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Kozo Sugano

This paper is the continuation of the author's previous papers [8] and [9]. Our aim is to give some improvements and generalizations of results in those papers. Therefore we will use the same notation as [8] and [9]. In particular throughout this paper A is a ring with the identity 1 , B is a subring of A containing 1 , C is the center of A and $D = V_A(B)$, the centralizer of B in A . In addition when we write $\{A/B, S/T\}$, S is a ring containing A as a subring with the common identity, and T is a subring of S containing B . In this case we denote the center of S and the centralizer of T in S by \tilde{C} and \tilde{D} , respectively. In the case where $A^* = \text{Bic}(A_M)$ and $B^* = \text{Bic}(B_M)$ for some left A -module M , we have always that $V_A(A) = V_A(A^*)$ and $V_A(B) = V_A(B^*)$. If furthermore A is an H-separable extension of B , we have $B^* = V_A(V_A(B^*))$ (See Theorem 1 [9]). Hence it is natural to consider the general situation where $\{A/B, S/T\}$ satisfy the above conditions. So we will define

Definition. We say that $\{A/B, S/T\}$ satisfy the centralizer property in the case $V_S(A) = V_S(S)$ ($= \tilde{C}$), $V_S(B) = V_S(T)$ ($= \tilde{D}$) and $T = V_S(V_S(T))$. In the case where only the first two conditions are satisfied, we say that $\{A/B, S/T\}$ satisfy the condition BC (= Bicommutator).

Lemma 1. Let $\{A/B, S/T\}$ satisfy the condition BC. Then we have

- (1) $V_T(B)$ coincides with $V_T(T)$, the center of T .
- (2) For any subring P of S containing A , $\{A/B, P/Q\}$ and $\{P/Q, S/T\}$ satisfy the condition BC, where $Q = P \cap T$.
- (3) $\{A/B, S/V_S(V_S(T))\}$ satisfy the centralizer property.

Proof. (1) and (3) are due to $V_T(B) = T \cap V_S(B) = T \cap V_S(T) = V_T(T)$ and $V_S(V_S(V_S(T))) = V_S(T)$. Let $C' = V_P(P)$ and $D' = V_P(Q)$. Then we have $C' \subset V_P(A) = P \cap V_S(A) = P \cap \tilde{C} \subset C'$ and $V_P(Q) \subset V_P(B) = P \cap V_S(B) = P \cap V_S(T) \subset P \cap V_S(Q) = V_P(Q)$. Thus we have $V_P(A) = C'$ and $V_P(B) = D'$. The other part of (2) is obvious.

As is pointed out in Remark 2 [9] some beautiful results about the bicommutator rings of modules over H-separable extension rings hold also in the general case where $\{A/B, S/T\}$ satisfy the centralizer property. Here we will introduce them without proof.

Theorem 1. Let $\{A/B, S/T\}$ satisfy the centralizer property, and assume that A is an H-separable extension of B. Then we have

(1) $D \otimes_C \tilde{C}$ is isomorphic to \tilde{D} , via $d \otimes \tilde{c} \longrightarrow d\tilde{c}$ ($d \in D, \tilde{c} \in \tilde{C}$), and \tilde{D} is \tilde{C} -finitely generated projective.

(2) $V_A(V_A(B)) = T \cap A, V_S(D) = T$ and $D = \tilde{D} \cap A$.

(3) If A is left (resp. right) B-finitely generated projective, S is an H-separable extension of T and S is left (resp. right) T-finitely generated projective.

(4) If B is a left (resp. right) B-direct summand of A, T is a left (resp. right) T-direct summand of S.

Theorem 2. Let $\{A/B, S/T\}$ satisfy the centralizer property, and assume that both A and S are H-separable extensions of B and T, respectively. Then we have,

(1) If $\{\sum x_{ij} \otimes y_{ij}, d_i\}$ is an H-system of A over B, then $\{\sum x_{ij} \otimes y_{ij}, d_i\}$ is an H-system of S over T.

(2) $A \otimes_B S$ is isomorphic to $S \otimes_T S$, via $a \otimes s \longrightarrow a \otimes s$ ($a \in A, s \in S$). Similarly we have $S \otimes_B A \cong S \otimes_T S$.

(3) If T is a left (resp. right) T-direct summand of S, we have $S = AT \cong A \otimes_B T$ (resp. $S = TA \cong T \otimes_B A$).

