On a Lower Bound for the Extinction Time of Surfaces Moved by Mean Curvature

Y. Giga and K. Yama-uchi

Series ♯185. March 1993

HOKKAIDO UNIVERSITY
PREPRINT SERIES IN MATHEMATICS

♯ 157: A. Arai, Momentum operators with gauge potentials, local quantization of magnetic flux, and representation of canonical commutation relations, 11 pages. 1992.

♯ 158: S. Izumiya, W.L. Marar, The Euler number of a topologically stable singular surface in a 3-manifold, 11 pages. 1992.

♯ 159: T. Hibi, Cohen-Macaulay types of Cohen-Macaulay complexes, 26 pages. 1992.

♯ 160: A. Arai, Properties of the Dirac-Weyl operator with a strongly singular gauge potential, 26 pages. 1992.

♯ 161: A. Arai, Dirac operators in Boson-Fermion Fock spaces and supersymmetric quantum field theory, 30 pages. 1992.

♯ 162: S. Albeverio, K. Iwata, T. Kolsrud, Random parallel transport on surfaces of finite type, and relations to homotopy, 8 pages. 1992.

♯ 163: S. Albeverio, K. Iwata, T. Kolsrud, Moments of random fields over a family of elliptic curves, and modular forms, 9 pages. 1992.

♯ 164: Y. Giga, M. Sato, Neumann problem for singular degenerate parabolic equations, 12 pages. 1992.

♯ 165: J. Wierzbicki, Y. Watatani, Commuting squares and relative entropy for two subfactors, 18 pages. 1992.

♯ 166: Y. Okabe, A new algorithm driven from the view-point of the fluctuation-dissipation theorem in the theory of KM_2O-Langevin equations, 13 pages. 1992.

♯ 167: Y. Okabe, H. Mano and Y. Itoh, Random collision model for interacting populations of two species and its strong law of large numbers, 14 pages. 1992.

♯ 168: A. Inoue, On the equations of stationary precesses with divergent diffusion coefficients, 25 pages. 1992.

♯ 169: T. Ozawa, Remarks on quadratic nonlinear Schrödinger equations, 19 pages. 1992.

♯ 170: T. Fukui, Y. Giga, Motion of a graph by nonsmooth weighted curvature, 11 pages. 1992.

♯ 171: J. Inoue, T. Nakazi, Finite dimensional solution sets of extremal problems in H^1, 10 pages. 1992.

♯ 172: S. Izumiya, A characterization of complete integrability for partial differential equations of first order, 6 pages. 1992.

♯ 173: T. Suwa, Unfoldings of codimension one complex analytic foliation singularities, 49 pages. 1992.

♯ 174: T. Ozawa, Wave propagation in even dimensional spaces, 15 pages. 1992.

♯ 175: S. Izumiya, Systems of Clairaut type, 7 pages. 1992.

♯ 176: A. Hoshiga, The initial value problems for quasi-linear wave equations in two space dimensions with small data, 25 pages. 1992.

♯ 177: K. Sugano, On bicommutators of modules over H-separable extension rings III, 9 pages. 1993.

♯ 178: T. Nakazi, Toeplitz operators and weighted norm inequalities, 17 pages. 1993.

♯ 179: O. Ogurisu, Existence and structure of infinitely degenerate zero-energy ground states of a Wess-Zumino type model in supersymmetric quantum mechanics, 26 pages. 1993.

♯ 180: O. Ogurisu, Ground state of a spin 1/2 charged particle in an even dimensional magnetic field, 9 pages. 1993.

♯ 181: K. Sugano, Note on H-separable Galois extension, 6 pages. 1993.

♯ 182: M. Yamada, Distance formulas of asymptotic Toeplitz and Hankel operators, 13 pages. 1993

♯ 183: G. Ishikawa, T. Ozawa, The genus of a connected compact real algebraic surface in the affine three space, 11 pages. 1993.

♯ 184: T. Hibi, Canonical modules and Cohen-Macaulay types of partially ordered sets, 6 pages. 1993.

On a Lower Bound for the Extinction Time of Surfaces Moved by Mean Curvature

Yoshikazu Giga* and Kazuyuki Yama-uchi

Department of Mathematics, Hokkaido University, Sapporo 060, Japan

1. Introduction

We consider a one-parameter family of bounded open sets $\{D_t\}_{t\geq 0}$ in $\boldsymbol{R}^n$ $(n \geq 2)$ whose boundary $\Gamma_t = \partial D_t$ is moving by its mean curvature. The initial value problem describing this motion is

$$\begin{cases} V = H \quad \text{on} \quad \Gamma_t \\ \Gamma_t|_{t=0} = \Gamma_0 = \partial D_0. \end{cases} \tag{E}$$

Here $V = V(t,x)$ and $H = H(t,x)$ are, respectively, the inward normal velocity and the inward ($n-1$ times) mean curvature of Γ_t at a point x of Γ_t.

By the level set method it is known that the problem (E) admits a unique global generalized solution $\{\Gamma_t\}$ (or $\{D_t\}$) provided that D_0 is a bounded open set [CGG1], [ES1]. (The paper [CGG3] includes corrections of technical errors in [CGG1]). It is also known that D_t becomes empty in a finite time. For example if D_0 is an open ball $B^n(R)$ of radius R (centered at the origin),

$$D_t = B^n(R(t)) \quad \text{with} \quad R(t)^2 = R^2 - 2(n-1)t$$

for $t < t_R = R^2/2(n-1)$ and D_t is empty for $t \geq t_R$. For a general initial data D_0 we consider a large ball $B^n(R)$ containing D_0. Applying the comparison principle in [CGG1], [ES1], we now observe that D_t becomes empty by the time $t = t_R$. Thus the *extinction time* defined by

$$t_* = t_*(D_0) = \inf\{t; D_t = \emptyset\} \tag{1.1}$$

is finite since $t_* \leq t_R$ provided that D_0 is bounded; here D_t is a unique generalized solution of (E).

