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Toeplitz Operators And Weighted Norm Inequalities

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Abstract. The symbols of invertible Toeplitz operators from $H^p(wd\theta/2\pi)$ to $L^p(wd\theta/2\pi)/e^{-i\theta}\bar{H}^p(wd\theta/2\pi)$ are described completely where $H^p(wd\theta/2\pi)$ denotes a weighted Hardy space. The result is strongly related with a weighted norm inequality. If the weight w satisfies the condition (A_p) then $L^p(wd\theta/2\pi)/e^{-i\theta}\bar{H}^p(wd\theta/2\pi) = H^p(wd\theta/2\pi)$ with equivalent norms.

§ 1. Introduction

Let m be a normalized Lebesgue measure on the unit circle T and let w be a non-negative integrable function on T which does not vanish identically. Suppose $1 \leqq p \leqq \infty$. Let $L^p(w) = L^p(wdm)$ and $L^p(w) = L^p$ when $w = 1$. Let $H^p(w)$ denote the closure of $L^p(w)$ of the set $\mathcal{P}$ of all analytic polynomials. We will write $H^p(w) = H^p$ when $w = 1$ and then this is a usual Hardy space. P denotes the projection from the set $\mathcal{C}$ of all trigonometric polnomials to $\mathcal{P}$. This densely defined operator P may not be extended to a bounded map of $L^p(w)$ onto $H^p(w)$ for some weight w. P can be extended to a bounded map of $L^p(w)$ onto $H^p(w)$ if and only if w satisfies the condition :

$$(A_p) \quad \sup_I \left(\frac{1}{|I|} \int_I w \, dm \right) \left(\frac{1}{|I|} \int_I w^{-\frac{1}{p-1}} \, dm \right)^{p-1} < \infty$$

where the supremum is over all intervals of T. This is the well known theorem of Hunt, Muckenhoupt and Wheeden [6] which is a generalization of the theorem of Helson and Szegő [4] .

For ϕ in L^∞, the Toeplitz operator T_ϕ is defined as a densely defined map from $H^p(w)$ to $H^p(w)$ by

$$T_\phi f = P(\phi f).$$

When w satisfies the condition (A_p), Rochberg [8] gave a nice characterization of the symbol ϕ for the invertible T_ϕ. If $p = 2$ and $w = 1$, this reduces to the theorem of Widom and Devinatz [2] .

In this paper we give a simple proof of the Rochberg's theorem and we study the Toeplitz operators for arbitrary weights w. For ϕ in L^∞, the Toeplitz operator T_ϕ is defined as a bounded map from $H^p(w)$ to $L^p(w)/\bar{H}^p_0(w)$ by

$$T_\phi f = \phi f + \bar{H}^p_0(w),$$

where $H^p_0(w) = zH^p(w)$ and $z = e^{i\theta}$. We give a characterization of the symbol ϕ for the invertible T_ϕ . If w satisfies the condition (A_p) then this shows the Rochberg's theorem. It is known that the symbols of invertible singular integral operators can be described by the Rochberg's theorem. Recently Yamamoto [9] studied singular integral operatore for p = 2 and w without the condition (A_p). He used a lifting theorem of Cotlar and Sadosky [1] . Clearly this argument does not apply to $p \neq 2$. We study singular integral operators for arbitrary p and w as a simple corollary of a previous section.

In this paper, a linear operator on a Banach space is called left invertible when it is bounded below. If V is a real function in L^1 then $\tilde{V}$ denotes the harmonic conjugate function with V(0) = 0.

§ 2. T_ϕ and $L^p(w)/\bar{H}^p(w)$

In this section, we will give a characterization of the symbol ϕ for the invertible Toeplitz operator T_ϕ from $H^p(w)$ onto $L^p(w)/\bar{H}_0^p(w)$. If logw is not integrable, then by a theorem of Szegő [5, p49] $L^p(w) = \overline{H_0^p(w)}$. Hence for the research of T_ϕ it is reasonable to assume the integrability of logw and hence $w = |h|^p$ for some outer function h in H^p. We need several lemmas to get the theorem.

Lemma 1. Suppose $1 \leqq p < \infty$ and $w = |h|^p$ for some outer function h in H^p. Let ϕ be a nonzero function in L^∞ and $\phi_0 = \phi\bar{h}/h$.