Theorems 1 and 2 can be proven in the same way as Proposition 1, Theorem 1 [8] and Lemma 1, Proposition 1 [9]. Some part of them hold under the weaker condition that $\{A/B, S/T\}$ satisfy the condition BC. But in this case $\{A/B, S/T'\}$ satisfy the centralizer property, and $V_S(T') = \tilde{D}$, where $T' = V_S(V_S(T))$. In addition, if furthermore S is an H-separable extension of T, then S is an H-separable extension of T' , $S \otimes_T S \cong S \otimes_{T'} S$ and $\text{End}(\tau S) = \text{End}(\tau' S)$ (See [3]). Therefore it is meaningless to consi-

der under the weaker condition.

In [10] H. Tominaga showed that, if A is an H -separable extension of B and is left B -projective, then A is left B -finitely generated projective. So hereafter we will say that A is a left (resp. right) projective H -separable extension of B when A is left (resp. right) B -finitely generated projective and an H -separable extension of B .

As an immediate consequence of Theorem 1 (2) and Lemma 1 we have

Proposition 1. Let $\{A/B, S/T\}$ satisfy the centralizer property, and assume that A is an H -separable extension of B . Then for any subring P of S containing A both $\{A/B, P/Q\}$ and $\{P/Q, S/T\}$ satisfy the centralizer property, where $Q = P \cap T$. Thus if furthermore A is left B -finitely generated projective, P is a left projective H -separable extension of Q .

Proof. Let C' be the center of P , and $D' = V_P(Q)$. We have $D' = DC'$ ($\cong D \otimes C'$) by Theorem 1 and the above remark. Then we have $V_P(V_P(Q)) = V_P(D') = V_P(D) \cap V_P(C') = V_P(D) \cap P = V_P(D) = V_S(D) \cap P = T \cap P = Q$.

Lemma 2. Let $\{A/B, S/T\}$ satisfy the centralizer property, and assume that A and S are H -separable extensions of B and T , respectively. Then for any left S -module N and any A -submodule M of N there exists a canonical monomorphism ι of $\text{Hom}({}_B A, {}_B M)$ to $\text{Hom}({}_T S, {}_T N)$ such that $\iota(f)$ is an extension of f for each $f \in \text{Hom}({}_B A, {}_B M)$. In particular each left B -endomorphism of A is extended to a left T -endomorphism of S .

Proof. As is shown on page 46 [5] there exists an isomorphism η_M of $D \otimes M$ to $\text{Hom}({}_B A, {}_B M)$ such that $\eta_M(d \otimes m)(a) = dam$ for $a \in A$, $d \in D$ and $m \in M$. Similarly there exists an isomorphism $\tilde{\eta}_N$ of $\tilde{D} \otimes N$ to $\text{Hom}({}_T S, {}_T N)$. But $\tilde{D} \otimes N = D \otimes C \otimes N = D \otimes N$ with D C -flat. Hence we have a commutative diagram

$$\begin{array}{ccc} D \otimes M & \xrightarrow{\quad} & \text{Hom}({}_B A, {}_B M) \\ \downarrow 1_D \otimes i & & \downarrow \iota \\ D \otimes N & \xrightarrow{\quad} & \text{Hom}({}_T S, {}_T N) \end{array}$$

where i is the inclusion map and η_N is same as η_M . Note that ι is given by $\iota(f)(s) = \sum e_i s m_i$ for each $f \in \text{Hom}({}_B A, {}_B M)$ and $s \in S$, where $e_i \in D$

and $m_i \in M$ are such that $f(a) = \sum e_i a m_i$ for each $a \in A$.

Theorem 3. Let $\{A/B, S/T\}$ satisfy the centralizer property, and assume that A is a left projective H -separable extension of B . Then,

(1) For any dual basis $\{x_i, f_i\}$ for ${}_B A$ there exist $\tilde{f}_i \in \text{Hom}(\tau S, \tau T)$ such that $\{x_i, \tilde{f}_i\}$ is a dual basis for τS and $\tilde{f}_i|_A$, the restrictions of \tilde{f}_i on A , coincide with f_i .

(2) $T \otimes_B A \cong S$, via $t \otimes a \rightarrow ta$, for $t \in T$ and $a \in A$. Thus we have $S = A$ if and only if $T = V_A(V_A(B))$.

(3) If $\{x_i\}$ is a left B -free basis of A , $\{x_i\}$ is a left T -free basis of S .

(4) Each left B -endomorphism of A is extended uniquely to a left T -endomorphism of S .