*Partly supported by the Inamori Foundation

As for upper bounds of t_*, besides the trivial estimate $t_* \leq t_R$ L.C. Evans and J. Spruck [ES3] proved an interesting estimate

$$t_* \leq C\mathscr{H}^{n-1}(\Gamma_0)^{2/(n-1)} \tag{1.2}$$

with some constant C depending only on n. Here $\mathscr{H}^{n-1}$ denotes the $n-1$ dimensional Hausdorff measure. Actually, their estimate is still valid even if t_* is replaced by the extinction time of Γ_t which may be greater than t_* because generalized interface evolution Γ_t may fatten for nonsmooth Γ_0 [ES1].

Our goal in this note is to derive a lower bound of t_* by geometric quantities of initial data. A trivial estimate for t_* is obtained by comparison with balls contained in D_0. However, as far as the authors know, our estimate provides the first nontrivial lower bound for t_*. Let $\mathscr{L}^n$ denote the Lebesgue measure in $\boldsymbol{R}^n$. Then our main result is

Theorem 1. *Let D_0 be a smoothly bounded open set in $\boldsymbol{R}^n$ and $t_*(D_0)$ be the extinction time defined in (1.1). Then $t_* = t_*(D_0)$ is estimated as*

$$t_* \geq 2(\mathscr{L}^n(D_0)/\mathscr{H}^{n-1}(\Gamma_0))^2. \tag{1.3}$$

The constant 2 is optimal for this estimate.

Remark. We may weaken the assumption on the regularity of ∂D_0. As observed later, it suffices to assume that ∂D_0 is $n-1$ rectifiable in the sense of Federer [F].

If $D_0 = B^n(R)$, then the right hand side of (1.3) equals $2R^2/n^2$. Since $t_R = R^2/2(n-1)$, for $n=2$ the equality in (1.3) holds for $D_0 = B^2(R)$. However $n \geq 3$ the estimate (1.3) is not sharp for $B^n(R)$. As discussed later, (1.3) is sharper than the trivial estimate if D_0 is a "thin bar" containing no large balls.

The estimate (1.3) is easily proved when $\{\Gamma_t\}_{0 \leq t < t_*}$ is a smooth family of smooth surfaces. Indeed, set

$$v(t) = \mathscr{L}^n(D_t) \qquad \text{(the volume of } D_t\text{)}$$
$$a(t) = \mathscr{H}^{n-1}(\Gamma_t) \quad \text{(the area of } \Gamma_t\text{)}.$$

We calculate the time derivative of a and v. Since $V = H$, we observe (see e.g. [H]) that

$$-a'(t) = \int VH d\mathcal{H}^{n-1} = \int H^2 d\mathcal{H}^{n-1} \geq 0 \tag{1.4}$$

$$-v'(t) = \int V d\mathcal{H}^{n-1} = \int H d\mathcal{H}^{n-1}, \tag{1.5}$$

where the integrals are over Γ_t. Applying the Schwarz inequality now yields

$$-v'(t) \leq (\int 1 d\mathcal{H}^{n-1} \int H^2 d\mathcal{H}^{n-1})^{1/2} = a^{1/2}(-a')^{1/2} = (-(a^2)'/2)^{1/2}.$$

Integrating this inequality over $(0, t_*)$ yields

$$\begin{aligned} v(0) - v(t_*) &\leq \int_0^{t_*} (-(a^2)'/2)^{1/2} dt \\ &\leq (\int_0^{t_*} 1 dt \int_0^{t_*} (-(a^2)'/2) dt)^{1/2} \\ &\qquad\qquad \text{(by the Schwarz inequality)} \\ &= (t_*/2)^{1/2}[a^2(0) - a^2(t_*)]^{1/2}. \end{aligned} \tag{1.6}$$

Since $a(t_*) = v(t_*) = 0$, this yields (1.3).

For $n = 2$ the equation (E) admits a unique smooth solution Γ_t until it shrinks to a point at $t = t_*$. This follows from results of [Gr1] and [GH]. Since smooth solution $\{\Gamma_t\}$ is also a generalized solution (see [ES1]), the formal proof of (1.3) given in the preceeding paragraph is sufficient to show (1.3) provided $n = 2$. Even if $n \geq 3$ the formal proof is sufficient if D_0 is convex, since D_t stays smooth and convex until it disappears at time $t = t_*$ [H].

Unfortunately, for $n \geq 3$ the surface $\{\Gamma_t\}$ may become singular at some time before t_* even if the initial surface Γ_0 is smooth. In fact if Γ_0 is a barbell with thin neck, it pintches in the middle before it disappears, as shown in [Gr2]. To show (1.3) for generalized solutions we should modify the formal proof for solutions of the level set equation, where each level set of solutions moves by its mean curvature. This is a main technical part of this note.

The bibiography of [AAG] includes many references to recent work on the generalized solution of the mean curvature flow equation (E). We take the opportunity to note some other, related articles not cited there and not mentioned elsewhere in this introduction. In

[I] Ilmanen proved that the singular limit of solution for the Allen -Cahn equation yields a Brakke's varifold solution to (E) and that its support is contained in our generalized interface evolution. In [ES4] Evans and Spruck proved that the almost every set of solution for the level set equation (L) is a unit density varifold solution in the sense of Brakke [B].

