Title	Distance formulas of asymptotic Toeplitz and Hankel operators
Author(s)	Yamada, M.
Citation	Hokkaido University Preprint Series in Mathematics, 182, 1-11
Issue Date	1993-02
DOI	10.14943/83326
Doc URL	http://hdl.handle.net/2115/68928
Type	bulletin (article)
File Information	pre182.pdf

Distance Formulas of Asymptotic Toeplitz and Hankel Operators

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Series ♯182. February 1993

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Distance Formulas

of

Asymptotic Toeplitz and Hankel Operators

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Key words; Toeplitz operator, Hankel operator, distance formula,
commutator ideal, Nehari's theorem

Abstract. An asymptotic Hankel operator on the Lebesgue space L^2 is defined and a distance formula is obtained. From it, two distance formulas on the Hardy space H^2 follow. One of them is known and concerned with Hankel operators. Another one is new and related with a commutator ideal of Toeplitz operators.

§ 1. Introduction

For $1 \leqq p \leqq \infty$, let L^p be the usual Lebesgue space $L^p(T)$ of the functions on the unit circle T, and the Hardy space H^p be the class of all L^p-functions whose negative Fourier coefficients are vanished. Moreover, $\mathcal{B}(L^2)$ and $\mathcal{B}(H^2)$ denote the class of all bounded linear operators on L^2 and H^2 respectively.

Given a function $\phi \in L^\infty$, the Toeplitz operator T_ϕ with symbol ϕ on H^2 is given by

$$T_\phi = P M_\phi | H^2,$$

where P is the orthogonal projection of L^2 onto H^2, and M_ϕ is the multiplication operator on L^2 corresponding to ϕ. Further, let unitary operator V on L^2 be

$$(V f)(z) = f(\bar{z}) \qquad (f \in L^2).$$

We define the Hankel operator H_ϕ with symbol ϕ by

$$H_\phi = P V M_\phi | H^2,$$

and we call the operator $V M_\phi$ on L^2 by "Hankel operator on L^2" in order to avoid a confusion.

Put $U = M_z$, and let P_n be the orthogonal projection of L^2 onto $[z^i ; |i| \leqq n]$. Given $A \in \mathcal{B}(L^2)$, for $n \geqq 0$, we define the operators σ_n by

$$\sigma_n(A) = P_n U^n A U^n.$$

If the sequence of finite rank operators $\{\sigma_n(A)\}$ converges weakly, then we will write its limit by $\sigma(A)$. Let $\{a_{ij} ; i, j \in Z\}$ be the set of the matrix elements of A with respect to an orthonormal basis $\{z^i ; i \in Z\}$. We prove that $\sigma(A)$ is a Hankel operator on L^2. Hence it is reasonable to call A an asymptotic Hankel operator on L^2 when $\{\sigma_n(A)\}$ converges weakly.

In § 2., we show that an asymptotic Hankel operator on L^2 is written to the direct sum of a Hankel operator and a vanishing operator(the definition of vanishing operators is in § 2.). Moreover, we give a distance formula which is concerned with the class of all asymptotic operators on L^2.

In § 3., we apply the distance formula in § 2. to Toeplitz operators, and obtain an interesting corollary. In § 4., we observe that this corollary is closely related to Douglas' theorem (c.f.[3;p179]), which shows the distance of a Toeplitz operator and a commutator ideal in a Toeplitz algebra is just equal to the norm of the Toeplitz operator. Barría and Halmos define an asymptotic Toeplitz operator[1], and give a result of that an extended symbol map for these operators is contractive. The definition of the asymptotic Hankel operator on L^2 is similar to that of the asymptotic Toeplitz operator, however our definition is weaker than the original definition which is due to Barría and Halmos. Moreover, our theorem in § 3. shows the existence of an isometrical correspondence between extended symbol functions and cosets in a certain quotient space of $\mathcal{B}(H^2)$. Hence we can say that our result is wider and sharper than that of them.

In the final section, § 5., we give two distance formulas for operators on H^2. One is related with Toeplitz operators. This was already obtained in the theorem of § 3., however again we give it changing an expression. Another one is Feintuch's distance formula which is concerned with Hankel operators on H^2. Feintuch represents the norm of a Hankel operator on H^2 by the distance of operators on H^2. On the other hand, we express it by the distance of operators on L^2. Therefore, we obtain an expression of the norm of a Hankel operator, which is more analogous to Nehari's theorem[6] than Feintuch's one.