(1) T_ϕ is a left invertible operator from $H^p(w)$ into $L^p(w)/\bar{H}_0^p(w)$ if and only if T_{ϕ_0} is a left invertible operator from H^p into $L^p/\bar{H}_0^p$.

(2) T_ϕ is an invertible operator from $H^p(w)$ onto $L^p(w)/\bar{H}_0^p(w)$ if and only if T_{ϕ_0} is an invertible operator from H^p onto $L^p/\bar{H}_0^p$.

Proof. (1) Let ε be a positive constant. For any $f \in \mathcal{P}$ and $g \in \mathcal{P}_0 = e^{i\theta}\mathcal{P}$,

$$\int |\phi f + \bar{g}|^p w dm \geqq \varepsilon \int |f|^p w\, dm$$

if and only if

$$\int |\phi_0 hf + \bar{h}\bar{g}|^p dm \geqq \varepsilon \int |hf|^p dm.$$

Since $hH^p(w) = H^p$, this equivalence shows (1).

(2) If T_ϕ is invertible then there exist two functions f and g such that $\phi f = 1 + \bar{g}$, $f \in H^p(w)$ and $g \in H^p(w)$. Hence

$$\phi_0 hf = \bar{h} + \bar{h}\bar{g}.$$

Since h is outer, there exists a function F in H^p such that $T_{\phi_0} F = 1 + \bar{H}^p_0$ This implies that $\phi_0 F = 1 + \bar{G}$ for some function G in H^p_0 and hence

$$\phi_0 zF = z + \overline{\hat{G}(1)} + z\overline{(G - \hat{G}(1)\bar{z})}$$

and $z\overline{(G - G(1)\bar{z})} \in \bar{H}^p_0$. Therefore there exists a function F_1 in H^p such that $T_{\phi_0} F_1 = z + \bar{H}^p_0$ because $1 + \bar{H}^p_0$ is in the range of T_{ϕ_0}. Proceeding similarly, we obtain $T_{\phi_0} H^p \ni z^\ell + \bar{H}^p_0$ for any nonnegative integer ℓ and hence T_{ϕ_0} has a dense range. By this and (1), T_{ϕ_0} is invertible. The proof is reversible and hence the converse is valid.

Lemma 2. Suppose $1 \leqq p < \infty$. T_ϕ is a left invertible operator from H^p to $L^p/\bar{H}^p_0$ if and only if $\phi = \varphi k$ and T_φ is left invertible, where φ is unimodular and k is an invertible function in H^∞.

Proof. If T_ϕ is a left invertible operator then for any $f \in H^p$

$$\int |\phi f|^p dm \geqq \varepsilon \int |f|^p dm$$

where ε is a positive constant. This implies that ϕ^{-1} is bounded. Hence ϕ has the form : $\phi = \varphi k$ where $|\phi| = |k|$ and k is an invertible function in H^∞. The following inequalities show the 'only if' part. For any $f \in H^p$ and $g \in H^p_0$

$$\int |\phi f + \bar{g}|^p dm = \int |\phi(kf) + \bar{g}|^p dm$$

$$\geqq \varepsilon \int |f|^p dm \geqq \varepsilon \|k\|_\infty^{-p} \int |kf|^p dm.$$

The proof above shows the 'if' part because $\phi = \phi k^{-1}$ and k^{-1} is an invertible function in H^∞.

The following lemma reduces the hard part of the problem to a theory of weighted norm inequalities.

Lemma 3. Suppose $1 \leqq p < \infty$ and $\phi = \bar{h}_0/h_0$ where h_0 is an outer function in H^p. T_ϕ is a left invertible operator from H^p into $L^p/\bar{H}_0^p$ if and only if for any $f \in H^p(w)$ and $g \in H_0^p(w)$

$$\int |f + \bar{g}|^p w\, dm \geqq \varepsilon \int |f|^p w\, dm$$

where $w = |h_0|^p$ and ε is a positive constant.

Proof. If T_ϕ is left invertible then for any $F \in H^p$ and $G \in H_0^p$

$$\int |\phi F + \bar{G}|^p dm \geqq \varepsilon \int |F|^p dm$$

where ε is a positive constant. Hence

$$\int |(h_0^{-1}F) + \bar{h}_0^{-1}\bar{G}|^p |h_0|^p dm \geqq \varepsilon \int |h_0^{-1}F|^p |h_0|^p dm$$

because $\phi = \bar{h}_0/h_0$, and this implies the 'only if' part. The proof is reversible and the 'if' part follows.