2. Extinction time for the level set equation

We recall generalized solutions of (E) constructed by [ES1] and [CGG1]. For a bounded open set D_0 we take $g \in K_\alpha$ such that $D_0 = \{g > 0\}$, the *super zero-level set* of g. Here α is a positive number and K_α is the space of all continuous functions on $\boldsymbol{R}^n$ whose values equal to $-\alpha$ outside some compact subset of $\boldsymbol{R}^n$. We then consider the *level set equation* for (E):

$$\begin{cases} u_t - |\nabla u| \operatorname{div}(\nabla u/|\nabla u|) = 0 & \text{in } (0,\infty) \times \boldsymbol{R}^n \\ u|_{t=0} = g & \text{on } \boldsymbol{R}^n. \end{cases} \tag{L}$$

The equation (L) admits a unique global solution in viscosity sense [CGG1], [ES1]. As stated in [AAG, Theorem 3.1] (see also [ES1] and [CGG1]) the super zero-level set of $u(t,\cdot)$, *i.e.*,

$$D_t = \{x; u(t,x) > 0\}$$

is independent of the choice of g and depends only on D_0. The family $\{D_t\}_{t\geq 0}$ is called the *generalized solution* (or *inner evolution*) with initial data D_0. We shall state a lower bound of $t_* = t_*(D_0)$ in terms of g which is crucial in proving (1.3). We set

$$\int_U |\nabla g| \equiv \sup\{\int_U g \operatorname{div} \varphi dx; \varphi \in C_0^\infty(U, \boldsymbol{R}^n), |\varphi(x)| \leq 1\}$$

which is the total variation of ∇g in an open set U in $\boldsymbol{R}^n$. If g is smooth, $\int |\nabla g| = \int |\nabla g| dx$ where the integrals are over U.

Theorem 2. *Assume that $g \in K_\alpha$ for some $\alpha > 0$ such that D_0 is a super zero-level set of g. Let $t_* = t_*(D_0)$ be the extinction time defined by (1.1). Then*

$$2(\int_{D_0} g(x)dx)^2 \leq t_*(\int_{D_0} |\nabla g|)^2.$$

Proof of (1.3) admitting Theorem 2. Let d denote a signed distance function of ∂D_0 defined by

$$d(x) = \begin{cases} \operatorname{dist}(x, \partial D_0), & \text{for } x \in D_0; \\ -\operatorname{dist}(x, \partial D_0), & \text{otherwise}. \end{cases}$$

We approximate the signiture function by a Lipschitz function θ_m $(m = 1, 2, \cdots)$ defined by

$$\theta_m(\xi) = \begin{cases} 1, & \text{if } \xi \geq 1/m; \\ m\xi, & \text{if } |\xi| \leq 1/m; \\ -1 & \text{if } \xi \leq -1/m. \end{cases}$$

By definition the function $g_m = \theta_m(d)$ belongs to K_1 and the super zero-level set of g_m agrees with D_0. Applying Theorem 2 yields

$$2(\int_{D_0} g_m dx)^2 \leq t_*(\int_{D_0} |\nabla g_m|)^2.$$

Letting $m \to \infty$ yields (1.3), if we prove

$$\lim_{m\to\infty} \int_{D_0} g_m dx = \mathscr{L}^n(D_0), \quad \lim_{m\to\infty} \int_{D_0} |\nabla g_m| = \mathscr{H}^{n-1}(\partial D_0).$$

The first convergence is clear from the monotone convergence theorem.

It remains to prove the last convergence. Since d and θ_m are Lipschitz, so is g_m. Moreover, $\nabla g_m = \theta'_m(d)\nabla d$ in the sense of distribution and $|\nabla d| = 1$ almost everywhere. We thus observe that

$$\int_{D_0} |\nabla g_m| = \int_{D_0} |\nabla g_m| dx = \int_{T_m} \theta'_m(d)|\nabla d| dx = \int_{T_m} m \cdot 1 dx = m\mathscr{L}^n(T_m).$$

with $T_m = \{x \in D_0; 0 \leq d(x) \leq 1/m\}$. It now suffices to prove

$$\lim_{m\to\infty} m\mathscr{L}^n(T_m) = \mathscr{H}^{n-1}(\partial D_0),$$

which is easy to prove when ∂D_0 is smooth enough. According to [F], this convergence is also valid even if ∂D_0 is merely rectifiable. Since this is the only part we use regularity of ∂D_0 we have proved (1.3) with the remark to Theorem 1.

Proof of the second assertion of Theorem 1. If $n = 2$, the equality in (1.3) holds for $D_0 = B^2(R)$. We shall assume $n \geq 3$ and prove that the constant 2 in (1.3) is optimal even if we restrict D_0 as a smooth, strictly convex, bounded open set. For each $\ell > 1$ let V_ℓ denote

$$V_\ell = \{x = (y, z) \in \boldsymbol{R}^{n-2} \times \boldsymbol{R}^2; (|y|/\ell)^{2\ell} + |z|^{2\ell} < 1\}.$$

Let us calculate the volume of V_ℓ by a change of variables:

$$\begin{aligned}\mathscr{L}^n(V_\ell) &= \int_{|y|\le \ell} \left(\int_0^{2\pi}\int_0^{s(|y|)} s\,ds\,d\theta\right)dy = \pi \int_{|y|\le \ell} s^2(|y|)dy, \ y = \ell\eta \\ &= \pi \ell^{n-2} \int_{|\eta|\le 1} \sigma^2(|\eta|)d\eta \quad \text{with} \quad \sigma(\rho) = (1-\rho^{2\ell})^{1/2\ell},\ s(r) = \sigma(r/\ell).\end{aligned}$$

Since $\sigma(\rho) \ge 1 - \rho^{2\ell}$, we observe that

$$\mathscr{L}^n(V_\ell) \ge \pi \ell^{n-2} \int_{|\eta|\le 1} (1 - |\eta|^{2\ell})d\eta = \pi\omega\ell^{n-2}(1 - O(1/\ell)) \quad \text{as} \quad \ell \to \infty$$

where $\omega = \mathscr{L}^{n-2}(B^{n-2}(1))$. Since each cross section of V_ℓ at $y =$ const. is a disk of radius $s(|y|)$, we see

$$\begin{aligned}\mathscr{H}^{n-1}(\partial V_\ell) &= 2\pi \int_{|y|\le \ell} s(r)(1 + |\nabla_y s(r)|^2)^{1/2} dy, \qquad r = |y| \\ &= 2\pi \ell^{n-2} \int_{|\eta|\le 1} \sigma(\rho)(1 + |\nabla_\eta \sigma(\rho)|^2/\ell^2)^{1/2} d\eta, \quad \rho = |\eta|\end{aligned}$$