The author wishes to thank Professor T. Nakazi for his advice and indispensable help during the writing of this paper.

§ 2. The distance formula for operators on L^2

Proposition 2.1. If A is a bounded operator on L^2 then $\{\sigma_n(A)\}$ converges weakly if and only if $\{a_{-l\ \ l+k}\}_{l=0}^{\infty}$ is a convergent sequence for all k. Moreover, an operator $\sigma(A)$ is a Hankel operator on L^2.

Proof. Suppose that $\{\sigma_n(A)\}$ converges weakly to $\sigma(A)$. For each i, j fixed, put $n_0 = |i|$ and $n \geqq n_0$, then the sequence $\{(\sigma_n(A)z^j, z^i) = a_{-l, l+i+j}\}$ converges to $(\sigma(A)z^j, z^i)$, where $l = n - i$. Since the matrix elements of $\sigma(A)$ depend only on $i + j$, therefore $\sigma(A)$ is the Hankel operator on L^2.

Conversely, if the limit of the sequence $\{a_{-l\ \ l+k}\}_{l=0}^{\infty}$ exists for all k, then the above computation shows that the bounded operator $\sigma(A)$ is well defined by

$$(\sigma(A)z^j, z^i) = \lim_{l} a_{-l, l+i+j},$$

since $\|\sigma_n(A)\| \leqq \|A\|$ for all $n \geqq 0$. ∎

The proof of the above proposition is parallel to that is shown by Feintuch[4].

Remark 2.2. If A is a Hankel operator on L^2, then $\sigma(A) = A$. Since an operator A is a Hankel operator on L^2 if and only if $AU = U^*A$, therefore

$$\begin{aligned}\sigma_n &= P_n U^n A U^n \\ &= P_n U^n U^{*n} A \\ &= P_n A.\end{aligned}$$

Hence $\{\sigma_n(A)\}$ converges strongly to A.

Set

$$\mathcal{AH} = \{A \in \mathcal{B}(L^2); \{\sigma_n(A)\} \text{ has a strong limit}\}$$

and

$$\mathcal{L} = \{ L \in \mathcal{AH} ; \sigma(L) = 0 \}.$$

Obviously, $\mathcal{AH}$ and $\mathcal{L}$ are the subspaces of $\mathcal{B}(L^2)$. And we call the elements of $\mathcal{L}$ by "vanishing operators".

Theorem 2.3. Both $\mathcal{AH}$ and $\mathcal{L}$ are norm closed.

Proof. See [4]. ∎

Theorem 2.4. If A is in $\mathcal{AH}$, then there exist a unique Hankel operator $\sigma(A)$ on L^2 and L in $\mathcal{L}$ such that $A = \sigma(A) + L$. Moreover, $\|\sigma(A)\| = \|A + \mathcal{L}\|$.

Proof. Evidently, from the remark 2.2., there exists L in $\mathcal{L}$ such that $A = \sigma(A) + L$, because

$$\sigma[A - \sigma(A)] = 0.$$

If A is written in the form $A = H + B$, where H is a Hankel operator on L^2 and B is in $\mathcal{L}$, then

$$\begin{aligned} \sigma(A) - H &= \sigma[\sigma(A) - H] \\ &= \sigma(B - L) \\ &= 0. \end{aligned}$$

Therefore, $\sigma(A) = H$ and $L = B$ respectively. Hence σ is a continuous projection in $\mathcal{AH}$ of norm 1.

Lastly, we will show that $\|\sigma(A)\| = \|A + \mathcal{L}\|$. Since $\ker \sigma = \mathcal{L}$ and σ is contractive, the left hand side is less than or equal to the right hand side. Conversely, by the above argument, there exists $L = (I - \sigma)(A)$ in $\mathcal{L}$ such that $\|\sigma(A)\| = \|A - L\|$. This completes the proof. ∎

§ 3. Application of the distance formula

Let $P_{+,0}$ be the orthogonal projection of L^2 onto zH^2. Put $\mathcal{AH}_r=\{A\in\mathcal{B}(H^2); VAP\in\mathcal{AH}\}$ and $\mathcal{L}_r=\{L\in\mathcal{B}(H^2); VLP\in\mathcal{L}\}$. Furthermore, an isometrical isomorphism Γ between L^∞ and $\mathcal{H}$ is defined by

$$\Gamma(\phi)=VM_\phi,$$

where $\mathcal{H}$ denotes the class of all Hankel operators on L^2. We will show the following theorem.

