



Title	Unitary equivalence between a spin 1/2 charged particle in a two-dimensional magnetic field and a spin 1/2 neutral particle with an anomalous magnetic moment in a two-dimensional electric field,
Author(s)	Ogurusu, O.
Citation	Hokkaido University Preprint Series in Mathematics, 187, 2-4
Issue Date	1993-03
DOI	10.14943/83331
Doc URL	http://hdl.handle.net/2115/68933
Type	bulletin (article)
File Information	pre187.pdf



[Instructions for use](#)

Unitary equivalence between a spin $1/2$
charged particle in a two-dimensional
magnetic field and a spin $1/2$ neutral
particle with an anomalous magnetic
moment in a two-dimensional
field electric

Osamu Ogurisu

Series #187. March 1993

HOKKAIDO UNIVERSITY
PREPRINT SERIES IN MATHEMATICS

- # 159: T. Hibi, Cohen-Macaulay types of Cohen-Macaulay complexes, 26 pages. 1992.
- # 160: A. Arai, Properties of the Dirac-Weyl operator with a strongly singular gauge potential, 26 pages. 1992.
- # 161: A. Arai, Dirac operators in Boson-Fermion Fock spaces and supersymmetric quantum field theory, 30 pages. 1992.
- # 162: S. Albeverio, K. Iwata, T. Kolsrud, Random parallel transport on surfaces of finite type, and relations to homotopy, 8 pages. 1992.
- # 163: S. Albeverio, K. Iwata, T. Kolsrud, Moments of random fields over a family of elliptic curves, and modular forms, 9 pages. 1992.
- # 164: Y. Giga, M. Sato, Neumann problem for singular degenerate parabolic equations, 12 pages. 1992.
- # 165: J. Wierzbicki, Y. Watatani, Commuting squares and relative entropy for two subfactors, 18 pages. 1992.
- # 166: Y. Okabe, A new algorithm driven from the view-point of the fluctuation-dissipation theorem in the theory of KM_2O -Langevin equations, 13 pages. 1992.
- # 167: Y. Okabe, H. Mano and Y. Itoh, Random collision model for interacting populations of two species and its strong law of large numbers, 14 pages. 1992.
- # 168: A. Inoue, On the equations of stationary processes with divergent diffusion coefficients, 25 pages. 1992.
- # 169: T. Ozawa, Remarks on quadratic nonlinear Schrödinger equations, 19 pages. 1992.
- # 170: T. Fukui, Y. Giga, Motion of a graph by nonsmooth weighted curvature, 11 pages. 1992.
- # 171: J. Inoue, T. Nakazi, Finite dimensional solution sets of extremal problems in H^1 , 10 pages. 1992.
- # 172: S. Izumiya, A characterization of complete integrability for partial differential equations of first order, 6 pages. 1992.
- # 173: T. Suwa, Unfoldings of codimension one complex analytic foliation singularities, 49 pages. 1992.
- # 174: T. Ozawa, Wave propagation in even dimensional spaces, 15 pages. 1992.
- # 175: S. Izumiya, Systems of Clairaut type, 7 pages. 1992.
- # 176: A. Hoshiga, The initial value problems for quasi-linear wave equations in two space dimensions with small data, 25 pages. 1992.
- # 177: K. Sugano, On bicommutators of modules over H-separable extension rings III, 9 pages. 1993.
- # 178: T. Nakazi, Toeplitz operators and weighted norm inequalities, 17 pages. 1993.
- # 179: O. Ogurusu, Existence and structure of infinitely degenerate zero-energy ground states of a Wess-Zumino type model in supersymmetric quantum mechanics, 26 pages. 1993.
- # 180: O. Ogurusu, Ground state of a spin 1/2 charged particle in an even dimensional magnetic field, 9 pages. 1993.
- # 181: K. Sugano, Note on H-separable Galois extension, 6 pages. 1993.
- # 182: M. Yamada, Distance formulas of asymptotic Toeplitz and Hankel operators, 13 pages. 1993.
- # 183: G. Ishikawa, T. Ozawa, The genus of a connected compact real algebraic surface in the affine three space, 11 pages. 1993.
- # 184: T. Hibi, Canonical modules and Cohen-Macaulay types of partially ordered sets, 6 pages. 1993.
- # 185: Y. Giga, K. Yama-uchi, On a lower bound for the extinction time of surfaces moved by mean curvature, 16 pages. 1993.
- # 186: Y. Kurokawa, On functional moduli for first order ordinary differential equations, 9 pages. 1993.