by a change of variable: $y = \ell\eta$. This yields the estimate

$$\begin{aligned}\mathscr{H}^{n-1}(\partial V_\ell) &\le 2\pi\ell^{n-2} \int_{|\eta|\le 1} (\sigma(\rho) + \ell^{-1}|\nabla_\eta \sigma(\rho)|\sigma(\rho))d\eta \\ &\le 2\pi\ell^{n-2}(\omega - \ell^{-1} \int_{|\eta|\le 1} \sigma'(\rho)\sigma(\rho)d\eta)\end{aligned}$$

since $\sigma(\rho) \le 1$ for $|\rho| \le 1$ and $|\nabla_\eta \sigma(\rho)| = -\sigma'(\rho)$. Note that

$$-2\int_0^1 \sigma'(\rho)\sigma(\rho)\rho^{n-3}d\rho = -\int_0^1 (\sigma^2(\rho))'\rho^{n-3}d\rho = -[\sigma^2(\rho)\rho^{n-3}]_0^1 + \int_0^1 \sigma^2(\rho)(\rho^{n-3})'d\rho$$

is bounded for $\ell \ge 1$. We thus observe that

$$\int_{|\eta|\le 1} \sigma'(\rho)\sigma(\rho)d\eta = O(1) \quad \text{as} \quad \ell \to \infty$$

which now yields

$$\mathscr{H}^{n-1}(\partial V_\ell) \le 2\pi \ell^{n-2}(\omega + O(1/\ell)) \quad \text{as} \quad \ell \to \infty.$$

Combining the estimate of $\mathscr{L}^n(V_\ell)$ yields

$$4(\mathscr{L}^n(V_\ell)/\mathscr{H}^{n-1}(V_\ell))^2 \ge 1 + O(1/\ell) \quad \text{as} \quad \ell \to \infty.$$

Let t_ℓ be the extinction time for V_ℓ, *i.e.* $t_\ell = t_*(V_\ell)$. Since V_ℓ is smooth and convex and since V_ℓ is contained in a cylinder $\boldsymbol{R}^{n-2} \times B^2(1)$, the classical comparison yields $t_\ell \le 1/2$; note that $B^2(1)$ extincts at the time $1/2$. Comparing the estimate of $(\mathscr{L}^n(V_\ell)/\mathscr{H}^{n-1}(V_\ell))^2$ we now observe that the constant 2 in (1.3) optimal by taking $\ell \to \infty$.

3. Approximate level set equations

The basic idea in proving Theorem 2 is found in the formal proof of (1.3) in Section 1. Although it is abuse of notation, we set

$$V = -u_t/|\nabla u| \quad \text{and} \quad H = -\operatorname{div}(\nabla u/|\nabla u|),$$

since they represent, respectively, the inward normal velocity and the inward mean curvatue on each level set of u. Suppose that u is a solution of (L). Then formally we have

$$-\frac{d}{dt}\int |\nabla u| dx = -\int \frac{\nabla u_t \cdot \nabla u}{|\nabla u|} dx = -\int u_t H dx = -\int VH|\nabla u| dx = \int H^2 |\nabla u| dx \tag{3.1}$$

$$-\frac{d}{dt}\int u dx = -\int u_t dx = \int V|\nabla u| dx = \int H|\nabla u| dx \tag{3.2}$$

by the integration by parts, where the integrals are $\boldsymbol{R}^n$. Identities (3.1) and (3.2) are regarded as a level set formulation of (1.4) and (1.5), respectively and are still important to prove Theorem 2 at least conceptually. Unfortunately, since (L) is degenerate parabolic, we cannot expect that u is differentiable so it is nontrivial to interpret (3.1) and (3.2)

To overcome this difficulty we consider the approximate level set equation for $\varepsilon > 0$:

$$\begin{cases} u_t - |\nabla u|_\varepsilon \operatorname{div} N^\varepsilon(\nabla u) = 0 & \text{in } (0,\infty) \times \boldsymbol{R}^n \\ u|_{t=0} = g & \text{on } \boldsymbol{R}^n, \end{cases} \tag{A$_\varepsilon$}$$

with $|p|_\varepsilon = (\varepsilon^2 + |p|^2)^{1/2}$, $N^\varepsilon(p) = p/|p|_\varepsilon$ for $p \in \boldsymbol{R}^n$. This equation is studied in [ES1] to construct a solution of (L) as a locally uniform limit of a solution of (A_ε) as $\varepsilon \to 0$. In fact, the equation (A_ε) admits a unique global solution u^ε (smooth for $t > 0$ and continuous for $t \geq 0$) for arbitrary $g \in K_\alpha$. Applying the maximum principle yields

$$\sup_{0<\varepsilon<1} \{||u^\varepsilon||_\infty, ||\nabla u^\varepsilon||_\infty, ||u_t^\varepsilon||_\infty\} \leq C||g||_{C^2} \tag{3.3}$$

for $g \in C^2$ with some constant $C = C(n)$ independent of g; here $||f||_\infty$ and $||f||_{C^2}$ denote, respectively, the maximum norm and C^2 norm of f. By the Ascoli-Arzela theorem u^ε converges to some function u locally uniformly by taking a subsequences. However since u solves (L), the limit u is unique so u^ε itself converges to u. Even if g is merely in K_α using the comparison principle for (L) ([ES1, CGG1]), we observe that

$$u^\varepsilon \to u \text{ locally uniform on } [0,\infty) \times \boldsymbol{R}^n \text{ as } \varepsilon \to 0 \tag{3.4}$$

for u which solves (L). We refer to [ES1] for details.

We still have a problem to interpret (3.1) and (3.2) even if u^ε and $|\nabla u^\varepsilon|_\varepsilon$ replace u and $|\nabla u|$ respectively, since $|\nabla u^\varepsilon|_\varepsilon = (\varepsilon^2 + |\nabla u^\varepsilon|^2)^{1/2}$ is not integrable on $\boldsymbol{R}^n$. We consider integrals over a bounded set. Then we should estimate the boundary terms which comes from integration by parts.

