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GROUND STATE OF A SPIN 1/2 CHARGED PARTICLE IN AN EVEN DIMENSIONAL MAGNETIC FIELD

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GROUND STATE OF A SPIN 1/2 CHARGED PARTICLE IN AN EVEN DIMENSIONAL MAGNETIC FIELD

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ABSTRACT

We investigate the ground state structure of the Schrödinger operator (Pauli Hamiltonian) H with a magnetic field b for a spin 1/2 particle in $\mathbb{R}^{2d} \cong \mathbb{C}^d$. We consider the case where b is given by the complex exterior derivative of a function W on $\mathbb{C}^d$ of the form $b = i(\bar{\partial} + \partial)(\bar{\partial} - \partial)W$. We found that $\dim \ker H$ is related to the asymptotic behavior of W at infinity. More precisely, if there exists a constant $C \in \mathbb{R}$ such that $W(z) \sim -C \log |z|$ as $z \to \infty$, then $\dim \ker H$ is equal to the number of all monomials f in d variables such that the degree of f is smaller than $|C| - d$. Moreover we clarify the structure of $\ker H$.

I. Introduction

We investigate the ground state structure of the Schrödinger operator H with a magnetic field for a spin 1/2 particle in the $2d$-dimensional Euclidean space $\mathbb{R}^{2d}, (d \geq 1)$. Let $r = 2^d$ and γ^i's be $r \times r$ Hermitian matrices (the so-called Dirac matrices) satisfying

$$\gamma^i \gamma^j + \gamma^j \gamma^i = 2\delta^{ij}, \qquad i, j = 1, \cdots, 2d, \tag{2.1}$$

where δ^{ij} is the Kronecker delta. Let $a(x) = \sum_{i=1}^{2d} a_i(x) dx^i$ be a real 1-form on $\mathbb{R}^{2d}$, which is called a vector potential. Define the Dirac operator $\not{D}(a)$ by

$$\not{D}(a) = \sum_{k=1}^{2d} \gamma^k \left(-i \frac{\partial}{\partial x^k} - a_k(x) \right)$$

acting in $L^2(\mathbb{R}^{2d}) \otimes \mathbb{C}^r$. The Schrödinger operator (the Pauli Hamiltonian) H we are going to study is given by

$$H = \not{D}(a)^2.$$

The relation (2.1) is the fundamental relation of the Clifford algebra associated with $\mathbb{R}^{2d}$. The representation space of the Clifford algebra in this formulation is $\mathbb{C}^r$. The 2-form

$$b = da$$

is called the magnetic field. Throughout this paper, we assume that a_i's are C^∞-functions. Then $\not{D}(a)$ and H are essentially self-adjoint operators on $C_0^\infty(\mathbb{R}^{2d}) \otimes \mathbb{C}^r$ (see [1]).

Shigekawa [2] studied the relation between spectral properties of H and the asymptotic behavior of b at infinity in any finite dimensions. In particular, in the even dimensional case, identifying $\mathbb{R}^{2d}$ with $\mathbb{C}^d$ and assuming that

(A) there exists a function $W \in \mathbb{C}^\infty(\mathbb{C}^d \to \mathbb{R})$ such that $b = id(\bar{\partial} - \partial)W$,

where ∂ and $\bar{\partial}$ are the complex exterior differential and its conjugation on $\mathbb{C}^d$, respectively, he proved that it has been proved that 0 is an infinitely degenerate eigenvalue of H unless $b(z)$ goes to 0 as $z \to \infty$ so rapidly (see [2]).

In this paper, under the same assumption (A), we prove that $\dim \ker H$ is related to the asymptotic behavior of W at infinity. More precisely, if there exists a constant $C \in \mathbb{R}$ such that $W(z) \sim -C \log |z|$ as $z \to \infty$, then $\dim \ker H$ is equal to the number of all monomials

f in d variables such that the degree of f is smaller than $|C| - d$. Moreover we clarify the structure of $\ker H$.

In Sec.II we give an expression of the Dirac operator $\not{D}(a)$ as an operator in a space of differential forms. In Sec.III we prove the main theorem (Theorem 3.1) and give several remarks.