Lemma 4. Suppose $1 < p < \infty$. T_ϕ is an invertible operator from H^p onto $L^p/\bar{H}_0^p$ if and only if $\phi = k\bar{h}_0/h_0$ where k is an invertible function and

h_0 is an outer function in H^p with $|h_0|^p$ satisfing the (A_p) condition.

Proof. It is known [3, p133] that $(L^p/\bar{H}_0^p)^* = H^q$ and $(H^p)^* = L^q/\bar{H}_0^q$ where $1/p + 1/q = 1$. This is a simple result of the Hahn-Banach theorem. T_ϕ^* is a bounded linear operator from H^q to $L^q/\bar{H}_0^q$ and

$$T_\phi^* g = \bar{\phi} g + \bar{H}_0^q.$$

In fact, for any $f \in H^p$ and $g \in H^q$

$$\langle f, T_\phi^* g \rangle = \langle T_\phi f, g \rangle = \int \phi f \bar{g} dm = \int f \overline{\bar{\phi} g} dm.$$

We can assume that ϕ is unimodular, by Lemma 2. If T_ϕ is invertible then there exist two functions $f_1 \in H^p$ and $f_2 \in H_0^p$ such that $\phi f_1 = 1 + \bar{f}_2$. Since T_ϕ^* is also invertible, there exist two functions $g_1 \in H^q$ and $g_2 \in H_0^q$ such that $\bar{\phi} g_1 = 1 + \bar{g}_2$. Thus $f_1 g_1 = (1 + \bar{f}_2)(1 + \bar{g}_2) = 1$ and hence both f_1 and $1 + f_2$ are outer. Therefore $\phi = \bar{h}_0/h_0$ for some outer function h_0 in H^p. By Lemma 3, if $w = |h_0|^p$ then for any $f \in H^p(w)$ and $g \in H_0^p(w)$

$$\int |f + \bar{g}|^p w \, dm \geqq \varepsilon \int |f|^p w \, dm$$

where ε is a positive constant. Hence $|h_0|^p$ satisfies the condition (A_p) by a theorem of Hunt, Muckenhoupt and Wheeden [6] . Conversely if $\phi = \bar{h}_0/h_0$ and $|h_0|^p$ satisfies the condition (A_p) then by Lemma 3 T_ϕ is left invertible. We may assume that $\int h_0 dm = 1$. Then $T_\phi h_0 = 1 + \bar{H}_0^p$ and hence by the proof of Lemma 1 T_ϕ has a dense range in $L^p/\bar{H}_0^p$.

Theorem 1. Suppose $1 < p < \infty$ and $w = |h|^p$ for some outer function h in H^p. Then the following conditions on ϕ and w are equivalent.

(1) T_ϕ is an invertible operator from $H^p(w)$ onto $L^p(w)/\bar{H}_0^p(w)$.

(2) $\phi = k \dfrac{\bar{h}_0}{h_0} \dfrac{h}{\bar{h}}$

where k is an invertible function in H^∞ and h_0 is an outer function in H^p with $|h_0|^p$ satisfying the condition (A_p).

(3) $\phi = \gamma \exp(U - i\tilde{V})$

where γ is constant with $|\gamma| = 1$, U is a bounded real function, V is a real function in L^1 and

$$w \exp\left(\frac{p}{2} V\right) \text{ satisfies } (A_p).$$

Proof. (1)$\Longrightarrow$(2). If T_ϕ is an invertible operator from $H^p(w)$ onto $L^p(w)/\bar{H}_0^p(w)$ then by Lemma 1 T_{ϕ_0} is an invertible operator from H^p onto $L^p/\bar{H}_0^p$ where $\phi_0 = \phi\bar{h}/h$. Then Lemma 4 implies (2).

(2)$\Longrightarrow$(1). By Lemma 4, T_{ϕ_0} is an invertible operator from H^p onto $L^p/\bar{H}_0^p$ where $\phi_0 = \phi\bar{h}/h$. Then Lemma 1 implies (1).