Proof. (1). Let $\iota : \text{Hom}({}_B A, {}_B A) \rightarrow \text{Hom}(\tau S, \tau S)$ be the same as is defined in Lemma 2. Put $B' = V_A(V_A(B))$. Since $\text{Hom}({}_B A, {}_B B) \subset \text{Hom}({}_B A, {}_B B') = \text{Hom}({}_B A, {}_B B')$ and $\text{Hom}({}_B A, {}_B B') \cong [D \otimes_C A]^D$ and $\text{Hom}(\tau S, \tau T) \cong [\tilde{D} \otimes_{\tilde{C}} S]^{\tilde{D}} = [D \otimes_C S]^{\tilde{D}}$, $\tilde{D} = D\tilde{C}$, we have $\iota(\text{Hom}({}_B A, {}_B B)) \subset \text{Hom}(\tau S, \tau T)$. Now it is easy to see that the following diagram is commutative

$$\begin{array}{ccc} \text{Hom}({}_B A, {}_B B) \otimes_B A & \xrightarrow{\theta} & \text{Hom}({}_B A, {}_B A) \cong D \otimes_C A \\ \downarrow \iota \otimes i & \tilde{\theta} & \downarrow \iota \quad \downarrow 1_D \otimes i \\ \text{Hom}(\tau S, \tau T) \otimes_{\tau S} & \xrightarrow{\tilde{\theta}} & \text{Hom}(\tau S, \tau S) \cong D \otimes_C S \end{array}$$

where i is the inclusion map and θ is defined by $\theta(f \otimes a)(x) = f(x)a$ for $f \in \text{Hom}({}_B A, {}_B B)$ and $a, x \in A$. $\tilde{\theta}$ is defined in the same way as θ . Since $\theta(\sum f_i \otimes x_i) = 1_A$, we have $\tilde{\theta}(\sum \iota(f_i) \otimes x_i) = 1_S$. This means that $\{x_i, \tilde{f}_i\}$ is a dual basis for τS , where $\tilde{f}_i = \iota(f_i)$. (2). By (1) we have $S = TA$.

Let π be the map of $T \otimes_B A$ to S such that $\pi(t \otimes a) = ta$ for $t \in T$ and $a \in A$. π is an epimorphism. Suppose $\sum t_j a_j = 0$ for some $t_j \in T$ and $a_j \in A$. Then in $S \otimes_{\tau S} S$ we have $\sum t_j \otimes a_j = \sum 1 \otimes t_j a_j = 0$. But $S \otimes_{\tau S} S \cong S \otimes_B A \supset T \otimes_B A$ by Theorem 2 (2) and our assumption. Hence $\sum t_j \otimes a_j = 0$ in $T \otimes_B A$, which means that $\text{Ker } \pi = 0$. Thus π is an isomorphism. The rest of (2) is obvious since $S = TA$. (3). If $\{x_i, f_i\}$ is a dual basis for ${}_B A$ such that $f_i(x_j) = \delta_{ij}$,

then $\tilde{f}_i(x_j) = f_i(x_j) = \delta_{ij}$, where \tilde{f}_i is defined in the same way as in (1). (4). Each $f \in \text{Hom}({}_B A, {}_B A)$ can be extended to an $\tilde{f} \in \text{Hom}({}_\tau S, {}_\tau S)$ by Lemma 2. If there exists another $\tilde{g} \in \text{Hom}({}_\tau S, {}_\tau S)$ such that $\tilde{g}|_A = \tilde{f}|_A = f$, then $(\tilde{f} - \tilde{g})(S) = (\tilde{f} - \tilde{g})(TA) = T(\tilde{f} - \tilde{g})(A) = 0$, and we have $\tilde{f} = \tilde{g}$.

Let A be a left (or right) projective H -separable extension of B . Then each two-sided ideal I of A is generated by $(I \cap B)$ (See Theorem 3.1 [5]). Therefore in this case a left A -module M is A -faithful, if and only if M is B -faithful. Then by Theorem 3 (2) we have the next theorem which is the improvement of Theorem 2 [8].

Theorem 4. Let A be a left or right projective H -separable extension of B such that $B = \bigvee_A (\bigvee_A (B))$. Then for any left A -module M , ${}_A M$ has the double centralizer property if and only if ${}_B M$ does.

We say that A is a localization of B in the case where $A \otimes_B A$ is isomorphic to A via, $a \otimes b \longrightarrow ab$ for $a, b \in A$. A localization A of B is called a left flat localization if A is flat as a left B -module.