Theorem 3.1. If Φ is defined as a map from $\mathcal{AH}_r/\mathcal{L}_r$ into $\mathcal{B}(L^2)$ by

$$\Phi(A+\mathcal{L}_r)=\sigma(VAP),$$

then the quotient space $\mathcal{AH}_r/\mathcal{L}_r$ is isometrically isomorphic to L^∞ by an isometric isomorphism $\Gamma^{-1}\circ\Phi$.

Proof. By the definition of $\mathcal{L}_r$, the linear mapping Φ is well defined. For any Hankel operator H on L^2 there exists a unique L^∞-function ϕ such that $H=VM_\phi$. Let T_ϕ be a Toeplitz operator on H^2 with symbol ϕ. We claim that T_ϕ is in $\mathcal{AH}_r$ and $\Phi(T_\phi+\mathcal{L}_r)=VM_\phi$.

For any i fixed, and n sufficiently large($n\geqq|i|$), then

$$\sigma_n(VT_\phi P)z^i=P_nU^nVPM_\phi U^nz^i.$$

Since $VU=U^*V$ and $VP=(I-P_{+,0})V$, we have

$$\sigma_n(VT_\phi P)z^i=P_nU^n(I-P_{+,0})U^{*n}(VM_\phi z^i)$$

$$=P_n(VM_\phi z^i) - P_nU^nP_{+,0}U^{*n}(VM_\phi z^i).$$

$$=P_n(VM_\phi z^i),$$

therefore

$$\sigma_n(VT_\phi P)z^i = P_n(VM_\phi)z^i$$

which converges to $VM_\phi z^i$. Hence, $\{\sigma_n(VT_\phi P)\}$ converges strongly to VM_ϕ. It follows that the mapping Φ is surjective.

Finally, we will show that Φ is isometric. Let A is in AH_r. By theorem 2.4., we have

$$\|\sigma(VAP)\| = \|VAP + \mathcal{L}\|.$$

Further, since V is unitary and $V\mathcal{L}_r P \subset \mathcal{L}$, it follows that

$$\|\sigma(VAP)\| \leqq \|VAP + V\mathcal{L}_r P\|$$

$$\leqq \|A + \mathcal{L}_r\|.$$

Here exists a function ϕ in L^∞ such that $\sigma(VAP) = VM_\phi$, and from the above argument, $\sigma(VT_\phi P) = VM_\phi$. This implies that $A - T_\phi \in \mathcal{L}_r$, and thus

$$\|\phi\|_\infty = \|VM_\phi\| = \|\sigma(VAP)\|$$

$$\leqq \|A + \mathcal{L}_r\|$$

$$\leqq \|T_\phi + \mathcal{L}_r\|$$

$$\leqq \|T_\phi\| = \|\phi\|_\infty.$$

This completes the proof. ▮

The following corollary is closely related to Douglas' theorem (c.f.[3;p179]). This discussion will be continued in the next section.

Corollary 3.2. For any ϕ in L^∞, the quotient norm of $T_\phi + \mathcal{L}_r$ is equal to the essential norm of ϕ; i.e.

$$\|T_\phi\| = \|T_\phi + \mathcal{L}_r\|.$$

Proof. Let ϕ be in L^∞. From the proof of theorem 3.1., T_ϕ is in AH_r and $\Gamma^{-1}\circ\Phi(T_\phi+\mathcal{L}_r)=\phi$. ∎

§ 4. Extension of Douglas' theorem

$\mathcal{T}^+$ denotes the Toeplitz algebra which is the norm closed algebra generated by the set of all Toeplitz operators, and Q denotes the commutator ideal in $\mathcal{T}^+$. Douglas(c.f.[3;p179]) shows that the mapping ξ_C from L^∞ to $\mathcal{T}^+/Q$ induced by

$$\xi_C(\phi)=T_\phi+Q$$

is a *-isometrical isomorphism. This gives that $\|T_\phi\|=\|T_\phi+Q\|$. In this case, "isomorphism" means linear and multiplicative, on the other hand in theorem 3.1., "isomorphism" means not multiplicative. But corollary 3.2. implies that the mapping $\Phi^{-1}\circ\Gamma;\phi\to T_\phi+\mathcal{L}_r$ is isometric. In the following proposition, we will observe that corollary 3.2. is an extension of the result which is due to Douglas.