Let θ be a nondecreasing C^2 function on $\boldsymbol{R}$ and Ω be a smoothly bounded open set in $\boldsymbol{R}^n$. We consider "approximate" area and volume functions:

$$a^\varepsilon(t) = \int_\Omega \theta'(u^\varepsilon)|\nabla u^\varepsilon|_\varepsilon dx, \quad |p|_\varepsilon = (\varepsilon^2 + |p|^2)^{1/2} \tag{3.5}$$

$$v^\varepsilon(t) = \int_\Omega \theta(u^\varepsilon) dx. \tag{3.6}$$

To express error terms we set $U^\varepsilon = \theta(u^\varepsilon)$ and, for $0 \leq t_0 < \infty$,

$$B^\varepsilon(t_0) = \int_0^{t_0} a^\varepsilon(t) b^\varepsilon(t) dt, \quad b^\varepsilon(t) = \int_{\partial\Omega} U_t^\varepsilon N^\varepsilon(\nabla u^\varepsilon) \cdot \nu d\mathscr{H}^{n-1} \tag{3.7}$$

$$E^\varepsilon(t_0) = \int_0^{t_0} a^\varepsilon(t) e^\varepsilon(t) dt, \quad e^\varepsilon(t) = \int_\Omega |\theta''(u^\varepsilon) u_t^\varepsilon| dx, \tag{3.8}$$

where ν is the unit exterior normal vector of $\partial\Omega$ and $N^\varepsilon(p) = p/|p|_\varepsilon$. The next result is interpreted as a variant of (1.6).

Lemma 1. *Assume that g is a C^2 function and $g \in K_\alpha$ for some $\alpha > 0$. Let u^ε be the solution of (A_ε) and Ω be a smoothly open set in $\boldsymbol{R}^n$. Suppose that θ is a nondecreasing C^2 function on $\boldsymbol{R}$. Let a^ε, v^ε, B^ε, E^ε be defined as (3.5)-(3.8). Then for t_0, $0 \le t_0 < \infty$*

$$v^\varepsilon(0) - v^\varepsilon(t_0) \le t_0^{1/2}[a^\varepsilon(0)^2/2 - a^\varepsilon(t_0)^2/2 + B^\varepsilon(t_0) + \varepsilon E^\varepsilon(t_0)]^{1/2}.$$

Proof. Using the equation (A_ε) we observe that

$$v^\varepsilon(0) - v^\varepsilon(t_0) = \int_{t_0}^{0}\int_\Omega \theta' u_t^\varepsilon dxdt = \int_0^{t_0}\int_\Omega \theta'|\nabla u^\varepsilon|_\varepsilon H^\varepsilon dxdt$$

with $H^\varepsilon = -\operatorname{div}(N^\varepsilon(\nabla u^\varepsilon))$ with $\theta' = \theta'(u^\varepsilon)$. Applying the Schwarz inequality yields

$$v^\varepsilon(0) - v^\varepsilon(t_0) \le \int_0^{t_0} (a^\varepsilon)^{1/2}\Big(\int_\Omega \theta'|\nabla u^\varepsilon|_\varepsilon (H^\varepsilon)^2 dx\Big)^{1/2} dt. \tag{3.9}$$

We next calculate the time derivative of $a^\varepsilon(t)$:

$$da^\varepsilon/dt = \int_\Omega \theta'' u_t^\varepsilon |\nabla u^\varepsilon|_\varepsilon dx + \int_\Omega \theta' \nabla u_t^\varepsilon \cdot N^\varepsilon(\nabla u^\varepsilon) dx = I_1 + I_2,$$

where $\theta'' = \theta''(u^\varepsilon)$. Since $U_t^\varepsilon = \theta' u_t^\varepsilon$, integrating the second term by parts yields

$$I_2 = -\int_\Omega \theta'' u_t^\varepsilon N^\varepsilon(\nabla u^\varepsilon)\cdot \nabla u^\varepsilon dx + \int_\Omega \theta' u_t^\varepsilon H^\varepsilon dx + b^\varepsilon(t).$$

Note that

$$|p|_\varepsilon - N^\varepsilon(p)\cdot p = \varepsilon^2/|p|_\varepsilon = \varepsilon^2/(\varepsilon^2 + |p|^2)^{1/2} \le \varepsilon$$

to get

$$\begin{aligned} da^\varepsilon/dt &= \int_\Omega \theta'' u_t^\varepsilon \varepsilon^2/|\nabla u^\varepsilon|_\varepsilon dx + \int \theta' u_t^\varepsilon H^\varepsilon dx + b^\varepsilon(t) \\ &\le \varepsilon e^\varepsilon(t) - \int_\Omega \theta'(H^\varepsilon)^2|\nabla u^\varepsilon|_\varepsilon dx + b^\varepsilon(t) \end{aligned}$$

since $u_t^\varepsilon = -H^\varepsilon|\nabla u^\varepsilon|_\varepsilon$ by (A_ε). Pluging this inequality in the right hand side of (3.9) yields

$$\begin{aligned} v^\varepsilon(0) - v^\varepsilon(t_0) &\le \int_0^{t_0} (a^\varepsilon)^{1/2}\Big(-\frac{da^\varepsilon}{dt} + \varepsilon e^\varepsilon(t) + b^\varepsilon(t)\Big)^{1/2} dt \\ &\le \Big(\int_0^{t_0} dt\Big)^{1/2}\Big(\int_0^{t_0}\Big(-a^\varepsilon\frac{da^\varepsilon}{dt} + \varepsilon a^\varepsilon e^\varepsilon + a^\varepsilon b^\varepsilon\Big)dt\Big)^{1/2} \end{aligned}$$

by the Schwarz inequality. Since $d(a^\varepsilon)^2/dt = 2a^\varepsilon da^\varepsilon/dt$ this inequality completes the proof of Lemma 1 by recalling definitions of B^ε and E^ε.

