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On bicommutators of modules over H-separable extension rings III

Kozo Sugano

This paper is the continuation of the author's previous papers [8] and [9]. Our aim is to give some improvements and generalizations of results in those papers. Therefore we will use the same notation as [8] and [9]. In particular throughout this paper A is a ring with the identity 1, B is a subring of A containing 1, C is the center of A and $D = V_A(B)$, the centralizer of B in A. In addition when we write $\{A/B, S/T\}$, S is a ring containing A as a subring with the common identity, and T is a subring of S containing B. In this case we denote the center of S and the centralizer of T in S by $\tilde{C}$ and $\tilde{D}$, respectively. In the case where $A^* = \mathrm{Bic}({}_AM)$ and $B^* = \mathrm{Bic}({}_BM)$ for some left A-module M, we have always that $V_{A^*}(A) = V_{A^*}(A^*)$ and $V_{A^*}(B) = V_{A^*}(B^*)$. If furthermore A is an H-separable extension of B, we have $B^* = V_{A^*}(V_{A^*}(B^*))$ (See Theorem 1 [9]). Hence it is natural to consider the general situation where $\{A/B, S/T\}$ satisfy the above donditions. So we will define

Definition. We say that $\{A/B, S/T\}$ satisfy the centralizer property in the case $V_S(A) = V_S(S)$ $(= \tilde{C})$, $V_S(B) = V_S(T)$ $(= \tilde{D})$ and $T = V_S(V_S(T))$. In the case where only the first two conditions are satisfied, we say that $\{A/B, S/T\}$ satisfy the condition BC (= Bicommutator).

Lemma 1. Let $\{A/B, S/T\}$ satisfy the condition BC. Then we have

(1) $V_T(B)$ coinsides with $V_T(T)$, the center of T.

(2) For any subring P of S containing A, $\{A/B, P/Q\}$ and $\{P/Q, S/T\}$ satisfy the condition BC, where $Q = P \cap T$.

(3) $\{A/B, S/V_S(V_S(T))\}$ satisfy the centralizer property.

Proof. (1) and (3) are due to $V_T(B) = T \cap V_S(B) = T \cap V_S(T) = V_T(T)$ and $V_S(V_S(V_S(T))) = V_S(T)$. Let $C' = V_P(P)$ and $D' = V_P(Q)$. Then we have $C' \subset V_P(A) = P \cap V_S(A) = P \cap \tilde{C} \subset C'$ and $V_P(Q) \subset V_P(B) = P \cap V_S(B) = P \cap V_S(T) \subset P \cap V_S(Q) = V_P(Q)$. Thus we have $V_P(A) = C'$ and $V_P(B) = D'$. The other part of (2) is obvious.

As is pointed out in Remark 2 [9] some beautiful results about the bicommutator rings of modules over H-separable extension rings hold also in the general case where {A/B, S/T} satisfy the centralizer property. Here we will introduce them without proof.

Theorem 1. Let {A/B, S/T} satisfy the centralizer property, and assume that A is an H-separable extension of B. Then we have

(1) $D \otimes_C \tilde{C}$ is isomorphic to $\tilde{D}$, via $d \otimes \tilde{c} \longrightarrow d\tilde{c}$ ($d \in D$, $\tilde{c} \in \tilde{C}$), and $\tilde{D}$ is $\tilde{C}$-finitely generated projective.

(2) $V_A(V_A(B)) = T \cap A$, $V_S(D) = T$ and $D = \tilde{D} \cap A$.

(3) If A is left (resp. right) B-finitely generated projective, S is an H-separable extension of T and S is left (resp. right) T-finitely generated projective.

(4) If B is a left (resp. right) B-direct summand of A, T is a left (resp. right) T-direct summand of S.

Theorem 2. Let {A/B, S/T} satisfy the centralizer property, and assume that both A and S are H-separable extensions of B and T, respectively. Then we have,

(1) If $\{\Sigma x_{ij} \otimes y_{ij}, d_i\}$ is an H-system of A over B, then $\{\Sigma x_{ij} \otimes y_{ij}, d_i\}$ is an H-system of S over T.

