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Unitary equivalence between a spin 1/2 charged particle in a two-dimensional magnetic field and a spin 1/2 neutral particle with an anomalous magnetic moment in a two-dimensional electric field

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Abstract. We prove the unitary equivalence between the Dirac Hamiltonian H_D for a relativistic spin 1/2 neutral particle with an anomalous magnetic moment in a two-dimensional electrostatic field $\mathbf{E} = (E_1, E_2)$ and the direct sum of the Dirac-Weyl operators $D(\pm\mathbf{A})$ for a spin 1/2 charged particle in two-dimensional magnetic fields $\pm d\mathbf{A}$ with the vector potential $\mathbf{A} = E_2 dx^1 - E_1 dx^2$, $(x^1, x^2) \in \mathbb{R}^2$. As applications, we investigate the ground state and the spectra of H_D.

1. Introduction

Recently V. V. Semenov [1] stated that a relativistic spin 1/2 neutral particle with an anomalous magnetic moment μ in a two-dimensional electrostatic field has $(N-1)$-fold degeneracy of the ground state, where the integer N is determined by μ and the total charge in the field. (In fact, the degeneracy of the ground state is equal to $2(N-1)$. See Corollary 3.4 and Remark 3.5.) He pointed out that this phenomenon is similar to the phenomenon Y. Aharonov and A. Casher discussed in [2]; A non-relativistic spin 1/2 charged particle in a two-dimensional magnetic field has $(N-1)$-fold degeneracy of the ground state, where N is determined by the charge and the total magnetic flux.

In this Letter we clarify why the Dirac Hamiltonian H_D discussed by Semenov [1] has $2(N-1)$-fold degeneracy of the ground state: We prove that H_D is unitary equivalent to the direct sum of two Dirac-Weyl operators, each of which has $(N-1)$-fold degeneracy of the ground state. This is done in section 2. In section 3, we apply the result in section 2 to investigating the ground state and the spectra of H_D.

2. The unitary equivalence

We consider a quantum system of a relativistic spin 1/2 neutral particle with an anomalous magnetic moment $\mu \in \mathbb{R} \setminus \{0\}$ in a two-dimensional electrostatic field $\mathbf{E}(\mathbf{r}) = (E_1(\mathbf{r}), E_2(\mathbf{r}))$, $\mathbf{r} = (x^1, x^2) \in \mathbb{R}^2$, with $E_j \in C^\infty(\mathbb{R}^2 \to \mathbb{R})$, $j = 1, 2$. As usual, we denote $p_j = -i\partial/\partial x^j, j = 1, 2$, and the Pauli's spin matrices by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $m \geq 0$ be a constant denoting the mass of the particle. Define the Dirac Hamiltonian H_D [1, 3] by

$$H_D = \begin{pmatrix} m & A^* \\ A & -m \end{pmatrix}$$

acting in $L^2(\mathbb{R}^2; \mathbb{C}^4)$, where

$$A = \sum_{j=1,2} \sigma^j (p_j + i\mu E_j)$$

acting in $L^2(\mathbb{R}^2; \mathbb{C}^2)$. Since E_1 and E_2 are C^∞-functions, it follows from Chernoff's theorem [4] that H_D is essentially selfadjoint on C_0^∞, where C_0^∞ is $C_0^\infty(\mathbb{R}^2; \mathbb{C}^4)$. We denote the closure of $H_D \upharpoonright C_0^\infty$ by the same symbol.

Let $\mathbf{A} = A_1 dx^1 + A_2 dx^2$ on $\mathbb{R}^2$ be a real 1-form denoting a vector potential. The Dirac-Weyl operator $D(q\mathbf{A})$ for a spin 1/2 charged particle with charge q is given by

$$D(q\mathbf{A}) = \sum_{j=1,2} \sigma^j (p_j - qA_j)$$

acting in $L^2(\mathbb{R}^2; \mathbb{C}^2)$. If $A_j \in C^\infty(\mathbb{R}^2 \to \mathbb{R})$, $j = 1, 2$, then $D(q\mathbf{A})$ is essentially selfadjoint on $C_0^\infty(\mathbb{R}^2; \mathbb{C}^2)$ [4]. In this case we denote the closure of $D(q\mathbf{A}) \upharpoonright C_0^\infty(\mathbb{R}^2; \mathbb{C}^2)$ by the same symbol. The operator U on $L^2(\mathbb{R}^2; \mathbb{C}^4)$ given by

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

is unitary. Now, we can state the main theorem of this Letter.

Theorem 2.1. *Define a 1-form* $\mathbf{A} = E_2 dx^1 - E_1 dx^2$. *Then the operator equality*

$$U^* H_D U = \begin{pmatrix} D(\mu \mathbf{A}) + m\sigma^3 & 0 \\ 0 & D(-\mu \mathbf{A}) - m\sigma^3 \end{pmatrix} \tag{2.1}$$

holds.