A is said to be left relatively separable extension of B in S in the case where the map of $A \otimes_B S$ to S defined by $a \otimes s \longrightarrow as$ for $a \in A$ and $s \in S$ splits as A - S -map. If A is a separable extension of B , A is both left and right relatively separable over B in S .

Proposition 2. Let $\{A/B, S/T\}$ satisfy the condition BC, and assume that S is an H -separable extension of T . Then we have

(1) If A is a left relatively separable extension of B in S , and T is an H -separable extension of B , then both S and T are localization of A and B , respectively.

(2) Let A be an H -separable extension of B and T a left T -direct summand of S . Assume furthermore that T is left B -flat, or T is a right T direct summand of S . Then S is a localization of A , if and only if T is a localization of B .

Proof. (1). In this case S is an H -separable extension of A by Proposition 2.2 [4] and Proposition 1.6 [3]. In addition we have $\bigvee_S (A) = \tilde{C}$

and $Vr(B) = Vr(T)$ by Lemma 1. Therefore S and T are localizations of A and B , respectively, by Lemma 1 [8]. (2). By Theorem 2 we have $A \otimes_B T \cong S$. Then if S is a localization of A , we have $S \otimes_B T = S \otimes_A (A \otimes_B T) = S \otimes_A S \cong S$. But under our assumption we have $T \otimes_B T \subset S \otimes_B T$. Then we have $T \otimes_B T \cong T$.

Now we drop the condition BC on $\{A/B, S/T\}$, and consider the relation between the flatness of ${}_e A$ and ${}_r S$.

Proposition 3. Under the situation $\{A/B, S/T\}$ we have

(1) Suppose that A is a separable extension of B , and S is left T -flat. Then if T is left B -flat, S is left A -flat.

(2) Suppose that A is left B -flat, and T is a left T -direct summand of S . Then if S is left A -flat, T is left B -flat.

Proof. (2) is easy to prove. (1) is an immediate consequence of the next lemma.

Lemma 3. Let M be a left A -module. If M is (A, B) -projective and B -flat, then M is A -flat.

Proof. Let X and Y be right A -modules and ι an A -monomorphism of X to Y . Let π be the map of $A \otimes_B M$ to M such that $\pi(a \otimes m) = am$ for $a \in A$ and $m \in M$. Since M is (A, B) -projective, there exists a left A -monomorphism ϕ of M to $A \otimes_B M$ such that $\pi \phi = 1_M$. Then we have the following commutative diagram

$$\begin{array}{ccccc}
 X \otimes_A M & \longrightarrow & X \otimes_A A \otimes_B M & \longrightarrow & X \otimes_B M \\
 \downarrow \iota \otimes M & & \downarrow \iota \otimes (A \otimes_B M) & & \downarrow \iota \otimes M \\
 Y \otimes_A M & \longrightarrow & Y \otimes_A A \otimes_B M & \longrightarrow & Y \otimes_B M
 \end{array}$$

Since $\iota \otimes M$ is a monomorphism and $X \otimes \phi$ is a split monomorphism, we see that ι is a monomorphism.

Note that in Proposition 3 (1) we can replace the condition that A is a separable extension of B with a weaker condition that S is left (A, B) -projective. Now we have come to our main Theorem

Theorem 5. Let $\{A/B, S/T\}$ satisfy the centralizer property, and assume that A is a left projective H -separable extension of B , and T is a

left T -direct summand of S . Then we have

(1) S is a localization of A , if and only if T is a localization of B .

(2) S is left A -flat, if and only if T is left B -flat

(3) A is a left flat localization of B , if and only if T is a left flat localization of B .

Proof. By the assumption and Theorem 1 (3) S is a left projective H -separable extension of T . Therefore we can apply Propositions 2 and 3. We need only to prove the 'only if' part of (1). By Theorem 3 (2) we have $T \otimes_B A \cong S$. Then if $S \otimes_B S \cong S$, we have $T \otimes_B T \subset T \otimes_B S \cong S$ as in the proof of Proposition 2 (2). Then we have $T \otimes_B T \cong T$.

Let A be an Azumaya C -algebra, M a left A -module, $A^* = \text{Bic}(A_M)$ and $C^* = \text{Bic}(C_M)$. In Proposition 3 [9] we gave a one to one correspondence between the class of subrings P of A^* containing A and the class of C -subalgebras Q of C^* by $P \rightarrow P \cap C^*$, and $Q \rightarrow QA (= AQ)$. Now we will generalize this theory to the case of H -separable extension rings.

