On functional moduli for first order ordinary differential equations

Yasuhiro Kurokawa

Series ♯186. March 1993

HOKKAIDO UNIVERSITY
PREPRINT SERIES IN MATHEMATICS

♯ 158: S. Izumiya, W.L. Marar, The Euler number of a topologically stable singular surface in a 3-manifold, 11 pages. 1992.

♯ 159: T. Hibi, Cohen-Macaulay types of Cohen-Macaulay complexes, 26 pages. 1992.

♯ 160: A. Arai, Properties of the Dirac-Weyl operator with a strongly singular gauge potential, 26 pages. 1992.

♯ 161: A. Arai, Dirac operators in Boson-Fermion Fock spaces and supersymmetric quantum field theory, 30 pages. 1992.

♯ 162: S. Albeverio, K. Iwata, T. Kolsrud, Random parallel transport on surfaces of finite type, and relations to homotopy, 8 pages. 1992.

♯ 163: S. Albeverio, K. Iwata, T. Kolsrud, Moments of random fields over a family of elliptic curves, and modular forms, 9 pages. 1992.

♯ 164: Y. Giga, M. Sato, Neumann problem for singular degenerate parabolic equations, 12 pages. 1992.

♯ 165: J. Wierzbicki, Y. Watatani, Commuting squares and relative entropy for two subfactors, 18 pages. 1992.

♯ 166: Y. Okabe, A new algorithm driven from the view-point of the fluctuation-dissipation theorem in the theory of KM_2O-Langevin equations, 13 pages. 1992.

♯ 167: Y. Okabe, H. Mano and Y. Itoh, Random collision model for interacting populations of two species and its strong law of large numbers, 14 pages. 1992.

♯ 168: A. Inoue, On the equations of stationary precesses with divergent diffusion coefficients, 25 pages. 1992.

♯ 169: T. Ozawa, Remarks on quadratic nonlinear Schrödinger equations, 19 pages. 1992.

♯ 170: T. Fukui, Y. Giga, Motion of a graph by nonsmooth weighted curvature, 11 pages. 1992.

♯ 171: J. Inoue, T. Nakazi, Finite dimensional solution sets of extremal problems in H^1, 10 pages. 1992.

♯ 172: S. Izumiya, A characterization of complete integrability for partial differential equations of first order, 6 pages. 1992.

♯ 173: T. Suwa, Unfoldings of codimension one complex analytic foliation singularities, 49 pages. 1992.

♯ 174: T. Ozawa, Wave propagation in even dimensional spaces, 15 pages. 1992.

♯ 175: S. Izumiya, Systems of Clairaut type, 7 pages. 1992.

♯ 176: A. Hoshiga, The initial value problems for quasi-linear wave equations in two space dimensions with small data, 25 pages. 1992.

♯ 177: K. Sugano, On bicommutators of modules over H-separable extension rings III, 9 pages. 1993.

♯ 178: T. Nakazi, Toeplitz operators and weighted norm inequalities, 17 pages. 1993.

♯ 179: O. Ogurisu, Existence and structure of infinitely degenerate zero-energy ground states of a Wess-Zumino type model in supersymmetric quantum mechanics, 26 pages. 1993.

♯ 180: O. Ogurisu, Ground state of a spin 1/2 charged particle in an even dimensional magnetic field, 9 pages. 1993.

♯ 181: K. Sugano, Note on H-separable Galois extension, 6 pages. 1993.

♯ 182: M. Yamada, Distance formulas of asymptotic Toeplitz and Hankel operators, 13 pages. 1993

♯ 183: G. Ishikawa, T. Ozawa, The genus of a connected compact real algebraic surface in the affine three space, 11 pages. 1993.

♯ 184: T. Hibi, Canonical modules and Cohen-Macaulay types of partially ordered sets, 6 pages. 1993.

♯ 185: Y. Giga, K. Yama-uchi, On a lower bound for the extinction time of surfaces moved by mean curvature, 16 pages. 1993.

On functional moduli
for first order ordinary differential equations

Yasuhiro KUROKAWA

Department of Mathematics, Hokkaido University

ABSTRACT. – We reduce some normal forms, which have functional moduli, for a certain class of mapping diagrams. This class is associated to completely integrable first order ordinary differential equations. The reduction is given relative to the equivalence relation among the differential equations under the group of point transformations in the sense of S. Lie.