II. An Expression of the Dirac Operator

In this section, we realize the Dirac operator $\not{D}(a)$ as an operator on a space of differential forms (see [2]). From now on, we work in the space $\mathbb{C}^d$. For $z = (z^1, \ldots, z^d) \in \mathbb{C}^d$, we write $z^k = x^k + iy^k, x^k, y^k \in \mathbb{R}$ and as usual we define tangent and cotangent vectors by

$$\begin{aligned}
\frac{\partial}{\partial z^k} &= \frac{1}{2}\left(\frac{\partial}{\partial x^k} - i\frac{\partial}{\partial y^k}\right), \\
\frac{\partial}{\partial \bar{z}^k} &= \frac{1}{2}\left(\frac{\partial}{\partial x^k} + i\frac{\partial}{\partial y^k}\right), \\
dz^k &= dx^k + i dy^k, \\
d\bar{z}^k &= dx^k - i dy^k.
\end{aligned}$$

We denote by $\wedge^{0,q}(T^*\mathbb{C}^d)$ the space of all $(0,q)$-type differential forms. Any element $\omega \in \wedge^{0,q}(T^*\mathbb{C}^d)$ is expressed as

$$\omega = \sum_I \omega_I d\bar{z}^I,$$

where $I = \{1 \le i_1 < \cdots < i_q \le d\}$ and $d\bar{z}^I = d\bar{z}^{i_1} \wedge \cdots \wedge d\bar{z}^{i_q}$.

We take the exterior algebra $\wedge^{0,*}(\mathbb{C}^d)^* = \oplus_{q=0}^{d} \wedge^{0,q} (\mathbb{C}^d)^*$ as a representation space of Clifford algebra. We define an operator $\mathrm{ext}(d\bar{z}^i)$ on $\wedge^{0,*}(\mathbb{C}^d)^*$ by

$$\mathrm{ext}(d\bar{z}^i)\eta = d\bar{z}^i \wedge \eta, \quad \eta \in \wedge^{0,*}(\mathbb{C}^d)^*.$$

Let $\mathrm{int}(dz^i)$ be the adjoint operator of $\mathrm{ext}(d\bar{z}^i)$. Then, we have that for $I = \{i_1, \ldots, i_q\}$

$$\mathrm{int}(dz^i)d\bar{z}^I = \begin{cases} 0, & i \neq i_k \quad \text{for all } k, \\ (-1)^{k-1} d\bar{z}^{i_1} \wedge \cdots \wedge \overset{\vee}{d\bar{z}^{i_k}} \wedge d\bar{z}^{i_q}, & i = i_k \quad \text{for some } k, \end{cases}$$

where $\overset{\vee}{d\bar{z}^{i_k}}$ means that $d\bar{z}^{i_k}$ is removed. Let

$$\gamma^{2i-1} = i(\mathrm{ext}(d\bar{z}^i) - \mathrm{int}(dz^i)),$$
$$\gamma^{2i} = -(\mathrm{ext}(d\bar{z}^i) + \mathrm{int}(dz^i)).$$

We denote by $[\cdot,\cdot]_+$ the anti-commutator. The well-known fact that

$$[\mathrm{ext}(d\bar{z}^i), \mathrm{ext}(d\bar{z}^j)]_+ = [\mathrm{int}(dz^i), \mathrm{int}(dz^j)]_+ = 0,$$
$$[\mathrm{ext}(d\bar{z}^i), \mathrm{int}(dz^j)]_+ = \delta^{ij},$$

implies that $[\gamma^i, \gamma^j]_+ = 2\delta^{ij}$, which is the fundamental relation (2.1) of Clifford algebra.

As usual, the complex exterior differential $\bar{\partial}$ on $\wedge^{0,*}(T^*\mathbb{C}^d) = \oplus_{q=0}^d \wedge^{0,q}(T^*\mathbb{C}^d)$ is given by

$$\bar{\partial}\omega = \sum_I \sum_{i=1}^d \frac{\partial \omega_I}{\partial \bar{z}^i} d\bar{z}^i \wedge d\bar{z}^I.$$

Now we identify $\wedge^{0,*}(T^*\mathbb{C}^d)$ and $C^\infty(\mathbb{C}^d) \otimes \wedge^{0,*}(\mathbb{C}^d)^*$, so $\bar{\partial}$ is expressed as follows:

$$\bar{\partial} = \sum_{i=1}^d \mathrm{ext}(d\bar{z}^i) \frac{\partial}{\partial \bar{z}^i}.$$