Unitary equivalence between a spin 1/2 charged particle
in a two-dimensional magnetic field and a spin 1/2 neutral particle
with an anomalous magnetic moment in a two-dimensional electric field

OSAMU OGURISU

Department of Mathematics, Hokkaido University
Sapporo 060, Japan

26, February 1993

Abstract. We prove the unitary equivalence between the Dirac Hamiltonian H_D for a relativistic spin 1/2 neutral particle with an anomalous magnetic moment in a two-dimensional electrostatic field $\mathbf{E} = (E_1, E_2)$ and the direct sum of the Dirac-Weyl operators $D(\pm\mathbf{A})$ for a spin 1/2 charged particle in two-dimensional magnetic fields $\pm d\mathbf{A}$ with the vector potential $\mathbf{A} = E_2 dx^1 - E_1 dx^2$, $(x^1, x^2) \in \mathbb{R}^2$. As applications, we investigate the ground state and the spectra of H_D .

1. Introduction

Recently V. V. Semenov [1] stated that a relativistic spin 1/2 neutral particle with an anomalous magnetic moment μ in a two-dimensional electrostatic field has $(N - 1)$ -fold degeneracy of the ground state, where the integer N is determined by μ and the total charge in the field. (In fact, the degeneracy of the ground state is equal to $2(N - 1)$. See Corollary 3.4 and Remark 3.5.) He pointed out that this phenomenon is similar to the phenomenon Y. Aharonov and A. Casher discussed in [2]; A non-relativistic spin 1/2 charged particle in a two-dimensional magnetic field has $(N - 1)$ -fold degeneracy of the ground state, where N is determined by the charge and the total magnetic flux.

In this Letter we clarify why the Dirac Hamiltonian H_D discussed by Semenov [1] has $2(N - 1)$ -fold degeneracy of the ground state: We prove that H_D is unitary equivalent to the direct sum of two Dirac-Weyl operators, each of which has $(N - 1)$ -fold degeneracy of the ground state. This is done in section 2. In section 3, we apply the result in section 2 to investigating the ground state and the spectra of H_D .

2. The unitary equivalence

We consider a quantum system of a relativistic spin 1/2 neutral particle with an anomalous magnetic moment $\mu \in \mathbb{R} \setminus \{0\}$ in a two-dimensional electrostatic field $\mathbf{E}(\mathbf{r}) = (E_1(\mathbf{r}), E_2(\mathbf{r}))$, $\mathbf{r} = (x^1, x^2) \in \mathbb{R}^2$, with $E_j \in C^\infty(\mathbb{R}^2 \rightarrow \mathbb{R})$, $j = 1, 2$. As usual, we denote $p_j = -i\partial/\partial x^j$, $j = 1, 2$, and the Pauli's spin matrices by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $m \geq 0$ be a constant denoting the mass of the particle. Define the Dirac Hamiltonian H_D [1, 3] by

$$H_D = \begin{pmatrix} m & A^* \\ A & -m \end{pmatrix}$$

acting in $L^2(\mathbb{R}^2; \mathbb{C}^4)$, where

$$A = \sum_{j=1,2} \sigma^j (p_j + i\mu E_j)$$

acting in $L^2(\mathbb{R}^2; \mathbb{C}^2)$. Since E_1 and E_2 are C^∞ -functions, it follows from Chernoff's theorem [4] that H_D is essentially selfadjoint on C_0^∞ , where C_0^∞ is $C_0^\infty(\mathbb{R}^2; \mathbb{C}^4)$. We denote the closure of $H_D \upharpoonright C_0^\infty$ by the same symbol.

