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Canonical modules and Cohen-Macaulay types  
of partially ordered sets

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Every partially ordered set (or "poset" for short) to be studied is finite. We write  $\#(X)$  for the cardinality of a finite set  $X$ .

(1.1) Let  $P$  be a poset. Then we set  $P^\wedge = P \cup \{0^\wedge, 1^\wedge\}$  with  $0^\wedge < \alpha < 1^\wedge$  for every  $\alpha \in P$ . A chain  $C$  of the poset  $P^\wedge$  is a totally ordered subset of  $P^\wedge$ . From now on, we assume that  $P^\wedge$  is graded of rank  $d+1$  [6, p.99], that is to say, there exists a unique rank function  $\rho : P^\wedge \rightarrow \{0, 1, \dots, d+1\}$  such that  $\rho(0^\wedge) = 0$ ,  $\rho(1^\wedge) = d+1$ , and  $\rho(\beta) = \rho(\alpha) + 1$  if  $\alpha < \beta$  and  $\alpha < \gamma < \beta$  for no  $\gamma \in P^\wedge$ .

(1.2) Fix a field  $k$ . Let  $A = k[x; x \in P]$  be the polynomial ring over  $k$  with the "standard grading," i.e., each  $\deg x = 1$ , whose variables are the elements of  $P$ . We write  $I$  for the ideal of  $A$  generated by those quadratic monomials  $xy$  such that  $x$  and  $y$  are incomparable in  $P$ . The quotient algebra  $k[P] = A/I$  is called the *Stanley-Reisner ring* of  $P$  over  $k$ . Define  $\theta_i \in k[P]$  by  $\theta_i = \sum_{\rho(x)=i} x$  for each  $1 \leq i \leq d$ . Then the sequence  $\theta_1, \theta_2, \dots, \theta_d$  is a system of parameters for  $k[P]$ , in other words, (i)  $\theta_1, \dots, \theta_d$  are algebraically independent over  $k$  and (ii)  $k[P]$  is finitely generated as a module over the

subalgebra  $k[\theta] = k[\theta_1, \theta_2, \dots, \theta_d]$ . We say that  $k[P]$  is *Cohen-Macaulay* (or  $P$  is *Cohen-Macaulay over  $k$* ) if  $k[P]$  is a free module over  $k[\theta]$ .

For example, if  $L = P^\wedge$  is a semimodular lattice then  $k[P]$  is Cohen-Macaulay. We refer the reader to, e.g., [2] for basic information about Cohen-Macaulay posets.

(1.3) Suppose that the Stanley-Reisner ring  $k[P]$  of a poset  $P$  is Cohen-Macaulay. First, the *canonical module*  $\Omega(k[P])$  of  $k[P]$  is defined to be the graded module

$$\Omega(k[P]) = \text{Hom}_{k[\theta]}(k[P], k[\theta])$$

over  $k[P]$ . Secondly, the *socle*  $\text{Soc}(k[P]/(\theta))$  of  $k[P]/(\theta)$  is

$$\text{Soc}(k[P]/(\theta)) = \{ y \in k[P]/(\theta) ; xy = 0 \text{ for every } x \in P \},$$

where  $(\theta)$  is the parameter ideal  $(\theta_1, \theta_2, \dots, \theta_d)$  of  $k[P]$ . The dimension  $\dim_k \text{Soc}(k[P]/(\theta))$  of  $\text{Soc}(k[P]/(\theta))$  as a vector space over  $k$  is called the *Cohen-Macaulay type* of  $k[P]$ .

It is known, e.g., [3, Corollary 6.11] that the Cohen-Macaulay type of  $k[P]$  is equal to the minimal number of generators of  $\Omega(k[P])$  as a module over  $k[P]$ . Moreover, in the language of homological algebra, the Cohen-Macaulay type of  $k[P]$  is equal to  $\dim_k \text{Tor}_{v-d}^A(k[P], k)$  with  $v = \#(P)$ .