Sur les modules fonctionnels
pour les équations différentielles ordinaires du premier ordre

RÉSUMÉ. – *Nous donnons quelques formes normales, qui ont modules fonctionnels, pour une certaine classe de diagrammes d'application. Cette classe s'associe aux équations différentielles ordinaires du premier ordre complètement intégrables. Les formes normales s'obtiennent relativement à la relation d'équivalence pour les équations différentielles sous la groupe des transformations de points au sens de S. Lie.*

Version française abrégée. – Dans [5] il a été étudié la classification générique d'équations différentielles ordinaires du premier ordre complètement intégrables par la relation d'équivalence sous la groupe des transformations de points au sens de S. Lie. Premièrement on rappele leurs résultats pour énoncer notre objet et théorème.

Soit $\pi: PT^*\mathbf{R}^2 \to \mathbf{R}^2$ le faisceau cotangent projectif sur $\mathbf{R}^2$. Alors $PT^*\mathbf{R}^2$ a la structure de contact naturelle. Pour tout $z \in PT^*\mathbf{R}^2$ il y a un système coordonnées locales (x, y, p) au voisinage de z tel que $\pi(x, y, p) = (x, y)$ et la structure de contact soit donnée par la 1-form $\omega = dy - pdx$. *Un germe d'équation différentielle ordinaire du premier ordre* (ou brièvement, *un germe d'équation*) est donné par un germe d'immersion $f: (\mathbf{R}^2, 0) \to PT^*\mathbf{R}^2$. On dit que f est *complètement intégrable* s'il existe un germe de submersion $\mu: (\mathbf{R}^2, 0) \to (\mathbf{R}, 0)$ tel que $d\mu \wedge f^*\omega = 0$. Alors on appelle le couple (μ, f) *un germe d'équation à une intégral complète.* Soit (μ, g) un couple d'un germe d'application C^∞

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

$g\colon (\mathbf{R}^2,0) \to (\mathbf{R}^2,0)$ et un germe de submersion $\mu\colon (\mathbf{R}^2,0) \to (\mathbf{R},0)$. Alors le diagramme divergent

$$(\mathbf{R},0) \xleftarrow{\mu} (\mathbf{R}^2,0) \xrightarrow{g} (\mathbf{R}^2,0)$$

ou brièvement (μ,g), est appelé ***un diagramme intégral*** s'il existe un germe d'équation $f\colon (\mathbf{R}^2,0) \to PT^*\mathbf{R}^2$ tel que (μ,f) soit un germe d'équation à une intégral complète et que $g=\pi\circ f$. En outre on introduit la relation d'équivalence pour les diagrammes intégrals. Soient (μ,g) et (μ',g') deux diagrammes intégrals. Alors (μ,g) et (μ',g') sont *équivalents* (resp. *fortement équivalents*) si l'on a un diagramme commutatif

$$\begin{tikzcd}
(\mathbf{R},0) \arrow[d, "\lambda"'] & (\mathbf{R}^2,0) \arrow[l, "\mu"'] \arrow[r, "g"] \arrow[d, "k"] & (\mathbf{R}^2,0) \arrow[d, "h"] \\
(\mathbf{R},0) & (\mathbf{R}^2,0) \arrow[l, "\mu'"] \arrow[r, "g'"'] & (\mathbf{R}^2,0)
\end{tikzcd}$$

où h,k et λ sont des germes de difféomorphisme C^∞ (resp. $\lambda = id_{\mathbf{R}}$).