We shall choose a special θ. For fixed $\delta > 0$ let θ_δ be a nondecreasing C^2 function on $\boldsymbol{R}$ such that

$$\theta_\delta(\xi) = \begin{cases} \xi - \delta & \text{for} \quad \xi \geq 3\delta/2 \\ 0 & \text{for} \quad \xi \leq \delta/2 \end{cases}$$

and that $0 \leq \theta'_\delta \leq 1$. For this choice of $\theta = \theta_\delta$ the quantities defined by (3.5)-(3.8) are denoted with subscript δ, for example, a^ε_δ, v^ε_δ, B^ε_δ, E^ε_δ e.t.c.

Lemma 2. *Assume that g is a C^2 function on $\boldsymbol{R}^n$ and $g \in K_\alpha$ for some $\alpha > 0$. Let D_0 be a super zero-level set of g i.e., $\{g > 0\}$. Assume that Ω is a ball $B^n(R)$ containing D_0. Let $t_* = t_*(D_0)$ be the extinction time defined by (1.1). Let u^ε be the solution of (A_ε) so that $u = \lim_{\varepsilon\downarrow 0} u_\varepsilon$ is the solution of (L). For fixed $\delta > 0$ we take θ_δ as above and define a^ε_δ, v^ε_δ, B^ε_δ, E^ε_δ by (3.5)-(3.8). Then*

(i) $\overline{\lim_{\varepsilon\downarrow 0}}\, a^\varepsilon_\delta(t) \leq \underline{\lim_{\varepsilon\downarrow 0}} \int_{D_t} |\nabla u^\varepsilon(t,x)| dx$ *for t, $0 \leq t \leq t_*$ where $D_t = \{x; u(t,x) > 0\}$.*

(ii) There is a positive constant $\varepsilon_0 = \varepsilon_0(\delta, u, t_*, R, n)$ such that

$$B^\varepsilon_\delta(t_*) = v^\varepsilon_\delta(t_*) = 0 \quad \text{for} \quad 0 < \varepsilon < \varepsilon_0.$$

(iii) There is a positive constant $C_0 = C_0(\delta, g, R, n)$ such that

$$\sup_{0<\varepsilon<1} E^\varepsilon_\delta(t_*) \leq C_0 t_*.$$

Proof. (i) By the convergence (3.4) we observe that there is a small positive constant $\varepsilon_0 = \varepsilon_0(\delta, u, t_*, R, n)$ such that

$$u^\varepsilon \leq u + \delta/2 \quad \text{on} \quad M = [0, t_*] \times \bar{\Omega} \quad \text{for} \quad 0 < \varepsilon < \varepsilon_0.$$

Let M_- be the set of $(t,x) \in M$ with $u(t,x) \leq 0$. Clearly, $u^\varepsilon \leq \delta/2$ on M_- for $\varepsilon < \varepsilon_0$. Since $\theta_\delta(\xi) = 0$ for $\xi \leq \delta/2$, this implies

$$\theta'_\delta(u^\varepsilon)|\nabla u^\varepsilon|_\varepsilon = 0 \quad \text{on} \quad M_- \quad \text{for} \quad 0 < \varepsilon < \varepsilon_0.$$

Since D_t is always contained in Ω by the comparison principle ([ES1], [CGG1]), we now obtain, by $0 \le \theta'_\delta \le 1$, that

$$a^\varepsilon_\delta(t) = \int_{D_t} \theta'_\delta(u^\varepsilon)|\nabla u^\varepsilon|_\varepsilon dx \le \int_{D_t} |\nabla u^\varepsilon|_\varepsilon dx \le \int_{D_t} (\varepsilon + |\nabla u^\varepsilon|)dx$$

for $0 \le t \le t_*$ and $0 < \varepsilon < \varepsilon_0$. Sending ε to zero completes the proof of (i).

(ii) Take ε_0 as in (i). By the comparison principle we see that the lateral boundary $\Sigma = [0, t_*] \times \partial\Omega$ of M is contained in M_-. By the definition of the extinction time, $S = \{t_*\} \times \Omega$ is also contained in M_-. If $0 < \varepsilon < \varepsilon_0$, we see $u^\varepsilon \le \delta/2$ on Σ and S. This implies

$$U^\varepsilon_\delta = 0 \quad \text{on} \quad S \quad \text{and} \quad (U^\varepsilon_\delta)_t = 0 \quad \text{on} \quad \Sigma$$

since $\theta_\delta(\xi) = 0$ for $\xi \le \delta/2$. We thus obtain $v^\varepsilon_\delta(t_*) = 0$ and $b^\varepsilon_\delta(t) = 0$ for $0 \le t \le t_*$ so that $B^\varepsilon_\delta(t_*) = 0$.

(iii) Applying the estimate (3.3) we have

$$a^\varepsilon_\delta(t) \le C(1 + ||g||_{C^2})\mathscr{L}^n(\Omega) \quad \text{for} \quad 0 < \varepsilon < 1.$$

Since $|\theta''_\delta|$ is bounded with a bound $c = C(\delta)$ applying (3.3) to e^ε_δ yields

$$e^\varepsilon_\delta(t) \le \int_\Omega c||u_t||_\infty dx \le cC||g||_{C^2}\mathscr{L}^n(\Omega).$$

We thus observe that

$$E^\varepsilon_\delta(t) \le t \max\{a^\varepsilon_\delta e^\varepsilon_\delta(t); 0 \le t \le t_*\} \le C_0 t \quad \text{for} \quad 0 < \varepsilon < 1$$

with $C_0 = C^2||g||_{C^2}(1 + ||g||_{C^2})c(\mathscr{L}^n(\Omega))^2$. This completes the proof of Lemma 2.