(1.4) Now, if  $\alpha < \beta$  in  $P^\wedge$  with  $\rho(\beta) - \rho(\alpha) = r + 1$ , then the  $(r-1)$ -th reduced homology group  $H_{r-1}^\sim((\alpha, \beta); k)$  of the open interval  $(\alpha, \beta) = \{ x \in P^\wedge ; \alpha < x < \beta \}$  of  $P^\wedge$  can be imbedded in  $k[P]$ . Given a chain  $C: 0^\wedge = x_0 < x_1 < \dots < x_s < x_{s+1} = 1^\wedge$  of  $P^\wedge$  with each  $\rho(x_{i+1}) - \rho(x_i) = r(i) + 1$ , we write  $\mathfrak{R}(C)$  for the subspace of  $k[P]$  spanned by those polynomials

$$f_0 x_1^2 f_1 x_2^2 \dots f_{s-1} x_s^2 f_s$$

with  $f_i \in H_{r(i)-1}^\sim((x_i, x_{i+1}); k)$  for every  $0 \leq i \leq s$ . Here we

employ the convention that each monomial of  $f_i$  is of the form  $\alpha_1 \alpha_2 \dots \alpha_{r(i)}$  with  $x_i < \alpha_1 < \alpha_2 < \dots < \alpha_{r(i)} < x_{i+1}$ . Moreover, we define  $\mathcal{Q}(C)$  to be the subspace of  $k[P]/(\theta)$  which is the image of  $x_1^{-2} x_2^{-2} \dots x_s^{-2} \mathcal{R}(C)$  in  $k[P]/(\theta)$ .

(1.5) Let  $\mu = \mu_{P^\wedge}$  be the Möbius function of  $P^\wedge$  as defined in, e.g., [6, p.116]. If  $C: 0^\wedge = x_0 < x_1 < \dots < x_s < x_{s+1} = 1^\wedge$  is a chain of  $P^\wedge$ , then we set

$$\mu(C) = \mu(x_0, x_1) \mu(x_1, x_2) \dots \mu(x_s, x_{s+1}).$$

Thus, in particular,

$$\dim_k(\mathcal{R}(C)) = |\mu(C)|.$$

We say that a chain  $C: 0^\wedge = x_0 < x_1 < \dots < x_s < x_{s+1} = 1^\wedge$  of  $P^\wedge$  is *essential* if  $\mu(C) \neq 0$ . Let  $\mathcal{E}(P^\wedge)$  be the set of essential chains of  $P^\wedge$  and  $\mathcal{E}^*(P^\wedge)$  ( $\subset \mathcal{E}(P^\wedge)$ ) the set of minimal essential chains (or *fundamental chains* [4]) of  $P^\wedge$ .

We write  $\mathcal{J}^*(k[P])$  for the ideal of  $k[P]$  generated by all  $\mathcal{R}(C)$  with  $C \in \mathcal{E}^*(P^\wedge)$ . Moreover, we define  $\mathcal{J}(k[P])$  to be the ideal of  $k[P]$  which is generated by those square-free monomials  $x_1 x_2 \dots x_s$  such that  $x_i < x_{i+1}$  in  $P$  for every  $1 \leq i \leq s$  and  $0^\wedge = x_0 < x_1 < \dots < x_s < x_{s+1} = 1^\wedge \in \mathcal{E}^*(P^\wedge)$ .

We are now in the position to state our main result in the paper.

(1.6) THEOREM. Suppose that the Stanley-Reisner ring  $k[P]$  of a poset  $P$  is Cohen-Macaulay. Then the ideal  $\mathcal{J}^*(k[P])$  is isomorphic to the canonical module  $\Omega(k[P])$  of  $k[P]$ , up to shift in grading, if and only if the Cohen-Macaulay type of  $k[P]$  is equal to  $\sum_{C \in \mathcal{E}^*(P^\wedge)} |\mu(C)|$ .