Suivant Lie on dit que deux germes d'équation f et f' sont équivalents s'il existe germes de difféomorphisme C^∞ $k\colon (\mathbf{R}^2,0) \to (\mathbf{R}^2,0)$ et $h\colon (\mathbf{R}^2,\pi\circ f(0)) \to (\mathbf{R}^2,\pi\circ f'(0))$ tels que $\hat{h}\circ f(0) = f'(0)$ et $\hat{h}\circ f = f'\circ k$, où $\hat{h}$ est le relèvement contact de h. Dans [5] il est montré que deux germes d'équation complètement intégrable f et f' sont équivalents si et seulement si les diagrammes intégrals $(\mu,\pi\circ f)$ et $(\mu',\pi\circ f')$ sont équivalents pour (μ,f) et (μ',f') génériques. De plus il est montré que tout diagramme intégral générique (μ,g) est fortement équivalent à un des germes suivants:

(1) $\mu = v, g=(u,v)$; (2) $\mu = v-\frac{1}{3}u^3, g=(u^2,v)$; (3) $\mu = v-\frac{1}{2}u, g=(u,v^2)$;

(4) $\mu_\alpha = \frac{3}{4}u^4+\frac{1}{2}u^2v+\alpha\circ g, g=(u^3+uv,v)$; (5) $\mu_\alpha = v+\alpha\circ g, g=(u,v^3+uv)$;

(6) $\mu_\alpha = \frac{1}{2}v^2+\alpha\circ g, g=(u,v^3+uv^2)$; où, dans les cas trois derniers, $\alpha(x,y)$ est un germe de fonction C^∞ sur $(\mathbf{R}^2,0)$ avec $\alpha(0)=0$, et dans (4), (6) $\frac{\partial\alpha}{\partial y}(0)=\pm1$, $\frac{\partial\alpha}{\partial x}(0)=\pm1$ respectivement.

Le germe de fonction α qui apparaissent dans les formes normales (4), (5), (6) s'appelle *un module fonctionnel* de chaque forme normale.

Notre préoccupation est de classifier les diagrammes intégrals génériques par l'équivalence (non forte). On remarque que les germes de (1) à (6) ne sont pas équivalents. Donc notre objet se réduit de classifier les familles des germes (4), (5), (6) par l'équivalence.

D'autre part, la forme normale (5) est aussi une des formes normales pour les diagrammes divergents, appelés familles à un paramètre de courbes planes, étudiés par Dufour [3]. De plus il a classé la forme normale (5) par l'équivalence dans [4]. Dans cette Note on classifie les formes normales (4), (6) par l'équivalence, qui est un analogue du résultat de Dufour sur (5).

Indiquons par $\mathcal{A}_y$ (resp. $\mathcal{A}_x$) l'ensemble des modules fonctionnels de (4), (6).

Maintenant on énonce notre théorème.

THÉORÈME. — A) *Soit* (μ_α, g) *un diagramme intégral de (4). Alors, pour tout* $\alpha \in \mathcal{A}_y$ *il existe un* $\alpha' \in \mathcal{A}_y$ *tel que*

(i) (μ_α, g) *soit équivalent à* $(\mu_{\alpha'}, g)$,

(ii) $\alpha'(0,y) = \frac{\partial \alpha}{\partial y}(0,0)y + \frac{1}{2}\chi_\alpha y^2$ *pour tout* $y \leq 0$,

où $\chi_\alpha = \frac{\partial^2 \alpha}{\partial y^2}(0,0)$.

B$_j$) (j=1,2) *Soit* (μ_α, g) *un diagramme intégral de (6). Alors, pour tout* $\alpha \in \mathcal{A}_x$ *il existe un* $\alpha' \in \mathcal{A}_x$ *tel que*

(i) (μ_α, g) *soit équivalent à* $(\mu_{\alpha'}, g)$,

(ii) $\alpha' = \delta x + \frac{1}{2}\chi_\alpha x^2$ *sur* D_j,

où $\chi_\alpha = \frac{\partial^2 \alpha}{\partial x^2}(0,0)$, $\delta = \pm 1$, $D_1 = \{(x,y) | y = 0\}$ *et* $D_2 = \{(x,y) | 27y = 4x^3\}$.

Remarque. — χ_α est un invariant de (4), (6) relativement à l'équivalence.

Dufour a montré aussi la unicité pour les modules fonctionnels. Pourtant, nous ne pouvons pas montrer l'unicité comme les germes (4), (6) sont assez compliqués.

1. Introduction

In the recent article (Hayakawa et al [5]) it has been studied the generic classification of completely integrable first order ordinary differential equations by the equivalence relation under the group of point transformations in the sense of S. Lie. At first we recall their results to state our purpose and theorem.