(2) $A \otimes_B S$ is isomorphic to $S \otimes_T S$, via $a \otimes s \longrightarrow a \otimes s$ ($a \in A$, $s \in S$). Similarly we have $S \otimes_B A \cong S \otimes_T S$.

(3) If T is a left (resp. right) T-direct summand of S, we have $S = AT \cong A \otimes_B T$ (resp. $S = TA \cong T \otimes_B A$).

Theorems 1 and 2 can be proven in the same way as Proposition 1, Theorem 1 [8] and Lemma 1, Proposition 1 [9]. Some part of them hold under the weaker condition that {A/B, S/T} satisfy the condition BC. But in this case {A/B, S/T'} satisfy the centralizer property, and $V_S(T') = \tilde{D}$, where $T' = V_S(V_S(T))$. In addition, if furthermore S is an H-separable extension of T, then S is an H-separable extension of T', $S \otimes_T S \cong S \otimes_{T'} S$ and $\mathrm{End}({}_T S) = \mathrm{End}({}_{T'} S)$ (See [3]). Therefore it is meaningless to consider under the weaker condition.

In [10] H. Tominaga showed that, if A is an H-separable extension of B and is left B-projective, then A is left B-finitely generated projective. So hereafter we will say that A is a left (resp. right) projective H-separable extension of B when A is left (resp. right) B-finitely generated projective and an H-separable extension of B.

As an immediate consequence of Theorem 1 (2) and Lemma 1 we have

Proposition 1. Let {A/B, S/T} satisfy the centralizer property, and assume that A is an H-separable extension of B. Then for any subring P of S containing A both {A/B, P/Q} and {P/Q, S/T} satisfy the centralizer property, where $Q = P \cap T$. Thus if furthermore A is left B-finitly generated projective, P is a left projective H-separable extension of Q.

Proof. Let C' be the center of P, and $D' = V_P(Q)$. We have $D' = DC'$ ($\cong D \otimes_C C'$) by Theorem 1 and the above remark. Then we have $V_P(V_P(Q)) = V_P(D') = V_P(D) \cap V_P(C') = V_P(D) \cap P = V_P(D) = V_S(D) \cap P = T \cap P = Q$.

Lemma 2. Let {A/B, S/T} satisfy the centralizer property, and assume that A and S are H-separable extensions of B and T, respectively. Then for any left S-module N and any A-submodule M of N there exists a canonical monomorphism ι of $\mathrm{Hom}({}_BA, {}_BM)$ to $\mathrm{Hom}({}_TS, {}_TN)$ such that $\iota(f)$ is a extension of f for each $f \in \mathrm{Hom}({}_BA, {}_BM)$. In particular each left B-endomorphism of A is extended to a left T-endomorphism of S.

Proof. As is shown on page 46 [5] there exists an isomorphism η_M of $D \otimes_C M$ to $\mathrm{Hom}({}_BA, {}_BM)$ such that $\eta_M(d \otimes m)(a) = dam$ for $a \in A$, $d \in D$ and $m \in M$. Similarly there exists an isomorphism $\tilde{\eta}_N$ to $\tilde{D} \otimes_{\tilde{C}} N$ to $\mathrm{Hom}({}_TS, {}_TN)$. But $\tilde{D} \otimes_{\tilde{C}} N = D \otimes_C \tilde{C} \otimes_{\tilde{C}} N = D \otimes_C N$ with D C-flat. Hence we have a commutative diagram

$$\begin{array}{ccc} D \otimes_C M & \longrightarrow & \mathrm{Hom}({}_BA, {}_BM) \\ \downarrow {\scriptstyle 1_D \otimes i} & & \downarrow {\scriptstyle \iota} \\ D \otimes_C N & \longrightarrow & \mathrm{Hom}({}_TS, {}_TN) \end{array}$$

where i is the inclusion map and η_N is same as η_M. Note that ι is given by $\iota(f)(s) = \Sigma e_i s m_i$ for each $f \in \mathrm{Hom}({}_BA, {}_BM)$ and $s \in S$, where $e_i \in D$

and $m_i \in M$ are such that $f(a) = \Sigma e_i a m_i$ for each $a \in A$.