Proof. By direct computations, we can verify (2.1) on C_0^∞. Since E_1 and E_2 are C^∞-functions, the operator on the right hand side of (2.1) is essentially selfadjoint on C_0^∞ [4]. On the other hand, H_D is essentially selfadjoint on C_0^∞ and U is bijective on C_0^∞. Therefore, (2.1) as an operator equality follows. □

3. Applications

In this section, by employing known results for a spin 1/2 charged particle in a two-dimensional magnetic field, we obtain some corollaries of Theorem 2.1. We assume that there exists a function $\phi \in C^\infty(\mathbb{R}^2 \to \mathbb{R})$ such that

$$E_j = -\frac{\partial \phi}{\partial x^j}, \qquad j = 1, 2.$$

Throughout this section, we put $\mathbf{A} = E_2 dx^1 - E_1 dx^2$. We have $d\mathbf{A} = \Delta\phi dx^1 \wedge dx^2$. Note that $-\Delta\phi/4\pi$ is the charge density per unit volume. We denote by $\sigma(A)$ and $\sigma_e(A)$ the spectrum and the essentially spectrum of the operator A, respectively. We need the following lemma.

Lemma 3.1 (Ref. 5, Proposition 2.5). *Let* $H_j, j = 1, 2$, *be Hilbert spaces,* $S\colon H_1 \to H_2$ *be a densely defined closed linear operator and* m *be a non-negative constant. Let*

$$T = \begin{pmatrix} m & S^* \\ S & -m \end{pmatrix}$$

acting in $H_1 \oplus H_2$. *Then*

$$\sigma(T) = \{\sqrt{m^2 + s}\ ;\ s \in \sigma(S^*S)\} \cup \{-\sqrt{m^2 + s}\ ;\ s \in \sigma(SS^*)\},$$
$$\sigma_e(T) = \{\sqrt{m^2 + s}\ ;\ s \in \sigma_e(S^*S)\} \cup \{-\sqrt{m^2 + s}\ ;\ s \in \sigma_e(SS^*)\}.$$

We remark that, in general, $\sigma(SS^*)\backslash\{0\} = \sigma(S^*S)\backslash\{0\}$ and $\sigma_e(SS^*)\backslash\{0\} = \sigma_e(S^*S)\backslash\{0\}$ (see [6]). With the aid of Lemma 3.1, we can prove the following:

Corollary 3.2. *For simplicity, we put* $\mu = 1$.

(i) If $\Delta\phi(\mathbf{r}) \to 0$ *as* $|\mathbf{r}| \to \infty$, *then* $\sigma_e(H_D) = (-\infty, -m] \cup [m, \infty)$;

(ii) If $\Delta\phi(\mathbf{r}) \to 1$ *as* $|\mathbf{r}| \to \infty$, *then* $\sigma_e(H_D) = \{\pm\sqrt{2k + m^2}\ ;\ k \in \mathbb{N}\} \cup \{m\}$ *and* m *is isolated;*

(iii) If $\Delta\phi(\mathbf{r}) \to \infty$ *as* $|\mathbf{r}| \to \infty$, *then* $\sigma(H_D)$ *is discrete except for* m *and* m *is an isolated point of the essential spectrum.*

Proof. First, we treat the case (ii). From Example 4.1 and the proof of it in [5], we see that $\sigma_e(D(\pm\mathbf{A})^2) = \{2(k-1)\ ;\ k \in \mathbb{N}\}$, 0 is isolated and $\ker D(\pm\mathbf{A}) \subset \ker(\sigma^3 \mp 1)$. Hence, applying Lemma 3.1 to $D(\pm\mathbf{A}) \pm m\sigma^3$, we obtain that $\sigma_e(D(\pm\mathbf{A}) \pm m\sigma^3) = \{+\sqrt{2k+m^2}, -\sqrt{2k+m^2}\,;\, k \in \mathbb{N}\} \cup \{m\}$ and m is isolated. Thus, Theorem 2.1 implies the desired result. In the cases (i) and (iii), we can obtain the desired results in a way similar to the case of (ii). □

Remark 3.3. Corollary 3.2 gives us a classification of the spectrum of H_D by the asymptotic behavior of the charge density per unit volume at infinity. (cf. [5].)

We next investigate the ground state of H_D. We put $\epsilon(x) = 1$, $x \geq 0$, $\epsilon(x) = -1$, $x < 0$.

Corollary 3.4. *Suppose that the limit* $\nu = \lim_{|\mathbf{r}|\to\infty} \phi(\mathbf{r})/\log|\mathbf{r}|$ *exists. Let* N *be the largest integer strictly less than* $|\mu\nu|$. *Then*

$$\dim\ker(H_D^2 - m^2) = \max\{2(N-1), 0\} \quad \textit{and} \quad \ker(H_D^2 - m^2) = \ker(H_D - \epsilon(\mu\nu)m).$$

Proof. In general, for a selfadjoint operator T, $\ker T^2 = \ker T$. By Theorem 3.1 and Corollary 3.2 in [7], we can prove that $\dim\ker D(\pm\mu\mathbf{A}) = \max\{N-1, 0\}$ and $\ker D(\pm\mu\mathbf{A}) \subset \ker(\sigma^3 \mp \epsilon(\mu\nu))$. Hence, by Theorem 2.1, we can obtain the desired results. □

Remark 3.5. Semenov [1] also considered the ground state of H_D. But, counting up the ground state, he seems to have neglected spin components and so reached the wrong conclusion: "this particle has $(N-1)$-fold degeneracy of the ground state".

Remark 3.6. Note that the number N is determined by the asymptotic behavior at $|\mathbf{r}| = \infty$ of the scalar potential $\phi(\mathbf{r})$.

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