Let $\pi : PT^*\mathbf{R}^2 \to \mathbf{R}^2$ be the projective cotangent bundle over $\mathbf{R}^2$. Then $PT^*\mathbf{R}^2$ has the natural contact structure. For any $z \in PT^*\mathbf{R}^2$ there is a local coordinate system (x, y, p)

around z such that $\pi(x, y, p) = (x, y)$ and the contact structure is given by the 1-form $\omega = dy - pdx$. *A first order ordinary differential equation-germ* (or briefly, *an equation-germ*) is given by an immersion-germ $f\colon (\mathbf{R}^2, 0) \to PT^*\mathbf{R}^2$. We say that f is *completely integrable* if there exists a submersion-germ $\mu\colon (\mathbf{R}^2, 0) \to (\mathbf{R}, 0)$ such that $d\mu \wedge f^*\omega = 0$. Then we call the pair (μ, f) *an equation-germ with a complete integral.* Let (μ, g) be a pair of a C^∞map-germ $g : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ and a submersion-germ $\mu : (\mathbf{R}^2, 0) \to (\mathbf{R}, 0)$. Then the divergent diagram

$$(\mathbf{R}, 0) \xleftarrow{\mu} (\mathbf{R}^2, 0) \xrightarrow{g} (\mathbf{R}^2, 0)$$

or briefly (μ, g), is called *an integral diagram* if there exists an equation-germ $f : (\mathbf{R}^2, 0) \to PT^*\mathbf{R}^2$ such that (μ, f) is an equation-germ with a complete integral and that $g = \pi \circ f$. Moreover we introduce the equivalence relation among integral diagrams. Let (μ, g) and (μ', g') be integral diagrams. Then (μ, g) and (μ', g') are *equivalent* (resp. *strictly equivalent*) if the diagram

$$\begin{tikzcd}
(\mathbf{R}, 0) \arrow[d, "\lambda"'] & (\mathbf{R}^2, 0) \arrow[l, "\mu"'] \arrow[r, "g"] \arrow[d, "k"] & (\mathbf{R}^2, 0) \arrow[d, "h"] \\
(\mathbf{R}, 0) & (\mathbf{R}^2, 0) \arrow[l, "\mu'"] \arrow[r, "g'"'] & (\mathbf{R}^2, 0)
\end{tikzcd}$$

commutes for some C^∞diffeomorphism-germs λ, k and h (resp. $\lambda = id_{\mathbf{R}}$).

Following Lie, two equation-germs f and f' are called equivalent if there exist C^∞diffeomorphism-germs $k\colon (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ and $h\colon (\mathbf{R}^2, \pi \circ f(0)) \to (\mathbf{R}^2, \pi \circ f'(0))$ such that $\hat{h} \circ f(0) = f'(0)$ and $\hat{h} \circ f = f' \circ k$, where $\hat{h}$ is the contact lift of h. In [5] it has been shown that two completely integrable equation-germs f and f' are equivalent if and only if the integral diagrams $(\mu, \pi \circ f)$ and $(\mu', \pi \circ f')$ are equivalent for generic (μ, f) and (μ', f'). Furthermore they showed that any generic integral diagram (μ, g) is strictly equivalent to one of the following germs:

(1) $\mu = v, g = (u, v)$; (2) $\mu = v - \frac{1}{3}u^3, g = (u^2, v)$; (3) $\mu = v - \frac{1}{2}u, g = (u, v^2)$;
(4) $\mu_\alpha = \frac{3}{4}u^4 + \frac{1}{2}u^2 v + \alpha \circ g, g = (u^3 + uv, v)$; (5) $\mu_\alpha = v + \alpha \circ g, g = (u, v^3 + uv)$;
(6) $\mu_\alpha = \frac{1}{2}v^2 + \alpha \circ g, g = (u, v^3 + uv^2)$; where $\alpha(x, y)$ is a C^∞ function-germ on $(\mathbf{R}^2, 0)$ with $\alpha(0) = 0$, and additionally in the case (4), (6) $\frac{\partial \alpha}{\partial y}(0) = \pm 1$, $\frac{\partial \alpha}{\partial x}(0) = \pm 1$ respectively.