Theorem 3. Let $\{A/B,\ S/T\}$ satisfy the centralizer property, and assume that A is a left projective H-separable extension of B. Then,

(1) For any dual basis $\{x_i,\ f_i\}$ for ${}_BA$ there exist $\tilde{f}_i \in \mathrm{Hom}({}_TS, {}_TT)$ such that $\{x_i,\ \tilde{f}_i\}$ is a dual basis for ${}_TS$ and $\tilde{f}_i \mid A$, the restrictions of $\tilde{f}_i$ on A, coinside with f_i.

(2) $T \otimes_B A \cong S$, via $t \otimes a \longrightarrow ta$, for $t \in T$ and $a \in A$. Thus we have $S = A$ if and only if $T = V_A(V_A(B))$.

(3) If $\{x_i\}$ is a left B-free basis of A, $\{x_i\}$ is a left T-free basis of S.

(4) Each left B-endomorphism of A is extended uniquely to a left T-endomorphism of S.

Proof. (1). Let $\iota : \mathrm{Hom}({}_BA, {}_BA) \longrightarrow \mathrm{Hom}({}_TS, {}_TS)$ be the same as is defined in Lemma 2. Put $B' = V_A(V_A(B))$. Since $\mathrm{Hom}({}_BA, {}_BB) \subset \mathrm{Hom}({}_BA, {}_BB') = \mathrm{Hom}({}_{B'}A, {}_{B'}B') \cong [D \otimes_C A]^D$ and $\mathrm{Hom}({}_TS, {}_TT) \cong [\tilde{D} \otimes_{\tilde{C}} S]^{\tilde{D}} = [D \otimes_C S]^{\tilde{D}}$, $\tilde{D} = D\tilde{C}$, we have $\iota(\mathrm{Hom}({}_BA, {}_BB)) \subset \mathrm{Hom}({}_TS, {}_TT)$. Now it is easy to see that the following diagram is commutative

$$\begin{array}{ccccc}
\mathrm{Hom}({}_BA, {}_BB) \otimes_B A & \xrightarrow{\theta} & \mathrm{Hom}({}_BA, {}_BA) & \cong & D \otimes_C A \\
\downarrow \iota \otimes i & & \downarrow \iota & & \downarrow 1_D \otimes i \\
\mathrm{Hom}({}_TS, {}_TT) \otimes_T S & \xrightarrow{\hat{\theta}} & \mathrm{Hom}({}_TS, {}_TS) & \cong & D \otimes_C S
\end{array}$$