Let $\mathbf{A} = A_1 dx^1 + A_2 dx^2$ on \mathbb{R}^2 be a real 1-form denoting a vector potential. The Dirac-Weyl operator $D(q\mathbf{A})$ for a spin 1/2 charged particle with charge q is given by

$$D(q\mathbf{A}) = \sum_{j=1,2} \sigma^j (p_j - qA_j)$$

acting in $L^2(\mathbb{R}^2; \mathbb{C}^2)$. If $A_j \in C^\infty(\mathbb{R}^2 \rightarrow \mathbb{R})$, $j = 1, 2$, then $D(q\mathbf{A})$ is essentially selfadjoint on $C_0^\infty(\mathbb{R}^2; \mathbb{C}^2)$ [4]. In this case we denote the closure of $D(q\mathbf{A}) \upharpoonright C_0^\infty(\mathbb{R}^2; \mathbb{C}^2)$ by the same symbol. The operator U on $L^2(\mathbb{R}^2; \mathbb{C}^4)$ given by

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

is unitary. Now, we can state the main theorem of this Letter.

Theorem 2.1. Define a 1-form $A = E_2 dx^1 - E_1 dx^2$. Then the operator equality

$$U^* H_D U = \begin{pmatrix} D(\mu A) + m\sigma^3 & 0 \\ 0 & D(-\mu A) - m\sigma^3 \end{pmatrix} \quad (2.1)$$

holds.

Proof. By direct computations, we can verify (2.1) on C_0^∞ . Since E_1 and E_2 are C^∞ -functions, the operator on the right hand side of (2.1) is essentially selfadjoint on C_0^∞ [4]. On the other hand, H_D is essentially selfadjoint on C_0^∞ and U is bijective on C_0^∞ . Therefore, (2.1) as an operator equality follows. \square

3. Applications

In this section, by employing known results for a spin 1/2 charged particle in a two-dimensional magnetic field, we obtain some corollaries of Theorem 2.1. We assume that there exists a function $\phi \in C^\infty(\mathbb{R}^2 \rightarrow \mathbb{R})$ such that

$$E_j = -\frac{\partial \phi}{\partial x^j}, \quad j = 1, 2.$$

Throughout this section, we put $A = E_2 dx^1 - E_1 dx^2$. We have $dA = \Delta \phi dx^1 \wedge dx^2$. Note that $-\Delta \phi / 4\pi$ is the charge density per unit volume. We denote by $\sigma(A)$ and $\sigma_e(A)$ the spectrum and the essentially spectrum of the operator A , respectively. We need the following lemma.

Lemma 3.1 (Ref. 5, Proposition 2.5). Let $H_j, j = 1, 2$, be Hilbert spaces, $S: H_1 \rightarrow H_2$ be a densely defined closed linear operator and m be a non-negative constant. Let

$$T = \begin{pmatrix} m & S^* \\ S & -m \end{pmatrix}$$

acting in $H_1 \oplus H_2$. Then

$$\begin{aligned} \sigma(T) &= \{\sqrt{m^2 + s}; s \in \sigma(S^*S)\} \cup \{-\sqrt{m^2 + s}; s \in \sigma(SS^*)\}, \\ \sigma_e(T) &= \{\sqrt{m^2 + s}; s \in \sigma_e(S^*S)\} \cup \{-\sqrt{m^2 + s}; s \in \sigma_e(SS^*)\}. \end{aligned}$$

We remark that, in general, $\sigma(SS^*) \setminus \{0\} = \sigma(S^*S) \setminus \{0\}$ and $\sigma_e(SS^*) \setminus \{0\} = \sigma_e(S^*S) \setminus \{0\}$ (see [6]). With the aid of Lemma 3.1, we can prove the following:

Corollary 3.2. For simplicity, we put $\mu = 1$.