(2)$\Longrightarrow$(3). Since k, h_0 and h are outer, $k = \gamma_1 \exp(U + i\tilde{U})$, $h_0 = \gamma_2 \exp(v_0 + i\tilde{v}_0)$ and $h = \gamma_3 \exp(v + i\tilde{v})$ where U is bounded, $v_0 \in L^1$, $v \in L^1$ and γ_j is constant with $|\gamma_j| = 1$ $(j = 1, 2, 3)$. Hence $\phi = \gamma \exp(U - i\tilde{V})$ where $\gamma = \gamma_1(\bar{\gamma}_2 \gamma_3)^2$ and $V = -U + 2v_0 - 2v$. Therefore

$$|h|^p \exp(\tfrac{p}{2}V) = \exp(-\tfrac{p}{2}U)\exp(pv_0) = \exp(-\tfrac{p}{2}U)\,|h_0|^p.$$

Since U is bounded and $|h_0|^p$ satisfies the condition (A_p), $w\exp(\frac{-p}{2}V)$ satisfies (A_p).

$(3) \Longrightarrow (2)$. Since $w\exp(\frac{p}{2}V)$ satisfies the condition (A_p), $\exp(\frac{-p}{2}U)\,w\exp(\frac{p}{2}V)$ does the condition, too. Then its logarithm is integrable and hence $\exp(\frac{p}{2}U)\,w\exp(\frac{-p}{2}V) = |h_0|^p$ for some outer function h_0 in H^p. Put $k = \gamma\exp(U + i\tilde{U})$, $h_0 = \exp(v_0 + i\tilde{v}_0)$ and $h = \exp(v + i\tilde{v})$ then $V = -U + 2v_0 - 2v$ because $\exp(\frac{U}{2})\,|h|\,\exp(\frac{V}{2}) = |h_0|$. Hence

$$\phi = \gamma\exp(U - i\tilde{V}) = k\,\frac{\bar{h}_0}{h_0}\,\frac{h}{\bar{h}}$$

and this implies (2).

§ 3. T_ϕ and $H^p(w)$

In this section, we will show a theorem of Widom, Devinatz and Rochberg (cf. [8]) as a corollary of Theorem 1.

Lemma 5. Suppose $1 < p < \infty$. w satisfies the condition (A_p) if and only if

$$L^p(w)/\bar{H}^p_0(w) \cong H^p(w)$$

with the equivalent norms.

Proof. If w satisfies the condition (A_p) then P is bounded from $L^p(w)$ to $H^p(w)$ by [6] . Hence the mapping : $F + \bar{H}^p(w) \rightarrow PF$ is one-one from $L^p(w)/\bar{H}^p_0(w)$ onto $H^p(w)$. Since P is bounded,

$$\varepsilon^p \int | PF |^p \, w \, dm \leqq \inf \{\int | F + \bar{g} |^p \, w \, dm : g \in H^p(w)\}$$
$$\leqq \int | PF |^p \, w \, dm$$

where $\varepsilon \| P \| = 1$. This implies the 'only if' part. The proof is reversible and the 'if' part follows.

Lemma 6. Suppose $1 < p < \infty$ and w satisfies the condition (A_p). Then T_ϕ is an (left) invertible operator from $H^p(w)$ into $L^p(w)/\bar{H}^p_0(w)$ if and only if T_ϕ is an (left) invertible operator on $H^p(w)$.

Proof. By Lemma 5, for any $f \in H^p(w)$

$$\varepsilon^p \| T_\phi f \|_p \leqq \| T_\phi f \|_p \leqq \| T_\phi f \|_p$$

where $\varepsilon \| P \| = 1$ and $\| \cdot \|_p$ is the norm of $L^p(w)/\bar{H}^p_0(w)$ or $H^p(w)$. This implies the statement about the left invertibility. Moreover $T_\phi f = 1 + \bar{H}^p_0(w)$ if and only if $T_\phi f = 1$. This implies the statement about the invertibility.

Theorem 2. Suppose $1 < p < \infty$ and $w = | h |^p$ satisfies the condition (A_p) where h is an outer function in H^p.

(1) T_ϕ is invertible on $H^p(w)$ if and only if $\phi = \gamma \exp(U - i\tilde{V})$ where γ is constant with $|\gamma| = 1$, U is a bounded real function, V is a real function in L^1 and $w \exp(\frac{-p}{2} V)$ satisfies (A_p).

(2) T_ϕ is left invertible on $H^p(w)$ if and only if $\phi = k(h/\bar{h})\psi$ where k is an invertible function in H^∞, and T_ψ is left invertible on H^p and ψ is unimodular.