**Proof.** Let  $\mathcal{J}(k[P])$  be the ideal of  $k[P]$  generated by all  $\mathcal{R}(C)$  with  $C \in \mathcal{E}(P^\wedge)$ . The monomorphism  $\psi$  in [1, Theorem 1]

enables us to define a monomorphism  $\psi : \mathfrak{J}(k[P]) \rightarrow \Omega(k[P])$  of graded modules over  $k[P]$  in the analogous way. Note that the lowest degree of a non-zero homogeneous element of  $\Omega(k[P])$  (resp.  $\mathfrak{J}(k[P])$ ) is  $-d + \min\{\#(C); C \in \mathfrak{E}(P^\wedge)\}$  (resp.  $d + \min\{\#(C); C \in \mathfrak{E}(P^\wedge)\}$ ), while  $\psi$  has degree  $-2d$ . We write  $Q(k[P])$  for the subspace of  $k[P]$  which is spanned by those polynomials  $f_0 x_1^{n_1} f_1 x_2^{n_2} \dots f_{s-1} x_s^{n_s} f_s$  such that  $0^\wedge = x_0 < x_1 < \dots < x_s < x_{s+1} = 1^\wedge \in \mathfrak{E}(P^\wedge)$  with  $\rho(x_{i+1}) - \rho(x_i) = r(i) + 1$ ,  $f_i \in H_{r(i)-1}((x_i, x_{i+1}); k)$  for every  $0 \leq i \leq s$ , and each  $n_i \geq 2$  is an integer. Since  $Q(k[P]) \subset \mathfrak{J}(k[P])$ , it follows from [5, Theorem 7.1, p.80] that the above monomorphism  $\psi : \mathfrak{J}(k[P]) \rightarrow \Omega(k[P])$  is an isomorphism and  $Q(k[P]) = \mathfrak{J}(k[P])$ . Moreover, each subspace  $\mathfrak{R}(C)$  with  $C \in \mathfrak{E}^*(P^\wedge)$  is contained in the subspace of  $k[P]$  spanned by an arbitrary system of generators of the ideal  $\mathfrak{J}(k[P])$ . Hence  $\mathfrak{J}^*(k[P]) = \mathfrak{J}(k[P])$  if and only if the minimal number of generators of  $\Omega(k[P])$ , i.e., the Cohen-Macaulay type of  $k[P]$ , is equal to  $\sum_{C \in \mathfrak{E}^*(P^\wedge)} |\mu(C)|$  as desired. Q.E.D.

When  $\mu(0^\wedge, 1^\wedge) \neq 0$ , the above Theorem (1.6) essentially coincides with [1, Theorem 2]. Now, it is shown in [4] that, for a poset  $P$  such that  $L = P^\wedge$  is a modular lattice, the Cohen-Macaulay type of  $k[P]$  is equal to  $\sum_{C \in \mathfrak{E}^*(P^\wedge)} |\mu(C)|$ .

(1.7) COROLLARY. If  $L = P^\wedge$  is a modular lattice, then  $\mathfrak{J}^*(k[P])$  is isomorphic to the canonical module  $\Omega(k[P])$  of  $k[P]$ .

REMARK. Every subspace  $Q(C)$  of  $k[P]/(\theta)$  with  $C \in \mathfrak{E}^*(P^\wedge)$  is contained in  $\text{Soc}(k[P]/(\theta))$ . However, even though the Cohen-Macaulay type of  $k[P]$  is equal to  $\sum_{C \in \mathfrak{E}^*(P^\wedge)} |\mu(C)|$ , the subspace of  $\text{Soc}(k[P]/(\theta))$  spanned by all  $Q(C)$  with  $C \in \mathfrak{E}^*(P^\wedge)$  does not necessarily coincide with  $\text{Soc}(k[P]/(\theta))$ .

It might be of interest to give a similar result to Hochster's theorem [5, Theorem 7.3, p.81]. We refer the reader to, e.g., [5, p.28] for the definition of an orientable pseudo-manifold with boundary. It follows easily that the order complex  $\Delta(P)$

(see, e.g., [6, p.120]) of a Cohen-Macaulay poset  $P$  possesses the connectivity property [5, Definition 3.15 (c), p.28]. Moreover, if the order complex  $\Delta(P)$  of a Cohen-Macaulay poset  $P$  over a field  $k$  is an orientable pseudo-manifold with boundary, then the poset  $P$  satisfies  $(\star)$   $|\mu(x,y)| \leq 1$  for every  $0^{\wedge} \leq x \leq y \leq 1^{\wedge}$  in  $P^{\wedge}$ .