We regard $\bar{\partial}$ as an operator in the Hilbert space $L^2(\wedge^{0,*}(T^*\mathbb{C}^d)) = L^2(\mathbb{C}^d) \otimes \wedge^{0,*}(\mathbb{C}^d)^*$, the space of all square integrable sections with respect to the Lebesgue measure on $\mathbb{C}^d$. The adjoint operator $\bar{\partial}^*$ of $\bar{\partial}$ is expressed as

$$\bar{\partial}^* = -\sum_{i=1}^d \mathrm{int}(dz^i) \frac{\partial}{\partial z^i}.$$

Next we realize the Dirac operator $\not{D}(a)$ as an operator in $L^2(\mathbb{C}^d) \otimes \wedge^{0,*}(\mathbb{C}^d)^*$. We assume the following.

(H). *There exists a function $W \in C^\infty(\mathbb{C}^d \to \mathbb{R})$ such that $a = i(\bar{\partial} - \partial)W$.*

We remark that, if the real 2-form $b = da$ is of type $(1,1)$, then there exists a $W \in C^\infty(\mathbb{C}^d \to \mathbb{R})$ such that $a = i(\bar{\partial} - \partial)W$ (see [3, Lemma II.2.15]), and that in the real 2-dimensional case, any 2-form is always of type $(1,1)$.

We take a W as in (H) and fix it. Define an operator $\bar{\partial}_W$ in $L^2(\mathbb{C}^d) \otimes \wedge^{0,*}(\mathbb{C}^d)^*$ by

$$\bar{\partial}_W = \bar{\partial} + \mathrm{ext}(\bar{\partial} W) = \sum_{i=1}^{d} \mathrm{ext}(d\bar{z}^i) \left(\frac{\partial}{\partial \bar{z}^i} + \frac{\partial W}{\partial \bar{z}^i} \right).$$

Then the adjoint operator of $\bar{\partial}_W$ is given by

$$\bar{\partial}_W^* = \bar{\partial}^* + \mathrm{int}(\partial W) = \sum_{i=1}^{d} \mathrm{int}(dz^i) \left(-\frac{\partial}{\partial z^i} + \frac{\partial W}{\partial z^i} \right).$$

By a straightforward computation, we can show that

$$2(\bar{\partial}_W + \bar{\partial}_W^*) = \not{D}(a).$$

Thus the Dirac operator is realized as an operator in a space of differential forms.

III. Main Theorem

For a real number C, we denote by $N_d(C)$ the number of all monomials f in d variables such that the degree of f is smaller than $|C| - d$:

$$N_d(C) = \sharp\{(\alpha_1, \ldots, \alpha_d); 0 \le \alpha_1 + \cdots + \alpha_d < |C| - d, \ \alpha_j \in \mathbb{N} \cup \{0\}, j = 1, \ldots, d\}.$$

The following theorem is the main theorem in this paper.

Theorem 3.1. *Assume that the hypothesis* **(H)** *in Sec.II holds and that the limit*

$$C = -\lim_{z \to \infty} \frac{W(z)}{\log |z|}$$

exists. Then

$$\dim \ker H = N_d(C).$$

In particular, if W is bounded then $\dim \ker H = 0$.

Remark. We emphasize that $\dim \ker H$ is determined by the behavior of $W(z)$ at $z \sim \infty$. Note that, in [2], the relation between spectral properties of H and the asymptotic behavior of b at infinity is treated.

Proof. First, we treat the case where $C \ge 0$. We denote by m the Lebesgue measure on $\mathbb{C}^d$:

$$m(dz) = dx^1 dy^1 \cdots dx^d dy^d.$$

We consider a unitary operator $U\colon L^2(\mathbb{C}^d)\otimes\wedge^{0,*}(\mathbb{C}^d)^*\to L^2(\mathbb{C}^d;e^{-2W}m)\otimes\wedge^{0,*}(\mathbb{C}^d)^*$ given by

$$U\omega=e^W\omega.$$

Then we have that $U\bar\partial_W U^*=\bar\partial$. Let $\bar\partial^*$ be the adjoint operator of $\bar\partial$ with respect to the measure $e^{-2W}m$, and we put $\hat H=UHU^*$. Since $\bar\partial^2=(\bar\partial^*)^2=0$, we have

$$(\hat H\omega,\omega)=4\int_{\mathbb{C}^d}\left(|\bar\partial\omega(z)|^2+|\bar\partial^*\omega(z)|^2\right)e^{-2W(z)}m(dz). \tag{3.1}$$