Theorem 6. Let $\{A/B, S/T\}$ satisfy the centralizer property, and assume that A is a left projective H -separable extension of B , and B is a left B -direct summand of A . Let \mathcal{I} be the class of subrings of S containing A and \mathcal{J} the class of subrings Q of T such that $B \subset Q \subset T$ and $AQ = QA$. Then for each P in \mathcal{I} we have $P \cap T \in \mathcal{J}$ and $P = A(P \cap T)$, and for each Q in \mathcal{J} we have $AQ \in \mathcal{I}$ and $Q = AQ \cap T$. P is a (left flat) localization of A if and only if $P \cap T$ is a (left flat) localisation of B .

Proof. Let $P \in \mathcal{I}$ and $Q = P \cap T$. By Proposition 1 $\{A/B, P/Q\}$ satisfy the centralizer property. Then P is a left projective H -separable extension of Q , and Q is a left Q -direct summand of P by Theorem 1. Then by Theorems 2 and 3 we have $P = AQ = QA$, and consequently $Q \in \mathcal{J}$. Suppose $Q \in \mathcal{J}$. Then $AQ (= QA)$ is a subring of S containing A . Let $P = AQ$ and $Q' = P \cap T$. Then $Q \subset Q'$ and $Q' \otimes_B A \cong P$, while $S \cong T \otimes_B A (\supset Q \otimes_B A)$ by Theorem 3. Hence we have $Q \otimes_B A \cong QA = P \cong Q' \otimes_B A$, and $Q'/Q \otimes_B A = 0$. Then $Q = Q'$, since

B is a left B-direct summand of A. The last part of the theorem is obvious by Theorem 5.

Proposition 3. Assume A is an Azumaya C-algebra, and let $\{A/C, S/T\}$ satisfy the centralizer property. Then $T = \tilde{C}$, and S is an Azumaya T-algebra.

Proof. Since $C \subset V_s(A) = \tilde{C}$, S is a C-algebra. Hence $V_s(T) = V_s(C) = S$, and $T = V_s(V_s(T)) = V_s(S) = \tilde{C}$. By Theorem 1 S is H-separable over $T = \tilde{C}$. Hence S is \tilde{C} -Azumaya.

Finally we will introduce other example of $\{A/B, S/T\}$ which satisfy the centralizer property but is different from the bicommutator rings of modules.

A valuation v of a division ring is defined in the same way as a field. Here we assume that each valuation v satisfies the triangular inequality, that is, $v(x + y) \leq v(x) + v(y)$.

Lemma 4. Let S be a division ring with a valuation v, and $\{a_n\}$ a Cauchy sequence in S which converges to an element a of S. If an element x of S commute with all a_n , then x commute with a.

Proof. Let ε be any positive real number. There exists a natural number m such that $v(a_n - a) < \varepsilon$ for each $n > m$. Then we have

$$\begin{aligned} v(ax - xa) &= v(ax - a_n x + x a_n - xa) \leq v(ax - a_n x) + v(x a_n - xa) \\ &= v(a - a_n)v(x) + v(x)v(a_n - a) < 2v(x)\varepsilon \end{aligned}$$

Since ε is arbitrary, $v(ax - xa) = 0$, and we have $ax = xa$.

By the above lemma we have immediately

Theorem 7. Assume that A and B are division rings, and A is an H-separable extension of B. Let S be the completion of A with respect to a valuation v of A and T the closure of B. Then we have

- (1) $\{A/B, S/T\}$ satisfy the centralizer property.
- (2) S and T are division rings, and S is an H-separable extension of T
- (3) $S = TA = AT$, $T \cap A = B$ and $[S:T] = [A:B] = [D:C]$

(4) For any subset X of S $V_S(X)$ is complete.

Proof. It is well known that the completion of a division ring is a division ring (See e.g., [2] Chapter 17). By Lemma 4 it is easy to see that $\{A/B, S/T\}$ satisfy the centralizer property. On the other hand by Theorem 1 and Corollary 2 [6] we have $[A:B]_l < \infty$, $[D:C] < \infty$ and $B = V_A(D)$. It is also well known that under these situation $[A:B]_l = [A:B]_r = [D:C]$ by Artin-Whaple's results (See e.g., [2] § 12.7 Exercise 5). Then by Theorem 4 (3) we have $[S:T] = [A:B]$. The remaining part of Theorem 7 is obvious by the previous results.

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