The function-germ α which appear in the normal forms (4), (5), (6) is called *a functional modulus* of each normal form.

Our concern is to classify generic integral diagrams by the equivalence (not the strict equivalence). We remark that the germs (1) to (6) are not equivalent each other. Hence we would like to classify the families of the germs (4), (5), (6) by the equivalence. On the other hand, the normal form (5) is also one of the normal forms for generic divergent diagrams, which are called one-parameter families of plane curves, studied by Dufour [3]. Furthermore he classified the normal form (5) by the equivalence in [4]. In this paper we will classify the normal forms (4), (6) by the equivalence, which is an analogy of Dufour's result on (5).

Denote by $\mathcal{A}_y$ (resp. $\mathcal{A}_x$) as the set of functional moduli of (4) (resp. (6)).

Now we state our theorem.

THEOREM. – A) *Let* (μ_α, g) *be an integral diagram of (4). Then, for any* $\alpha \in \mathcal{A}_y$ *there exists an* $\alpha' \in \mathcal{A}_y$ *such that*

(i) (μ_α, g) *is equivalent to* $(\mu_{\alpha'}, g)$,

(ii) $\alpha'(0, y) = \frac{\partial \alpha}{\partial y}(0,0)y + \frac{1}{2}\chi_\alpha y^2$ *for all* $y \leq 0$,

where $\chi_\alpha = \frac{\partial^2 \alpha}{\partial y^2}(0,0)$.

B$_j$) (j=1,2) *Let* (μ_α, g) *be an integral diagram of (6). Then, for any* $\alpha \in \mathcal{A}_x$ *there exists an* $\alpha' \in \mathcal{A}_x$ *such that*

(i) (μ_α, g) *is equivalent to* $(\mu_{\alpha'}, g)$,

(ii) $\alpha' = \delta x + \frac{1}{2}\chi_\alpha x^2$ *on* D_j,

where $\chi_\alpha = \frac{\partial^2 \alpha}{\partial x^2}(0,0)$, $\delta = \pm 1$, $D_1 = \{(x,y)|y=0\}$ *and* $D_2 = \{(x,y)|27y = 4x^3\}$.

Dufour also have shown uniqueness of functional modulus. However the germs (4), (6) are so complicated that we can not obtain the uniqueness result in this paper.

In this paper, we assume that all mappings and diffeomorphisms are of class C^∞.

The author would like to express his sincere gratitude to Professors I. Nakai, S. Izumiya and G. Ishikawa for their valuable advices and constant encouragements. Also gratitude to Professor T. Morimoto for his useful comments.

2. Proof of Theorem

The proof is based on the following two propositions.

PROPOSITION 2.1.(Takens [6]) – *Let $\psi\colon \mathbf{R},0 \to \mathbf{R},0$ be a diffeomorphism such that ψ^2 has the form $\psi^2(x) = x + x^k F(x)$ with $F(0)\neq 0$ and $k \geq 2$. Then there is an orientation preserving diffeomorphism $\lambda\colon \mathbf{R},0 \to \mathbf{R},0$ such that, in some neighbourhood of $0 \in \mathbf{R}$,*

$$\lambda \circ \psi \circ \lambda^{-1}(x) = \pm x + \delta x^k + \chi x^{2k-1}$$

where $\delta = \pm 1$ and $\chi \in \mathbf{R}$.

The following is implicitly proved in [5].

PROPOSITION 2.2. – *Let (μ', g') be an integral diagram which is equivalent to (μ_α, g) of (4) (resp. (6)) for some $\alpha \in \mathcal{A}_y$ (resp. $\mathcal{A}_x$). Then (μ', g') is strictly equivalent to $(\mu_{\alpha'}, g)$ of (4) (resp. (6)) for some $\alpha' \in \mathcal{A}_y$ (resp. $\mathcal{A}_x$).*