4. Proof of Theorem 2

We first prove Theorem 2 by assuming that g is C^2. Let a^ε_δ, v^ε_δ e.t.c. be as in Lemma 2. Applying Lemma 1 with $t_0 = t_*$ yields

$$v^\varepsilon_\delta(0) - v^\varepsilon_\delta(t_*) \le t_*^{1/2}[a^\varepsilon(0)^2/2 + B^\varepsilon_\delta(t_*) + \varepsilon E^\varepsilon_\delta(t_*)]^{1/2}.$$

By Lemma 2 (ii), (iii) and (i) we send $\varepsilon \to 0$ to get

$$\overline{\lim_{\varepsilon\downarrow 0}} v^\varepsilon_\delta(0) \le (t_*/2)^{1/2} \overline{\lim_{\varepsilon\downarrow 0}} a^\varepsilon_\delta(0) \le (t_*/2)^{1/2} \int_{D_0} |\nabla g| dx.$$

Since $\theta_\delta(\xi)=\xi-\delta$ for $\xi\geq 3\delta/2$ and $0\leq\theta'_\delta\leq 1$, we see

$$\int_{D_0}(g-\delta)dx\leq\int_{D_0}\theta_\delta(g)dx=\int_\Omega\theta_\delta(g)dx=v_\delta^\varepsilon(0).$$

Combining these two inequalities yield

$$\int_{D_0}gdx\leq(t_*/2)^{1/2}\int_{D_0}|\nabla g|dx+\delta\mathscr{L}^n(D_0)$$

which yields the desired inequality in Theorem 2 since $\delta>0$ is arbitrary.

To complete the proof of Theorem 2 for arbitrary $g\in K_\alpha$ we need an approximation lemma.

Lemma 3. *Suppose that $g\in K_\alpha$ for some $\alpha>0$. Then there is a sequence $\{g_\varepsilon\}_{\varepsilon>0}$ of C^∞ functions such that $g_\varepsilon\in K_\alpha$ and that*

(a) $g-3\varepsilon/2\leq g_\varepsilon\leq g$

(b) $\overline{\lim_{\varepsilon\downarrow 0}}\int_U|\nabla g_\varepsilon|dx\leq\int_U|\nabla g|$ *for each open set U in $\boldsymbol{R}^n$.*

Completion of the proof of Theorem 2. Let $D_{0\varepsilon}$ be the super zero-level set of g_ε where g_ε is given in Lemma 3.. Applying Theorem 2 for smooth g_ε yields

$$\int_{D_{0\varepsilon}}g_\varepsilon dx\leq(\frac{t_{*\varepsilon}}{2})^{1/2}\int_{D_{0\varepsilon}}|\nabla g_\varepsilon|dx$$

where $t_{*\varepsilon}=t_*(D_{0\varepsilon})$ is the extinction time for $D_{0\varepsilon}$. By Lemma 3 (a) we see $D_{0\varepsilon}$ is contained in D_0. This implies $t_{*\varepsilon}\leq t_*$ by the comparison principle. We thus obtain

$$\int_{D_{0\varepsilon}}g_\varepsilon dx\leq(\frac{t_*}{2})^{1/2}\int_{D_0}|\nabla g_\varepsilon|dx.$$

Since $g_\varepsilon\to g$ uniformly by Lemma 3(a), sending ε to zero yields

$$\int_{D_0}gdx\leq(\frac{t_*}{2})^{1/2}\int_{D_0}|\nabla g|,$$

where Lemma 3(b) is now invoked. This completes the proof of Theorem 2.

Proof of Lemma 3. This is essentially known (cf. [Gi]); however we give a proof for completeness. We use a symmetric mollifier. Let ρ be a nonnegative smooth function whose support is contained in $B^n(1)$ such that $\rho(-x) = \rho(x)$ and $\int \rho dx = 1$. A symmetric mollifier ρ_δ is obtained by setting $\rho_\delta(x) = \delta^{-n}\rho(x/\delta)$. For $\varepsilon > 0$ we set

$$2\delta_\varepsilon = \inf\{\delta; \|g * \rho_\delta - g\|_\infty \geq \varepsilon/2\} \quad \text{and} \quad g_\varepsilon = g * \rho_\delta - \varepsilon \quad \text{with} \quad \delta = \delta_\varepsilon$$

Since $g \in K_\alpha$, we see that $g * \rho_\delta$ converges uniformly to g in $\boldsymbol{R}^n$ as $\delta \to 0$ so that $\delta_\varepsilon > 0$ for $\varepsilon > 0$. This guarantees that g_ε is well-defined. The property (a) is obvious.

It remains to prove (b). Integration by parts yields

$$\int g_\varepsilon \operatorname{div} \varphi dx = -\int (g * \nabla \rho_\delta) \cdot \varphi dx \quad \text{for} \quad \varphi \in C_0^\infty(U, \boldsymbol{R}^n)$$

with $\delta = \delta_\varepsilon$, where integrals are over $\boldsymbol{R}^n$. Since $\rho(-x) = \rho(x)$ implies that $(\nabla\rho_\delta)(x) = -(\nabla\rho_\delta)(-x)$, we observe that

$$\begin{aligned} -\int (g * \nabla\rho_\delta) \cdot \varphi dx &= \iint g(y) \nabla\rho_\delta(y - x) \cdot \varphi(x) dy dx \\ &= \int g(y) \operatorname{div}((\varphi * \rho_\delta)(y)) dy \quad \text{with} \quad \delta = \delta_\varepsilon. \end{aligned}$$

Since $\varphi * \rho_\delta$ $(\delta = \delta_\varepsilon)$ is compactly supported in U for sufficiently small $\varepsilon > 0$ and since $\|\varphi * \rho_\delta\|_\infty \leq \|\varphi\|_\infty$, the definition of the total variation of ∇g and ∇g_ε in U yields (b).