Suppose that $\omega\in\ker\hat H$. We show that ω is of type $(0,d)$. We can write $\omega=\omega_1+\omega_2$, where

$$\omega_1\in L^2(\mathbb{C}^d;e^{-2W}m)\otimes\bigoplus_{q=0}^{d-1}\wedge^{0,q}(\mathbb{C}^d)^*,$$
$$\omega_2\in L^2(\mathbb{C}^d;e^{-2W}m)\otimes\wedge^{0,d}(\mathbb{C}^d)^*.$$

The first term ω_1 can be expressed as

$$\omega_1(z)=\sum_I\omega_I(z)d\bar z^I,$$

where the summation is over all proper subsets I of $\{1,2,\dots,d\}$ and $\omega_I\in L^2(\mathbb{C}^d;e^{-2W}m)$. Since $\bar\partial\omega=\bar\partial\omega_1$, we see

$$\bar\partial\omega(z)=\sum_I\sum_{i\notin I}\frac{\partial\omega_I(z)}{\partial\bar z^i}d\bar z^i\wedge d\bar z^I.$$

By (3.1) we obtain

$$0=\int_{\mathbb{C}^d}|\bar\partial\omega(z)|^2e^{-2W(z)}m(dz)=\sum_I\sum_{i\notin I}\int_{\mathbb{C}^d}\left|\frac{\partial\omega_I(z)}{\partial\bar z^i}\right|^2e^{-2W(z)}m(dz).$$

Since I is a proper subset of $\{1,2,\dots,d\}$, we may assume that $d\notin I$ without loss of generality. Then we have

$$\int_{\mathbb{C}^d}\left|\frac{\partial\omega_I(z)}{\partial\bar z^d}\right|^2e^{-2W(z)}m(dz)=0.$$

Since $e^{-2W(z)}>0$, we obtain

$$\frac{\partial\omega_I(z)}{\partial\bar z^d}=0\quad\text{on}\quad\mathbb{C}^d.$$

This means that ω_I is an entire function in z^d. On the other hand, ω_I must be in $L^2(\mathbb{C}^d; e^{-2W}m)$. Since W is bounded from above, $\omega_I \in L^2(\mathbb{C}^d)$. By Fubini's theorem, it holds that

$$\int_{\mathbb{C}} |\omega_I(z_0, z^d)|^2 dx^d dy^d < \infty$$

for a.e. $z_0 = (z^1, z^2, \ldots, z^{d-1})$ with respect to the measure $dx^1 dy^1 \cdots dx^{d-1} dy^{d-1}$. Since $\omega_I(z_0, \cdot)$ is entire, $\omega_I(z_0, \cdot)$ must be equal to 0. Thus we see $\omega_I = 0$ a.e. with respect to m on $\mathbb{C}^d$. Consequently we have that $\omega_1 = 0$. Hence $\omega = \omega_2$.

Since $\ker \hat{H} = \ker \not{D}(a)U^*$, it suffices to compute the dimension of $\ker \not{D}(a)U^*$. We can write

$$(U^*\omega)(z) = \tilde{\omega}(z) d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^d,$$

where $\tilde{\omega} \in L^2(\mathbb{C}^d)$. Since $\bar{\partial}_W U^*\omega = 0$, it follows that

$$\not{D}(a)U^*\omega = 2\bar{\partial}_W^* U^*\omega = 2\sum_{i=1}^{d} \left(-\frac{\partial \tilde{\omega}}{\partial z^i} + \frac{\partial W}{\partial z^i}\tilde{\omega} \right) (-1)^{i-1} d\bar{z}^1 \wedge \cdots \wedge \overset{\vee}{d\bar{z}^i} \wedge \cdots \wedge d\bar{z}^d.$$

Therefore the equality $\not{D}(a)U^*\omega = 0$ implies

$$-\frac{\partial \tilde{\omega}}{\partial z^i} + \frac{\partial W}{\partial z^i}\tilde{\omega} = 0, \qquad i = 0, \ldots, d.$$

Let $f = e^{-W}\tilde{\omega}$. We obtain the equations

$$\frac{\partial f}{\partial z^i} = 0, \qquad i = 0, \ldots, d.$$

This means that f is an anti-holomorphic function in each $z^i, i = 1, \ldots, d$. By the assumption, we have that

$$e^{W(z)} \sim \frac{1}{|z|^C} \quad \text{as} \quad |z| \to \infty.$$

Hence $fe^W \in L^2(\mathbb{C}^d)$ if and only if f is a polynomial in $\bar{z}^1, \cdots, \bar{z}^d$ such that the degree of f is smaller than $C - d$. Therefore, $\dim \ker \hat{H}$ is equal to $N_d(C)$.