**(1.8) PROPOSITION.** Let  $P$  be a Cohen-Macaulay poset over  $k$ . Then there exists a homogeneous non-zero divisor  $\Theta \in k[P]$  of degree  $d$  with  $\mathfrak{J}^*(k[P]) \cdot \Theta = \mathfrak{J}(k[P]) (= \mathcal{G}(k[P]))$  if and only if  $\Delta(P)$  is an orientable pseudo-manifold with boundary.

**Proof.** Let  $\mathfrak{M}(P^{\wedge})$  be the set of maximal chains of  $P^{\wedge}$ . Given a function  $\varepsilon : \mathfrak{M}(P^{\wedge}) \rightarrow k - \{0\}$ , for each  $F : 0^{\wedge} = y_0 < y_1 < \dots < y_d < y_{d+1} = 1^{\wedge} \in \mathfrak{M}(P^{\wedge})$ , we set  $m(F;\varepsilon) = \varepsilon(F) \prod_{1 \leq i \leq d} y_i \in k[P]$ , and define  $\Theta(\varepsilon) \in k[P]$  by  $\Theta(\varepsilon) = \sum_{F \in \mathfrak{M}(P^{\wedge})} m(F;\varepsilon)$ , which is a homogeneous non-zero divisor on  $k[P]$  of degree  $d$ .

Now, let  $P$  be a Cohen-Macaulay poset over  $k$  and suppose the existence of a homogeneous non-zero divisor  $\Theta \in k[P]$  of degree  $d$  with  $\mathfrak{J}^*(k[P]) \cdot \Theta = \mathcal{G}(k[P])$ . Then  $\Theta$  must be of the form  $\Theta(\varepsilon)$  for some  $\varepsilon$ . Hence, the poset  $P$  satisfies the above condition  $(\star)$ , in particular,  $\Delta(P)$  is a pseudo-manifold with boundary. If  $F : 0^{\wedge} < \alpha_1 < \dots < \alpha_p < \gamma < \beta_1 < \dots < \beta_q < 1^{\wedge}$ ,  $F' : 0^{\wedge} < \alpha_1 < \dots < \alpha_p < \delta < \beta_1 < \dots < \beta_q < 1^{\wedge} \in \mathfrak{M}(P^{\wedge})$  with  $\gamma \neq \delta$ , then  $\varepsilon(F) = -\varepsilon(F')$  since  $C : 0^{\wedge} < \alpha_1 < \dots < \alpha_p < \beta_1 < \dots < \beta_q < 1^{\wedge} \in \mathfrak{E}(P^{\wedge})$  and  $\alpha_1^2 \dots \alpha_p^2 (\gamma - \delta) \beta_1^2 \dots \beta_q^2 \in \mathcal{R}(C)$ . Thus  $0 \neq \Theta \in H_{d-1}(\Delta(P), \partial\Delta(P); k)$ , hence  $\Delta(P)$  is orientable.

On the other hand, suppose that the order complex  $\Delta(P)$  of a Cohen-Macaulay poset  $P$  over  $k$  is an orientable pseudo-manifold with boundary. Since  $H_{d-1}(\Delta(P), \partial\Delta(P); k) \neq (0)$ , there exists  $\varepsilon : \mathfrak{M}(P^{\wedge}) \rightarrow k - \{0\}$  such that  $\varepsilon(F) = -\varepsilon(F')$  if  $\#(F \cap F') = d - 1$ . Then  $0 \neq x_1 x_2 \dots x_s \Theta(\varepsilon) \in \mathcal{R}(C)$  for every  $C : 0^{\wedge} = x_0 < x_1 < \dots < x_s < x_{s+1} = 1^{\wedge} \in \mathfrak{E}(P^{\wedge})$ . Thus, since  $\dim_k \mathcal{R}(C) = 1$ , the subspace  $\mathcal{R}(C)$  is spanned by  $x_1 x_2 \dots x_s \Theta$  over  $k$ . Hence  $\mathfrak{J}^*(k[P]) \cdot \Theta = \mathcal{G}(k[P])$  as required. Q.E.D.



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