where i is the inclusion map and θ is defined by $\theta(f \otimes a)(x) = f(x)a$ for $f \in \mathrm{Hom}({}_BA, {}_BB)$ and $a,\ x \in A$. $\hat{\theta}$ is defined in the same way as θ. Since $\theta(\Sigma f_i \otimes x_i) = 1_A$, we have $\hat{\theta}(\Sigma \iota(f_i) \otimes x_i) = 1_S$. This means that $\{x_i,\ \tilde{f}_i\}$ is a dual bais for ${}_TS$, where $\tilde{f}_i = \iota(f_i)$. (2). By (1) we have $S = TA$. Let π be the map of $T \otimes_B A$ to S such that $\pi(t \otimes a) = ta$ for $t \in T$ and $a \in A$. π is an epimorphism. Suppose $\Sigma t_j a_j = 0$ for some $t_j \in T$ and $a_j \in A$. Then in $S \otimes_T S$ we have $\Sigma t_j \otimes a_j = \Sigma 1 \otimes t_j a_j = 0$. But $S \otimes_T S \cong S \otimes_B A \supset T \otimes_B A$ by Theorem 2 (2) and our assumption. Hence $\Sigma t_j \otimes a_j = 0$ in $T \otimes_B A$, which means that $\mathrm{Ker}\,\pi = 0$. Thus π is an isomorphism. The rest of (2) is obvious since $S = TA$. (3). If $\{x_i,\ f_i\}$ is a dual basis for ${}_BA$ such that $f_i(x_j) = \delta_{ij}$, then $\tilde{f}_i(x_j) = f_i(x_j) = \delta_{ij}$, where $\tilde{f}_i$ is defined in the same way as in (1). (4). Each $f \in \mathrm{Hom}({}_BA, {}_BA)$ can be extended to an $\tilde{f} \in \mathrm{Hom}({}_TS, {}_TS)$ by Lemma 2. If there exists another $\tilde{g} \in \mathrm{Hom}({}_TS, {}_TS)$ such that $\tilde{g} \mid A = \tilde{f} \mid A = f$, then $(\tilde{f} - \tilde{g})(S) = (\tilde{f} - \tilde{g})(TA) = T(\tilde{f} - \tilde{g})(A) = 0$, and we have $\tilde{f} = \tilde{g}$.

Let A be a left (or right) projective H-separable extension of B. Then each two-sided ideal I of A is generated by $(I \cap B)$ (See Theorem 3.1 [5]). Therefore in this case a left A-module M is A-faithful, if and only if M is B-faithful. Then by Theorem 3 (2) we have the next theorem which is the improvement of Theorem 2 [8].

Theorem 4. Let A be a left or right projective H-separable extension of B such that $B = V_A(V_A(B))$. Then for any left A-module M, ${}_AM$ has the double centralizer property if and only if ${}_BM$ does.

We say that A is a localization of B in the case where $A \otimes_B A$ is isomorphic to A via, $a \otimes b \longrightarrow ab$ for $a, b \in A$. A localization A of B is called a left flat localization if A is flat as a left B-module.

A is said to be left relatively separable extension of B in S in the case where the map of $A \otimes_B S$ to S defined by $a \otimes s \longrightarrow as$ for $a \in A$ and $s \in S$ splits as A-S-map. If A is a separable extension of B, A is both left and right relatively separable over B in S.

Proposition 2. Let {A/B, S/T} satisfy the condition BC, and assume that S is an H-separable extension of T. Then we have

(1) If A is a left relatively separable extension of B in S, and T is an H-separable extension of B, then both S and T are localization of A and B, respectively.

(2) Let A be an H-separable extension of B and T a left T-direct summand of S. Assume furthermore that T is left B-flat, or T is a right T direct summand of S. Then S is a localization of A, if and only if T is a localization of B.

Proof. (1). In this case S is an H-separable extension of A by Proposition 2.2 [4] and Proposition 1.6 [3]. In addition we have $V_S(A) = \tilde{C}$

and $V_T(B) = V_T(T)$ by Lemma 1. Therefore S and T are localizations of A and B, respectively, by Lemma 1 [8]. (2). By Theorem 2 we have $A\otimes_B T \cong S$. Then if S ia a localization of A, we have $S\otimes_B T = S\otimes_A(A\otimes_B T) = S\otimes_A S \cong S$. But under our assumption we have $T\otimes_B T \subset S\otimes_B T$. Then we have $T\otimes_B T \cong T$.

Now we drop the condition BC on $\{A/B,\ S/T\}$, and consider the relation between the flatness of ${}_BA$ and ${}_TS$.

Proposition 3. Under the situation $\{A/B,\ S/T\}$ we have

(1) Suppose that A is a separable extension of B, and S is left T-flat. Then if T is left B-flat, S is left A-flat.

(2) Suppose that A is left B-flat, and T is a let T-direct summand of S. Then if S ia let A-flat, T is let B-flat.

Proof. (2) is easy to prove. (1) is an immediate consequence of the next lemma.

Lemma 3. Let M be a left A-module. If M is (A,B)-projective and B-flat, then M is A-flat.