Proof. (1) is a result of Theorem 1 and Lemma 6. (2) is a result of Lemmas 2 and 6.

By Theorem 2, if $\phi = q \exp(U - i\tilde{V})$ and q is inner then T_ϕ is left invertible on $H^p(w)$. If $p = 2$ the converse is also true. However unfortunately we don't know whether if the converse is true for $p \neq 2$. By (2) of Theorem 2, it is important to describe a unimodular symbol ϕ such that T is left invertible on H^p. Unfortunately we could not do it except $p = 2$. If $p = 2$, P is an orthogonal projection and hence T_ϕ is left invertible on H^2 if and only if $\| \phi + H^\infty \| < 1$. This is a theorem of Devinatz and Widom (cf. [2]). Then it is not difficult to see that $\phi = q\phi_0$ where q is inner and T_{ϕ_0} is invertible on H^2. Now we can conjecture the following.

Conjecture. Suppose ϕ is unimodular, $1 < p < \infty$ and $p \neq 2$. T_ϕ is left invertible on H^p if and only if $\phi = q\phi_0$ where q is inner and T_{ϕ_0} is invertible on H^p.

The 'if' part of this conjecture is true trivially. If this conjecture is true, we can describe the symbol of a left invertible Toeplitz operator on $H^p(w)$ when w satisfies the condition (A_p). It is more interesting to show Theorem without the condition (A_p). Unfortunately we could not do it. However we can give a sufficient condition.

Propositiion 3. Suppose $1 < p < \infty$ and $w = |h|^p$ for some outer function h in H^∞. If ϕ satisfies the condition of (1) in Theorem 2 then T_ϕ is left invertible.

Proof. Note that $H^p \subset H^p(w)$ because w is bounded. If $f \in \mathcal{P}$ then $P(\phi f) \in H^p \subset H^p(w)$ and hence $\phi f - P(\phi f) \in \overline{H}_0^p \subset \overline{H}_0^p(w)$. Therefore by Theorem 1

$$\int |P(\phi f)|^p w\, dm \geqq \inf_{g \in \mathcal{P}_0} \int |\phi f + \bar{g}|^p w\, dm \geqq \varepsilon \int |f|^p w\, dm$$

for some $\varepsilon > 0$.

§ 4. Singular integral operators

Put $Q = I - P$. For α, β in L^∞, the singular integral operator S is defined as a densely defined map on $L^p(w)$ by

$$S_{\alpha,\beta}\, f = \alpha Pf + \beta Qf.$$

If w satisfies the condition (A_p) and $1 < p < \infty$, then $S_{\alpha,\beta}$ is bounded and a characterization of symbols α, β of the invertible $S_{\alpha,\beta}$ are known. In fact, this is the easy corollary of a theorem of Devinatz, Widom and Rochberg (cf. [8]). We are interested in the general weight w. In this direction, Yamamoto [9] got a result when p = 2. In this section, we will prove it for arbitrary p with $1 < p < \infty$, using Theorem 1. For f in $\mathcal{P}$ and g in $\mathcal{P}_0$, put

$$\| f + g \|_{p,w} = \left(\int | f |^p w\, dm\right)^{1/p} + \left(\int | g |^p w\, dm\right)^{1/p}$$

then $\| \cdot \|_{p,w}$ is a norm on $\mathcal{C} = \mathcal{P} + \bar{\mathcal{P}}_0$. $L^p(w)$ denotes the completion of $\mathcal{C}$ by this norm $\| \cdot \|_{p,w}$. Then $L^p(w)$ is a Banach space. It is reasonable to assume that log w is integrable as in Section 2. If w satisfies the condition (A_p) and $1 < p < \infty$ then $L^p(w) \cong L^p(w)$ with equivalent norms. The converse is also true. Even if w does not satisfy the condition (A_p) $S_{\alpha,\beta}$ is a bounded operator from $L^p(w)$ into $L^p(w)$. When p = 2, Yamamoto and the author [7] determined when $S_{\alpha,\beta}$ is bounded on $L^p(w)$.

Lemma 7. Suppose $1 < p < \infty$. Then $S_{\phi,1}$ is an (left) invertible operator from $L^p(w)$ to $L^p(w)$ if and only if T_ϕ is an (left) invertible operator from $H^p(w)$ to $L^p(w)/\bar{H}^p_0(w)$.