Proposition 4.1. $\mathcal{L}_r$ contains the commutator ideal Q of an algebra $\mathcal{T}^+$.

To prove this, we need two lemmas.

Lemma 4.2. Let α, β be L^∞-functions, then the following relations (1), (2) hold;

$$(1)\quad T_\alpha T_\beta=T_{\alpha\beta}-H_{V\alpha}H_\beta$$
$$(2)\quad H_\alpha T_\beta=H_{\alpha\beta}-T_{V\alpha}H_\beta$$

Lemma 4.3. Let H_r be the class of all Hankel operators on H^2, then $\mathcal{L}_r$ contains $B(H^2)\times H_r$.

Lemma 4.2. is well known. We sketch the proofs of lemma 4.3. and proposition 4.1., because they are similar to the proof of theorem 4[1].

Proof of Lemma 4.3. Let H be a Hankel operator, then $HS = S^*H$ implies that $\{HS^n\}$ converges strongly to 0, where $S = T_z$. For any fixed $i < 0$, if $n \geqq |i|$, then

$$HPU^n z^i = HPU^{n+i}1 = HS^{n+i}1.$$

Therefore $\{HPU^n z^i\}$ converges to 0. Consequently, $\{HPU^n\}$ converges strongly to 0 in L^2. Hence, for any C in $\mathcal{B}(H^2)$ and the Hankel operator H, $\{\sigma_n[V(CH)P]\}$ converges strongly to 0. ∎

Proof of Proposition 4.1. Firstly, we claim that $\mathcal{L}_r$ is uniformly closed. In fact, let $\{L_k\}$ be the sequence of $\mathcal{L}_r$ which converges uniformly to an operator L, then

$$\|VLP - VL_kP\| \leqq \|V\| \cdot \|L - L_k\| \cdot \|P\|$$

$$\to 0 \qquad (k \to \infty)$$

Since $\{VL_kP\}$ is the sequence of vanishing operators and $\mathcal{L}$ is uniformly closed, therefore VLP is vanishing.

Let α, β be L^∞-functions, an operator D be in $\mathcal{T}^*$, and $\{T_k; 1 \leqq k \leqq m\}$ be a finite set of Toeplitz operators. Put

$$L = D(T_\alpha T_\beta - T_\beta T_\alpha)(\prod T_k).$$

It is sufficient to prove that L belongs to $\mathcal{L}_r$, because $\mathcal{L}_r$ is uniformly closed. By lemma 4.2.(1), the part of the commutator in L is been able to change to four Hankel operators, and using Barría-Halmos' technique[1], lemma 4.2.(2) is applied to L in inductively, then L is written in the form,

$$L = D(B_1 H_1 + B_2 H_2 + \cdots + B_l H_l),$$

where each B_k is the finite multiple of Hankel and Toeplitz operators, and H_k is a Hankel operator respectively. Consequently, lemma 4.3. shows that L belongs to $\mathcal{L}_r$. This completes the proof. ∎

Corollary 4.4.(Douglas) For any Toeplitz operator with symbol ϕ, it holds that

$$\|T_\phi\| = \|T_\phi + Q\|.$$

Remark 4.5. Q is a proper subspace of $\mathcal{L}_r$. Because lemma 4.3. implies that the class of all Hankel operators H_r is contained in $\mathcal{L}_r$, but a Hankel operator which dose not belong to commutator ideal Q exists. Further, lemma 4.3. shows that there are no Hankel operators that are left invertible. If there exists an operator C on the range of certain Hankel operator H such that $CH = I$, then the bounded operator CH on H^2 is in $\mathcal{L}_r$, but I is not in $\mathcal{L}_r$, since $\sigma(I)$ is a constant function 1. In fact, corollary 3.2. and again lemma 4.3. imply that $\|\phi\|_\infty = \|T_\phi - H\|$ for any Hankel operator H and fixed any Toeplitz operator. Here, put $\phi \equiv 1$, then $1 \leqq \|I - H\|$.