For each case A, B_1, B_2 we will define a map-germ $\gamma_\alpha\colon (\mathbf{R},0) \to (\mathbf{R}^2,0)$, as follows. In the case A: Put $\Delta = \{(x,y) | 27x^2 + 4y^3 < 0\}$. Note that $\Delta = \{(x,y) | \sharp(g^{-1}(x,y)) = 3\}$. Let $(u_1,y), (u_2,y)$ and (u_3,y) be the preimages of (x,y) by g for each $(x,y) \in \Delta$ near $(0,0)$, where $u_j = u_j(x,y)$ $(j = 1,2,3)$ are three real roots of the equation $U^3 + yU - x = 0$ and ordered by $u_1 < u_2 < u_3$. For each $\alpha \in \mathcal{A}_y$ set $c_j(x,y) = \mu_\alpha(u_j,y)$. We see clearly $c_1(0,y) = c_3(0,y) = \frac{1}{4}y^2 + \alpha(0,y)$, $c_2(0,y) = \alpha(0,y)$ for any $(0,y) \in \Delta$. We set $\gamma_\alpha(y) = (\alpha(0,y), \frac{1}{4}y^2 + \alpha(0,y))$ for each $\alpha \in \mathcal{A}_y$.

In the case B_1 (resp. B_2): Note that $\sharp(g^{-1}(x,y)) = 2$ for any $(x,y) \in D_1$ (resp. D_2). Let (x,v_1), (x,v_2) be the preimages by g for each $(x,y) \in D_1$ (resp. D_2), where $v_j = v_j(x,y)$ $(j = 1,2)$ are three real roots of $V^3 + xV^2 - y = 0$ (v_1 is the multiple root). For any $(x,y) \in D_1$ (resp. D_2), set $c_j(x,y) = \mu_\alpha(x,v_j)$. We see clearly $c_1(x,y) = \alpha(x,y)$, $c_2(x,y) = \frac{1}{2}x^2 + \alpha(x,y)$ (resp. $c_1(x,y) = \frac{2}{9}x^2 + \alpha(x,y)$, $c_2(x,y) = \frac{1}{18}x^2 + \alpha(x,y)$). We set $\gamma_\alpha(x) = (c_1(x,0), c_2(x,0))$ (resp. $\gamma_\alpha(x) = (c_1(x, \frac{4}{27}x^3), c_2(x, \frac{4}{27}x^3))$ for each $\alpha \in \mathcal{A}_x$.

LEMMA 2.3. — *Let $\theta : (\mathbf{R}, 0) \to \mathbf{R}$ be a function-germ such that*

$$\theta(0) = \theta'(0) = \theta''(0) = \theta'''(0) = 0. \tag{2.1}$$

In the case A for any $\alpha \in \mathcal{A}_y$ there exists an $\alpha' \in \mathcal{A}_y$ such that

(i) *(μ_α, g) and $(\mu_{\alpha'}, g)$ are equivalent,*

(ii) *$-y^2 + \alpha'(0,y)^2 - \chi_\alpha \alpha'(0,y)^3 + \theta(\alpha'(0,y)) = 0$ for all $y \leq 0$,*

where $\chi_\alpha = \frac{\partial^2 \alpha}{\partial y^2}(0,0)$.

In the case B_1 (resp. B_2) for any $\alpha \in \mathcal{A}_x$ there exists an $\alpha' \in \mathcal{A}_x$ such that

(i) *(μ_α, g) and $(\mu_{\alpha'}, g)$ are equivalent,*

(ii) *$-x^2 + \alpha'(x,0)^2 - \chi_\alpha \alpha'(x,0)^3 + \theta(\alpha'(x,0)) = 0$ on D_1*

(resp. $-x^2 + c^2 - (\frac{1}{9} + \chi_\alpha)c^3 + \theta(c) = 0$ on D_2)

where $c = \frac{1}{18}x^2 + \alpha'(x, \frac{4}{27}x^3)$, $\chi_\alpha = \frac{\partial^2 \alpha}{\partial x^2}(0,0)$.