Proof. Let X and Y be right A-modules and ι an A-monomorphism of X to Y. Let π be the map of $A\otimes_B M$ to M shch that $\pi(a\otimes m) = am$ for $a \in A$ and $m \in M$. Since M is (A,B)-projective, there exists a left A-monomorphism ϕ of M to $A\otimes_B M$ such that $\pi\phi = 1_M$. Then we have the following cmmutative diagram

$$\begin{tikzcd}
X\otimes_A M \arrow[r] \arrow[d, "\iota\otimes M"] & X\otimes_A A\otimes_B M \arrow[r] \arrow[d, "\iota\otimes(A\otimes_B M)"] & X\otimes_B M \arrow[d, "\iota\otimes M"] \\
Y\otimes_A M \arrow[r] & Y\otimes_A A\otimes_B M \arrow[r] & Y\otimes_B M
\end{tikzcd}$$

Since $\iota\otimes M$ is a monomorphism and $X\otimes\phi$ is a split monomorphism, we see that ι is a monomophism.

Note that in Proposition 3 (1) we can replace the condition that A is a separable extension of B with a weaker condition that S is left (A,B)-projective. Now we have come to our main Theorem

Theorem 5. Let $\{A/B,\ S/T\}$ satisfy the centralizer property, and assume that A is a left projective H-separable extension of B, and T is a left T-direct summand of S. Then we have

(1) S is a localization of A, if and only if T is a localization of B.

(2) S is let A-flat, if and only if T is left B-flat

(3) A is a left flat localization of B, if and only if T is a left flat localization of B.

Proof. By the assumption and Theorm 1 (3) S is a left projective H-separable extension of T. Therefore we can apply Propositions 2 and 3. We need only to prove the 'only if' part of (1). By Theorem 3 (2) we have $T \otimes_B A \cong S$. Then if $S \otimes_A S \cong S$, we have $T \otimes_B T \subset T \otimes_B S \cong S$ as in the proof of Proposition 2 (2). Then we have $T \otimes_B T \cong T$.

Let A be an Azumaya C-algebra, M a left A-module, $A^* = \mathrm{Bic}({}_AM)$ and $C^* = \mathrm{Bic}({}_CM)$. In Proposition 3 [9] we gave a one to one correspondence between the class of subrings P of A* containing A and the class of C-subalgebras Q of C* by $P \longrightarrow P \cap C^*$, and $Q \longrightarrow QA$ (= AQ). Now we will generalize this theory to the case of H-separable extension rings.

Theorem 6. Let {A/B, S/T} satisfy the centralizer property, and assume that A is a left projctive H-separable extesion of B, and B is a left B-direct summand of A. Let И be the class of subrings of S containing A and Ю the class of subrings Q of T such that $B \subset Q \subset T$ and AQ = QA. Then for each P in И we have $P \cap T \varepsilon$ Ю and $P = A(P \cap T)$, and for each Q in Ю we have $AQ \varepsilon$ И and $Q = AQ \cap T$. P ia a (left flat) localisation of A if and only if $P \cap T$ is a (left flat) localisation of B.

Proof. Let $P \varepsilon$ И and $Q = P \cap T$. By Proposition 1 {A/B, P/Q} satisfy the centralizer property. Then P is a left projective H-separable extension of Q, and Q is a left Q-direct summand of P by Theorem 1. Then by Theorems 2 and 3 we have P = AQ = QA, and consequently $Q \varepsilon$ Ю. Suppose Q ε Ю. Then AQ (= QA) is a subring of S containing A. Let P = AQ and $Q' = P \cap T$. Then $Q \subset Q'$ and $Q' \otimes_B A \cong P$, while $S \cong T \otimes_B A$ ($\supset Q \otimes_B A$) by Theorem 3. Hence we have $Q \otimes_B A \cong QA = P \cong Q' \otimes_B A$, and $Q'/Q \otimes_B A = 0$. Then $Q = Q'$, since

B is a left B-direct summand of A. The last part of the theorem is obvious by Theorem 5.

