal{F}(-\omega, -\rho) = \frac{1}{\sqrt{2\pi}} \int_\rho^\infty (s - \rho)^{-1/2} R_g(\omega, s) ds = 0,$$
$$\mathcal{F}(\omega, \rho) - \mathcal{F}(-\omega, -\rho) = -\frac{1}{\sqrt{2\pi}} \int_\rho^\infty (s - \rho)^{-1/2} \partial_s R_f(\omega, s) ds = 0.$$

Using Tichmarsh's Theorem (see [14], p.166), Radon problem (see [5], p.162) and integrating by parts, we find that f and g vanish identically. Therefore the condition "$H = 0$" implies either "$C(-1, \omega) = 0$ for any $\omega \in S^1$" or "f and g vanish identically".

Acknowledgement. The author is grateful to Professor Rentaro Agemi for his suggestion of this problem and his useful comments.

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