Proof. By Lemma 2 and the proof, we can assume that ϕ is unimodular. If T_ϕ is left invertible then for any $f \in \mathcal{D}$ and $g \in \mathcal{D}_0$

$$\int |\phi f + \bar{g}|^p w\, dm \geq \varepsilon^p \int |f|^p w\, dm$$

where $\varepsilon > 0$. Since $\bar{z}g \in \mathcal{D}$ and $zf \in \mathcal{D}_0$,

$$\int |\phi g + \bar{f}|^p w\, dm = \int |\phi(\bar{z}g) + \overline{zf}|^p w\, dm$$

$$\geq \varepsilon^p \int |\bar{z}g|^p w\, dm = \varepsilon^p \int |g|^p w\, dm.$$

Note that $|\phi f + \bar{g}| = |\phi g + \bar{f}|$ because ϕ is unimodular. Then by the two inequalities above, for any $f \in \mathcal{D}$ and $g \in \mathcal{D}_0$

$$\left(\int |\phi f + \bar{g}|^p w\, dm\right)^{1/p} \geq \varepsilon \left\{\left(\int |f|^p w\, dm\right)^{1/p} + \left(\int |g|^p w\, dm\right)^{1/p}\right\}$$

and this implies that $S_{\phi,1}$ is left invertible. The converse is trivial. Now we will show the statement of the invertibility.

$$T_\phi H^p(w) \text{ is dense in } L^p(w)/\bar{H}^p_0(w)$$

if and only if

$$\phi H^p(w) + \bar{H}^p_0(w) \text{ is dense in } L^p(w)$$

if and only if

$$S_{\phi,1} L^p(w) \text{ is dense in } L^p(w)$$

by the definition of $L^p(w)$. This completes the proof of the lemma.

Theorem 3. Suppose $1 < p < \infty$ and $w = |h|^p$ for some outer function h in H^p.

(1) $S_{\alpha,\beta}$ is an invertible operator from $L^p(w)$ onto $L^p(w)$ if and only if both α and β are invertible in L^∞ and $\alpha/\beta = \gamma \exp(U - i\tilde{V})$ where γ is constant with $|\gamma| = 1$, U is a bounded real function, V is a real function in L^1 and $w \exp(\frac{p}{2} V)$ satisfies (A_p).

(2) $S_{\alpha,\beta}$ is a left invertible operator from $L^p(w)$ into $L^p(w)$ if and only if both α and β are invertible in L^∞ and $\alpha/\beta = k(h/\bar{h})\phi$ where k is an invertible function in H^∞, ϕ is unimodular and $S_{\phi,1}$ is left invertible on L^p.

Proof. (1) is clear by Lemma 7 and Theorem 1. (2) is clear by Lemmas 1, 2 and 7.

As in [9], we can show that $S_{\alpha,\beta}$ is left invertible if and only if both $S_{\alpha,\beta}$ and $S_{\alpha,-\beta}$ is left invertible.

References

1. M. Cotlar and C. Sadosky, On the Helson-Szegő theorem and a related class of modefied Toeplitz kernels, Proc. Symp. Pure Math. Amer. Math. Soc., 35 : I (1979), 383-407.

2. A.Devinatz, Toeplitz operators on H^2 spaces, Trans. Amer. Math. Soc. 112(1964), 304-317.

3. J.B.Garnett, Bounded Analytic Functions, Academic Press, New York 1981.

4. H.Helson and G.Szegő, A problem in prediction theory, Ann. Mat. Pura Appl. 51(1960), 107-138

5. K.Hoffman, Banach Spaces Of Analytic Functions, Prentice Hall, Englewood Cliffs, New Jersey

6. R.Hunt, B.Muckenhoupt and R.Wheeden, Weighted norm inequalities for the conjugate function and Hilbert transform, Trans. Amer. Math. Soc. 176(1973), 227-251.

7. T.Nakazi and T.Yamamoto, Some singular integral operators and Helson-Szegő measures, J. Funct. Anal. 88(1990), 366-384.

8. R.Rochberg, Toeplitz operators on weighted H^p spaces, Indiana Univ. Math. J. 26(1977), 291-298.

9. T.Yamamoto, Invertibility of some singular integral operators and a lifting theorem, in preprint.