Proposition 3. Assume A is an Azumaya C-algebra, and let $\{A/C, S/T\}$ satify the centralizer property. Then $T = \tilde{C}$, and S is an Azumaya T-algebra.

Proof. Since $C \subset V_S(A) = \tilde{C}$, S is a C-algera. Hence $V_S(T) = V_S(C) = S$, and $T = V_S(V_S(T)) = V_S(S) = \tilde{C}$. By Theorem 1 S is H-separable over $T = \tilde{C}$. Hence S is $\tilde{C}$-Azumaya.

Finally we will introduce other example of $\{A/B, S/T\}$ which satisy the centralizer property but is different from the bicommmutator rings of modules.

A valuation v of a division ring is defined in the same way as a field. Here we assume that each valuation v satisfies the triangular inequality, that is, $v(x + Y) \leqq v(x) + v(y)$.

Lemma 4. Let S be a divisionring with a valuation v, and $\{a_n\}$ a Cauchy sequence in S which converges to an element a of S. If an element x of S commute with all a_n, then x commute with a.

Proof. Let ε be any positive real number. There exists a natural number m such that $v(a_n - a) < \varepsilon$ for each $n > m$. Then we have

$$\begin{aligned} v(ax - xa) &= v(ax - a_n x + xa_n - xa) \leqq v(ax - a_n x) + v(xa_n - xa) \\ &= v(a - a_n)v(x) + v(x)v(a_n - a) < 2v(x)\varepsilon \end{aligned}$$

Since ε is arbitrary, $v(ax - xa) = 0$, and we have $ax = xa$.

By the above lemma we have immediately

Theorem 7. Assume that A and B are division rings, and A is an H-separable extension of B. Let S be the completion of A with respective to a valuation v of A and T the closure of B. Then we have

(1) $\{A/B, S/T\}$ satisty the centralizer property.

(2) S and T are division rings, and S is an H-separable extension of T

(3) $S = TA = AT$, $T \cap A = B$ and $[S:T] = [A:B] = [D:C]$

(4) For any subset X of S $V_S(X)$ is complete.

Proof. It is well known that the comletion of a division ring is a division ring (See e.g., [2] Chapter 17). By Lemma 4 it is easy to see that {A/B, S/T} satisfy the centralizer property. On the other hand by Theorem 1 and Corollary 2 [6] we have $[A:B]_l < \infty$, $[D:C] < \infty$ and $B = V_A(D)$. It is also well known that under these situation $[A:B]_l = [A:B]_r = [D:C]$ by Artin-Whaple's results (See e.g., [2] § 12.7 Exercise 5). Then by Theorem 4 (3) we have [S:T] = [A:B]. The remaining part of Theorem 7 is obvious by the previous results.

References

[1] K. Hirata: Separable extensions and centralizers of rings, Nagoya Math. J., 35 (1969), 107-115.

[2] R. Pierce: Associative Algebras, Springer GTM, 1980.

[3] K. Sugano: Note on semisimple extensions and separable extensions, Osaka J. Math., 4 (1967), 265-270.

[4] K. Sugano: On centralizers in separable extensions, Osaka J. Math., 7 (1970), 29-40.

[5] K. Sugano: On projective H-separable extensions, Hokkaido Math. J., 5 (1976), 44-54.

[6] K. Sugano: On H-separable extensions of two-sided simple rings, Hokkaido Math. J., 11 (1982), 246-252.

[7] K. Sugano: On H-separable extensions of two-sided simple rings II, Hokkaido Math. J., 16 (1987), 71-74.

[8] K. Sugano: On Bicommutators of modules over H-separable extension rings, Hokkido Math. J., 20 (1991), 497-502.

[9] K. Sugano: On bicommutators of modules over H-separable extension rings II, Hokkaido Math. J., 20 (1991), 601-608.

[10] H. Tominaga: A note on H-separable extensions, Proc. Japan Acad., 50 (1974), 446-447.