Next, we consider the case where $C < 0$. A unitary operator $U \colon L^2(\mathbb{C}^d) \otimes \wedge^{0,*}(\mathbb{C}^d)^* \to L^2(\mathbb{C}^d; e^{2W}m) \otimes \wedge^{0,*}(\mathbb{C}^d)^*$ is given by $U\omega = e^{-W}\omega$. Then we have that

$$U\bar{\partial}_W^* U^* = \sum_{i=1}^{d} \operatorname{int}(dz^i)\frac{\partial}{\partial z^i}.$$

For $\omega \in \ker UHU^*$ we can write $\omega = \omega_1 + \omega_2$, where $\omega_1 \in L^2(\mathbb{C}^d; e^{2W}m) \otimes \Lambda^{0,0}(\mathbb{C}^d)^*$ and $\omega_2 \in L^2(\mathbb{C}^d; e^{2W}m) \otimes \oplus_{p=1}^d \Lambda^{0,p}(\mathbb{C}^d)^*$. Then, in a way similar to the case where $C \geq 0$, we can obtain that $\omega = \omega_1$ and that $\dim \ker H$ is equal to $N_d(C)$. Thus we obtain the desired result. □

From the proof of Theorem 3.1 we can also clarify the structure of $\ker H$.

Corollary 3.2. *Under the assumption of Theorem 3.1, the following holds: Let $C \geq 0$ (resp. $C < 0$). Then each vector in $\ker H$ has the form;*

$$fe^W d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^d \quad (resp. \quad fe^{-W}).$$

Here, f is a polynomial in $z^1, \ldots, z^d$ (resp. $\bar{z}^1, \ldots, \bar{z}^d$) such that the degree of f is smaller than $|C| - d$.

In the rest of this section we give several remarks

Remark 3.3. In the 2-dimensional case, Aharonov and Casher has proved Theorem 3.1 in another context (see [4]).

Remark 3.4. In the special case where there exist constants $C \in \mathbb{R}$ and $R > 0$ such that $W(z) = -C \log |z|$ for $|z| > R$, the magnetic field $b = id(\bar{\partial} - \partial)W$ goes to 0 as $z \to \infty$. Then, by Theorem 2.2 in [2], the spectrum of H is equal to $[0, \infty)$. Hence H is not a Fredholm operator (see [5]).

Remark 3.5. We can state Theorem 3.1 in terms of index theory. Let

$$\Gamma = (-i)^d \gamma^1 \gamma^2 \cdots \gamma^{2d}.$$

We can easily check that Γ is self-adjoint, $\sigma(\Gamma) = \sigma_p(\Gamma) = \{-1, 1\}$, $\ker(\Gamma - 1) = L^2(\mathbb{C}^d) \otimes \oplus_{q:\ \mathrm{even}} \Lambda^{0,q}(\mathbb{C}^d)^*$ and $\ker(\Gamma + 1) = L^2(\mathbb{C}^d) \otimes \oplus_{q:\ \mathrm{odd}} \Lambda^{0,q}(\mathbb{C}^d)^*$, where we denote by $\sigma(\Gamma)$ and $\sigma_p(\Gamma)$ the spectrum and the point spectrum of Γ, respectively. Moreover, we see

$$[\not{D}(a), \Gamma]_+ = 0.$$

Hence we can write

$$\not{D}(a) = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$$

with A being a densely defined closed linear operator from $\ker(\Gamma - 1)$ to $\ker(\Gamma + 1)$. As usual, we define the index of A by $Ind(A) = \dim\ker A - \dim\ker A^*$. By virtue of Corollary 3.2, we can verify that

$$Ind(A) = \begin{cases} N_d(C), & C < 0 \text{ or } d \text{ is even}, \\ -N_d(C), & \text{otherwise}. \end{cases}$$

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