§ 5. Two distance formulas

In this section, we obtain two distance formulas. One is the formula of Toeplitz operators, and another is that of Hankel operators which is obtained by Feintuch.

For any A in AH, put

$$H_{\sigma(A)} = P\sigma(A)|H^2$$

and

$$T_{\sigma(A)} = PV\sigma(A)|H^2.$$

Set $\mathcal{F} = \{F \in AH; P\sigma(F)|H^2 = 0\}$. Since $\sigma(A)$ is a Hankel operator on L^2, there exists a function ϕ in L^∞ such that $\sigma(A) = VM_\phi$. We note that $H_{\sigma(A)}$, $T_{\sigma(A)}$ are the Hankel operator and the Toeplitz operator with symbol ϕ respectively.

We give the following two distance formulas.

Theorem 5.1. For any A in $\mathcal{AH}$, the following is true;

$$(1)\ \|H_{\sigma(A)}\| = \|A + \mathcal{F}\|$$

$$(2)\ \|T_{\sigma(A)}\| = \|A + \mathcal{L}\|.$$

In fact, the subspace $\mathcal{F}$ is written in the form of $\mathcal{F} = V(zH^{\infty}) + \mathcal{L}$.

Proof. We show that $\mathcal{F} = V(zH^{\infty}) + \mathcal{L}$. For any $F \in \mathcal{F}$, there exist a L^{∞}-function ϕ and L in $\mathcal{L}$ such that $F = VM_{\phi} + L$. Since $VM_{\phi} = \sigma(F)$, then

$$0 = P\sigma(F)|H^2 = PVM_{\phi}|H^2 = H_{\phi}.$$

Therefore ϕ is in zH^{∞}. The converse inclusion is trivial.

By the definition of $\mathcal{F}$, $H_{\sigma(A)} = P\sigma(A - F)|H^2$ for any F in $\mathcal{F}$, and we have that $\|H_{\sigma(A)}\| \leqq \|A + \mathcal{F}\|$. Conversely, since $H_{\sigma(A)}$ is a Hankel operator on H^2, Nehari's theorem[6] shows that there exists ϕ in L^{∞} such that $P\sigma(A - VM_{\phi})|H^2 = 0$ and $\|H_{\sigma(A)}\| = \|\phi\|_{\infty}$. It follows that

$$\|A + \mathcal{F}\| = \|VM_{\phi} + \mathcal{F}\|$$

$$\leqq \|\phi\|_{\infty} = \|H_{\sigma(A)}\|.$$

Hence (1) is obtained.

(2) follows from proposition 2.1., theorem 2.4. and the fact that the norm of the Toeplitz operator is equal to the essential supremum norm of its symbol. ∎

Remark 5.2. For an operator T on H^2, Feintuch shows that the distance between T and certain subspace $\mathcal{L}_0$ of $\mathcal{B}(H^2)$ is equal to the norm of the Hankel operator $H(T)$ for which is constructed from own T. Let ϕ be a symbol of $H(T)$, Nehari's theorem[6] implies that

$\|\phi + H^\infty\| = \|T + \mathcal{L}_0\|$. Moreover, note that the constructed Hankel operator from T_ϕ with symbol ϕ is again equal to H_ϕ, it follows that $\|\phi + H^\infty\| = \|T_\phi + \mathcal{L}_0\|$. Therefore, in generally, Feintuch's distance formula implies that the distance from T_ϕ to $\mathcal{L}_0$ is less than the norm of T_ϕ. On the other hand, corollary 3.2. shows that the distance from T_ϕ to $\mathcal{L}_r$ is equal to the norm of T_ϕ. The reason of these differences is that $\mathcal{L}_r$ is the proper subspace of $\mathcal{L}_0$. Since for any $F \in \mathcal{F}$, $V\sigma(F)$ has a lower triangular matrix, then in a sense the subspace $\mathcal{L}_0$ is the restriction of $V\mathcal{F}$ to H^2. Therefore, Feintuch's distance formula on H^2 can be represented in the distance of operators on L^2. Hence, our result is more analogous to Nehari's theorem than Feintuch's formula.

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