Proof. — Since any $\alpha \in \mathcal{A}_y$ (resp. $\mathcal{A}_x$) has the condition $\frac{\partial \alpha}{\partial y}(0,0) = \pm 1$ (resp. $\frac{\partial \alpha}{\partial x}(0,0) = \pm 1$), by the implicit function theorem, there exists the function-germ $\psi_\alpha : (\mathbf{R}, 0) \to (\mathbf{R}, 0)$ such that $Im\gamma_\alpha = graph\psi_\alpha$ in each case. By direct calculations, we see respectively in the case A, B_1, B_2

$$\psi_\alpha(c) = c + \frac{1}{4}c^2 - \frac{1}{4}\chi_\alpha c^3 + o(|c|^3),$$

$$\psi_\alpha(c) = c + \frac{1}{2}c^2 - \frac{1}{2}\chi_\alpha c^3 + o(|c|^3),\ \psi_\alpha(c) = c + \frac{1}{6}c^2 - \frac{1}{6}(\frac{1}{9} + \chi_\alpha)c^3 + o(|c|^3).$$

In the case A, B_1, B_2 respectively, define the function-germ $\overline{\psi_\alpha} : (\mathbf{R}, 0) \to (\mathbf{R}, 0)$ by

$$\overline{\psi_\alpha}(c) = c + \frac{1}{4}c^2 - \frac{1}{4}\chi_\alpha c^3 + \frac{1}{4}\theta(c),$$

$$\overline{\psi_\alpha}(c) = c + \frac{1}{2}c^2 - \frac{1}{2}\chi_\alpha c^3 + \frac{1}{2}\theta(c),\ \overline{\psi_\alpha}(c) = c + \frac{1}{6}c^2 - \frac{1}{6}(\frac{1}{9} + \chi_\alpha)c^3 + \frac{1}{6}\theta(c).$$

Then by Proposition 2.1 there exists an orientation preserving diffeomorphism-germ $\lambda : (\mathbf{R}, 0) \to (\mathbf{R}, 0)$ such that $\overline{\psi_\alpha} = \lambda \circ \psi_\alpha \circ \lambda^{-1}$ in each case. Since $(\lambda \circ \mu_\alpha, g)$ is equivalent to (μ_α, g), by Proposition 2.2 there exists a functional modulus $\alpha' \in \mathcal{A}_y$ (resp. $\mathcal{A}_x$) in the case A (resp. B_1, B_2) such that the following diagram commute, hence (μ_α, g) and $(\mu_{\alpha'}, g)$ are equivalent:

$$\begin{array}{ccccc}
(\mathbf{R},0) & \xleftarrow{\mu_\alpha} & (\mathbf{R}^2,0) & \xrightarrow{g} & (\mathbf{R}^2,0) \\
\lambda\downarrow & & \| & & \| \\
(\mathbf{R},0) & \xleftarrow[\lambda\circ\mu_\alpha]{} & (\mathbf{R}^2,0) & \xrightarrow[g]{} & (\mathbf{R}^2,0) \\
\| & & \downarrow h & & \downarrow h \\
(\mathbf{R},0) & \xleftarrow[\mu_{\alpha'}]{} & (\mathbf{R}^2,0) & \xrightarrow[g]{} & (\mathbf{R}^2,0) .
\end{array}$$

In the case A since the set $\{(0,y)|y \leq 0\}$ is preserved by h and since λ is orientation preserving, the above commutative diagram implies

$$(2.2) \qquad \lambda \times \lambda(Im\gamma_\alpha|_{y\leq 0}) = graph\overline{\psi_\alpha} \cap \{\delta c \leq 0\} = Im(\gamma_{\alpha'}|_{y\leq 0})$$

where $\delta = \frac{\partial\alpha}{\partial y}(0,0)$. In the case B_1 (resp. B_2) since a discriminant set D_1 (resp. D_2) is preserved by h, the above commutative diagram implies $\lambda\times\lambda(Im\gamma_\alpha) = graph\overline{\psi_\alpha} = Im\gamma_{\alpha'}$. Therefore by definition of $\overline{\psi_\alpha}$, γ_α we have the equation in Lemma 2.3.(ii) in each case. This completes the proof.

Remark 2.4. – In the case A, by (2.2) if $\delta = 1$, then $c = \alpha'(0,y) \leq 0$, thus $\frac{\partial\alpha'}{\partial y}(0,0) = 1$. Similarly, if $\delta = -1$ then $\frac{\partial\alpha'}{\partial y}(0,0) = -1$. That is $\frac{\partial\alpha}{\partial y}(0,0) = \frac{\partial\alpha'}{\partial y}(0,0)$.