5. Miscellaneous remarks

Our estimate (1.3) is consistent with (1.2). In fact, combining (1.2) and (1.3) yields the isoperimetric inequality

$$\mathscr{L}^n(D_0) \leq (C/2)^{1/2} \mathscr{H}^{n-1}(\partial D_0)^{n/(n-1)}.$$

When $n = 2$, the equality in (1.3) is attained if and only if D_0 is a disk. For $n \geq 3$ the equality in (1.3) is never attained provided that D_0 is convex. indeed, the formal proof of (1.3) in Section 1 applies to both cases. To keep the equality in the proof, the mean curvature H must be a constant in space valiables. By Alexandorf's theorem [A] such an embedded surface must be a sphere. Thus D_0 must be a ball provided that the equality in (1.3) holds. For $n = 3$, the disks attain the equality in (1.3). However, for $n \leq 3$, a ball D_0 never attains the equality. This complete the proof. We believe that for $n \leq 3$ the equality in (1.3) attains for all smooth D_0, but we do not have the proof right now.

In [ES1] and [ES2] there are examples that the zero-level set Γ_t of solution u of (L) extinct instantaneously even if Γ_0 is non-empty. However, in all their examples there is no nontrivial bounded open set D_0 such that Γ_0 contains ∂D_0. So their observation does not conflict to our result (1.3).

Suppose that the inward mean curvature is positive everywhere on $\Gamma_0 = \partial D_0$. Then the generalized solution D_t of (E) is given as

$$D_t = \{x \in D_0; U(x) > t\}$$

with U solving

$$\begin{cases} -\Delta U + \sum_{1\le i,j\le n} \frac{U_i U_j}{|\nabla U|^2} U_{ij} = 1 & \text{in} \quad D_0 \\ U = 0 & \text{on} \quad \partial D_0 \end{cases}$$

in the viscosity sense [ES1]; here $U_i = \partial U/\partial x_j$, $U_{ij} = \partial^2 U/\partial x_i \partial x_j$. This equation is formally obtained from (L) by setting $u = U(x) - t$. The extinction time t_* is the maximum U_M of U over D_0. Our estimate (1.3) gives

$$U_M \ge 2(\mathscr{L}^n(D_0)/\mathscr{H}^{n-1}(\partial D_0))^2.$$

There are several results on a lower bound of the maximum u_M of solution u of semilinear elliptic boundary value problems (cf. [PSS] and [S]). For example if u solves

$$\begin{cases} -\Delta u = 1 & \text{in} \quad D_0 \\ u = 0 & \text{on} \quad \partial D_0 \end{cases}$$

then $u_M \ge 2^{-1}(\mathscr{L}^2(D_0)/\mathscr{H}^1(\partial D_0))^2$ when $n = 2$. This is found for example in [PSS, §5] where u is replaced by $2^{-1}u$. Note that U is a supersolution of this equation provided that U is concave. It would be interesting to compare our result with their various estimates.

Finally, it would be interesting to extend our estimates for motion driven by anisotropic curvatures; see a book of Gurtin [Gu] and references there. Generalized solutions are constructed for such a motion in [CGG1]. Approximate level set equations are studied for such motions in [CGG2].

References

[A] Aleksandrov, A.D.: *Uniquness theorems for surfaces in the large V*, AMS Translaton **21**, Ser.2, 412-416 (1962).

[AAG] Altschuler, S., Angenent, S., Giga, Y.: *Mean curvature flow through singularities for surfaces of rotation*, Hokkaido Univ. Preprint Series # 130, December 1991.

[B] Brakke, K.A.: The Motion of a Surface by its Mean Curvature. Princeton, Princeton University Press (1978).

[CGG1] Chen, Y.-G., Giga, Y., Goto, S.: *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations*, J. Differential Geometry **33**, 749-786 (1991).

[CGG2] Chen, Y.-G., Giga, Y., Goto, S.: *Analysis toward snow crystal growth*, Proceeding of a Symposium (Sapporo 1990), pp. 43-57. Functional Analysis and Related Topics (ed. S.Koshi), Singapore: World Scientific 1991.

[CGG3] Chen, Y.-G., Giga, Y., Goto, S.: *Remarks on viscosity solutions for evolution equations*, Proc. Japan Acad., **67**, Ser.A, 263-266 (1991).

[ES1] Evans, L.C., Spruck, J.: *Motion of level sets by mean curvature I*, J. Differential Geometry **33**, 635-681 (1991).

[ES2] Evans, L.C., Spruck, J.: *Motion of level sets by mean curvature II*, Trans. AMS (to appear)

[ES3] Evans, L.C., Spruck, J.: *Motion of level sets by mean curvature III*, J. Geometric Analysis **2**, 121-150 (1992).

[ES4] Evans, L.C., Spruck, J.: *Motion of level sets by mean curvature IV*, preprint.

[F] Federer, H.: Geometric Measure Theory. New York: Springer 1969.

[GaH] Gage, M., Hamilton, R.S.: *The heat equation shrinking of convex plane curves*, J. Differential Geometry **23**, 69-96 (1986).

[Gi] Giusti, E.: Minimal Surfaces and Functions of Bounded Variation. Boston-Basel-Stuttgart: Birkhauser 1984.

[Gr1] Grayson, M.A.: *The heat equation shrinking embedded plane curves to round points*, J. Differential Geometry **26**, 285-314 (1987).

[Gr2] Grayson, M.A.: *A short note on the evolution of a surface by its mean curvature*, Duke Math. J. **58**, 555-558 (1989).

[Gu] Gurtin, M.E.: Thermomechanics of Evolving Phase Boundaries in the Plane. Oxford, Clarendon Press 1993.

[H] Huisken, G.: *Flow by mean curvature of convex surfaces into sphere*, J. Differential Geometry **20**, 237-266 (1984).

[I] Ilmanen, T.: *Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature*, preprint.

[PSS] Payne, L.E., Sperb, R., Stakgold, I.: *On Hopf type maximum principles for convex domains*, Nonlinear Analysis, TMS, **1**, 547-559 (1977).

[S] Sperb, R. Maximum Principles and their Applications. New York, Academic Press 1981.