- (i) If $\Delta \phi(\mathbf{r}) \rightarrow 0$ as $|\mathbf{r}| \rightarrow \infty$, then $\sigma_e(H_D) = (-\infty, -m] \cup [m, \infty)$;
- (ii) If $\Delta \phi(\mathbf{r}) \rightarrow 1$ as $|\mathbf{r}| \rightarrow \infty$, then $\sigma_e(H_D) = \{\pm\sqrt{2k + m^2}; k \in \mathbb{N}\} \cup \{m\}$ and m is isolated;
- (iii) If $\Delta \phi(\mathbf{r}) \rightarrow \infty$ as $|\mathbf{r}| \rightarrow \infty$, then $\sigma(H_D)$ is discrete except for m and m is an isolated point of the essential spectrum.

Proof. First, we treat the case (ii). From Example 4.1 and the proof of it in [5], we see that $\sigma_e(D(\pm A)^2) = \{2(k-1); k \in \mathbb{N}\}$, 0 is isolated and $\ker D(\pm A) \subset \ker(\sigma^3 \mp 1)$. Hence,

applying Lemma 3.1 to $D(\pm A) \pm m\sigma^3$, we obtain that $\sigma_\epsilon(D(\pm A) \pm m\sigma^3) = \{+\sqrt{2k+m^2}, -\sqrt{2k+m^2}; k \in \mathbb{N}\} \cup \{m\}$ and m is isolated. Thus, Theorem 2.1 implies the desired result. In the cases (i) and (iii), we can obtain the desired results in a way similar to the case of (ii). \square

Remark 3.3. Corollary 3.2 gives us a classification of the spectrum of H_D by the asymptotic behavior of the charge density per unit volume at infinity. (cf. [5].)

We next investigate the ground state of H_D . We put $\epsilon(x) = 1, x \geq 0, \epsilon(x) = -1, x < 0$.

Corollary 3.4. *Suppose that the limit $\nu = \lim_{|r| \rightarrow \infty} \phi(r)/\log|r|$ exists. Let N be the largest integer strictly less than $|\mu\nu|$. Then*

$$\dim \ker(H_D^2 - m^2) = \max\{2(N-1), 0\} \quad \text{and} \quad \ker(H_D^2 - m^2) = \ker(H_D - \epsilon(\mu\nu)m).$$

Proof. In general, for a selfadjoint operator T , $\ker T^2 = \ker T$. By Theorem 3.1 and Corollary 3.2 in [7], we can prove that $\dim \ker D(\pm\mu A) = \max\{N-1, 0\}$ and $\ker D(\pm\mu A) \subset \ker(\sigma^3 \mp \epsilon(\mu\nu))$. Hence, by Theorem 2.1, we can obtain the desired results. \square

Remark 3.5. Semenov [1] also considered the ground state of H_D . But, counting up the ground state, he seems to have neglected spin components and so reached the wrong conclusion: "this particle has $(N-1)$ -fold degeneracy of the ground state".

Remark 3.6. Note that the number N is determined by the asymptotic behavior at $|r| = \infty$ of the scalar potential $\phi(r)$.

Acknowledgement

The author would like to thank Professor Asao Arai for helpful conversations and encouragement.

References

1. SEMENOV, V. V., Ground state of a spin 1/2 neutral particle with an anomalous magnetic moment in a two-dimensional electrostatic field, *J. Phys. A: Math. Gen.* **25** (1992), L619–L621.
2. AHARONOV, Y. AND CASHER, A., Ground state of a spin 1/2 charged particle in a two-dimensional magnetic field, *Phys. Rev. A* **19** (1979), 2461–2462.
3. PAULI, W. J., Relativistic Field Theories of Elementary Particles, *Rev. Mod. Phys.* **13** (1941), 203–232.
4. CHERNOFF, P. R., Essentially self-adjointness of powers of generators of hyperbolic equations, *J. Funct. Anal.* **12** (1973), 401–414.
5. SHIGEKAWA, I., Spectral properties of Schrödinger operators with magnetic fields for a spin 1/2 particle, *J. Funct. Anal.* **101** (1991), 255–285.
6. DEIFT, P., Applications of a commutation formula, *Duck Math. J.* **45** (1978), 267–310.
7. OGURISU, O., Ground state of a spin 1/2 charged particle in an even dimensional magnetic field, *submitted to Lett. Math. Phys.*