The functional moduli α' in Lemma 2.3 depends on θ and α. Now, by means of special choice of θ, we normalize α' such that $\alpha'(0,y)$ (resp. $\alpha'(x,0)$, $\alpha'(x,\frac{4}{27}x^3)$) is the polynomial as low degree as possible in the case A (resp. B_1, B_2). Note that the degree of the normalization is not zero for the condition of the functional moduli $\frac{\partial\alpha}{\partial y}(0,0) = \pm 1$ (resp. $\frac{\partial\alpha}{\partial x}(0,0) = \pm 1$). In the case of $\chi_\alpha = 0$, we have the normalization to degree one. In the case of $\chi_\alpha \neq 0$, we can not have the normalization to degree one because of the condition (2.1) for θ, thus we consider the normalization to degree two. In fact, it is possible as follows: For any $\chi \in \mathbf{R}$ we define the function-germ $\theta_\chi : (\mathbf{R},0) \to (\mathbf{R},0)$ by

$$\theta_\chi = \begin{cases} 0 & \text{if} \quad \chi = 0 \\ a\circ\xi & \text{if} \quad \chi \neq 0 \end{cases}$$

where $\xi(t) = \frac{-1+\sqrt{1+2\chi t}}{\chi}$, $a(t) = \frac{5}{4}\chi^2t^4 + \frac{3}{4}\chi^3t^5 + \frac{1}{8}\chi^4t^6$.

Then the θ_χ satisfy the condition (2.1) in Lemma 2.3. Moreover we define the function-germ $h_\chi: (\mathbf{R} \times \mathbf{R}, (0,0)) \to (\mathbf{R}, 0)$ by $h_\chi(t,c) = -t^2 + c^2 - \chi c^3 + \theta_\chi(c)$ for any $\chi \in \mathbf{R}$. By the definition of θ_χ it can be directly shown that $h_\chi(\pm t, t + \frac{\chi}{2}t^2) = 0$ for all $t \in (\mathbf{R}, 0)$. Hence we can easily have

LEMMA 2.5. — *If* $(t,c) \in h_\chi{}^{-1}(0)$, *then* $c = \pm t + \frac{\chi}{2}t^2$ *for any* $\chi \in \mathbf{R}$.

For any $\alpha \in \mathcal{A}_y$ (resp. $\mathcal{A}_x$), set $\chi = \frac{\partial^2 \alpha}{\partial y^2}(0,0)$ (resp. $\chi = \frac{\partial^2 \alpha}{\partial x^2}(0,0)$, $\frac{1}{9} + \frac{\partial^2 \alpha}{\partial x^2}(0,0)$). Then, by Lemma 2.3, Lemma 2.5 (and Remark 2.4 in the case A), we obtain Theorem A), B_1), B_2).

Remark. — χ_α is an invariant of (4), (6) relative to the equivalence because we can see that by the direct calculation the third coefficient of the Taylor expansion at the origin for ψ_α is invariant under the conjugate $\psi_\alpha = \lambda^{-1} \circ \psi_{\alpha'} \circ \lambda$ for a diffeomorphism-germ λ.

REFERENCES

[1] V. I. Arnol'd, *Mathematical methods of classical mechanics*, Springer-Verlag, 1978.

[2] M. J. Dias Carneiro, *Singularities of envelopes of families of submanifolds in* $\mathbf{R}^n$, Ann. Sci. Ecole. Norm. Sup. (4)**16** (1983), 173–192.

[3] J. P. Dufour, *Familles de courbes planes différentiables*, Topology **22-4** (1983), 449–474.

[4] ———, *Modules pour les familles de courbes planes*, Ann. Inst. Fourier, Grenoble. **39-1** (1989), 225–238.

[5] A. Hayakawa, G. Ishikawa, S. Izumiya and K. Yamaguchi, *Classification of generic integral diagram and first order ordinary differential equations*, Preprint, Hokkaido University.

[6] F. Takens, *Normal forms for certain singular vector fields*, Ann. Inst. Fourier, Grenoble. **23-2** (1973), 163–195.

Department of Mathematics, Faculty of Science, Hokkaido University, Sapporo 060, Japan
E-mail address: y-kurokw@math.hokudai.ac.jp