Multipliers Of Invariant Subspaces In The Bidisc

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Multipliers Of Invariant Subspaces In The Bidisc

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Abstract. For any nonzero invariant subspace M in $H^2(T^2)$, set $M^{\times} = [\bigcup_{n=0}^{\infty} \bar{z}^n M] \cap [\bigcup_{n=0}^{\infty} \bar{w}^n M]$ then $M^{\times}$ is also an invariant subspace of $H^2(T^2)$ that contains M. If M is of finite codimension in $H^2(T^2)$ then $M^{\times} = H^2(T^2)$ and if $M = qH^2(T^2)$ for some inner function q then $M^{\times} = M$. In this paper invariant subspaces with $M^{\times} = M$ are studied. If $M = q_1 H^2(T^2) \cap q_2 H^2(T^2)$ and q_1, q_2 are inner functions then $M^{\times} = M$. However in general this invariant subspace may not be of the form : $qH^2(T^2)$ for some inner function q. Put $\mathcal{M}(M) = \{\phi \in L^{\infty} : \phi M \subseteq H^2(T^2)\}$ then $\mathcal{M}(M)$ is described and $\mathcal{M}(M) = \mathcal{M}(M^{\times})$ is shown. This is the set of all multipliers of M in the title. A necessary and sufficient condition for $\mathcal{M}(M) = H^{\infty}(T^2)$ is given. It is noted that the kernel of a Hankel operator is an invariant subspace M with $M^{\times} = M$. The argument applies to the polydisc case.

§1. Introduction

Let T^2 be the torus that is the Cartesian product of 2 unit circles in $\mathbb{C}$. The usual Lebesgue spaces, with respect to the Haar measure m of T^2, are denoted by $L^p = L^p(T^2)$, and the Hardy spaces $H^p = H^p(T^2)$ are spaces of all $f \in L^p(T^n)$ whose Fourier coefficients

$$\hat{f}(j,\ell) = \int_{T^2} f(z,w)\bar{z}^j \bar{w}^\ell dm(z,w)$$

are 0 as soon as at least one component of (j,ℓ) is negative where $1 \leqq p \leqq \infty$. Let U^2 be the unit bidisc that is the Cartesian product of 2 open unit discs in $\mathbb{C}$. Any function f in H^p has an analytic extension to U^2 which is also denoted by f.

A closed subspace M of H^2 is said to be invariant if

$$zM \subset M \text{ and } wM \subset M.$$

Put $M_1 = [\bigcup_{n=0}^{\infty} \bar{z}^n M]$ and $M_2 = [\bigcup_{n=0}^{\infty} \bar{w}^n M]$ where $[\bigcup_{n=0}^{\infty} \bar{z}^n M]$ is the closed linear span of $\bigcup_{n=0}^{\infty} \bar{z}^n M$ in L^2. Set

$$M^{\times} = M_1 \cap M_2$$

then M is also an invariant subspace of H^2. Put

$$H_1 = [\bigcup_{n=0}^{\infty} \bar{z}^n H^2] \quad \text{and } H_2 = [\bigcup_{n=0}^{\infty} \bar{w}^n H^2]$$

then $H^2 = H_1 \cap H_2$ and hence $(H^2)^{\times} = H^2$. Therefore it is desirable to know invariant subspaces M which have the following property : $M^{\times} = H^2$ or $M^{\times} = M$. An invariant subspace M of H^2 has full range if $M_1 = H_1$ and

$M_2 = H_2$. Such an invariant subspace has been studied by Agrawal, Clark and Douglas [2]. It is clear that M has full range if and only if $M^\times = H^2$. An invariant subspace M with $M^\times = M$ has not been studied. In this paper we study invariant subspaces with $M^\times = M$. For an invariant subspace M of H^2 set

$$\mathfrak{M}(M) = \{\phi \in L^\infty : \phi M \subseteq H^2\}.$$

It is essentially known [2] that if $M^\times = M$ then $\mathfrak{M}(M) = H^\infty$. In this paper we give a necessary and sufficient condition for $\mathfrak{M}(M) = H^\infty$.

Let K_0^2 denote the orthogonal complement of $\bar{H}^2 = \{\bar{f} : f \in H^2\}$ in L^2. The invariant subspace qH^2 for an inner function q is called a Beurling subspace.

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§ 2. Intersection of Beurling subspaces

A Beurling subspace M satisfies $M^\times = M$ trivially. In this section we show that the intersections of Beurling subspaces have this property.

Proposition 1. If M is a nonzero invariant subspace of H^2 then there exist two unimodular functions Q_1 in H_1 and Q_2 in H_2 such that $M_1 = Q_1 H_1$ and $M_2 = Q_2 H_2$, and hence

$$M^{\times} = Q_1 H_1 \cap Q_2 H_2.$$

Proof. By [7 , p164 ~ p165] , $[\bigcup_{n=0}^{\infty} \bar{z}^n M] = \chi_E Q_1 H_1 + (1 - \chi_E) L^2$ where χ_E is a characteristic function of some measurable set E in T^2, $\chi_E \in H_1$ and $|Q_1| = 1$ a.e.. Since $M \subset H^2$ and H_1 has no reducing invariant subspaces under the multiplication of w, this implies that $M_1 = Q_1 H_1$ and $Q_1 \in H_1$ because $M_1 \subset H_1$. The same argument shows that $M_2 = Q_2 H_2$ and $Q_2 \in H_2$.

Theorem 2. Let ℓ be a finite positive integer. If $M = \bigcap_{j=1}^{\ell} q_j H^2$ and $\{q_j\}_{j=1}^{\ell}$ are inner functions then $M^{\times} = M$. Moreover

$$M^{\times} = Q_1 H_1 \cap Q_2 H_2$$

where $\{Q_i\}_{i=1}^{2}$ are unimodular functions in $\{H_i\}_{i=1}^{2}$ and

$$M_i = Q_i H_i = \bigcap_{j=1}^{\ell} q_j H_i \qquad (i = 1, 2).$$

Proof. Let $N = \bar{M}^{\perp}$, that is, N is the orthogonal complement of $\bar{M} = \{\bar{f} \ ; \ f \in M\}$. Then $N = [\sum_{j=1}^{n} q_j K_0^2]$ because $(\bar{H}^2)^{\perp} = K_0^2$. Put $N_1 = [\sum_{j=1}^{\ell} q_j w H_1]$ and $N_2 = [\sum_{j=1}^{\ell} q_j z H_2]$ then $\{N_i\}_{i=1}^{2}$ are invariant subspaces and $N = [N_1 + N_2]$ because $K_0^2 = wH_1 + zH_2$. It is clear that $zN_1 = N_1$ and $wN_2 = N_2$. If $wN_1 = N_1$ then $N_1 \supset \bar{q}_j \bar{w}^n H_1$ for any positive n and so $N_1 = L^2$. This contradicts that $M \supset (\prod_{j=1}^{n} q_j) H^2$. Thus $wN_1 \neq N_1$ and similarly $zN_2 \neq N_2$. By [7] there exist unimodular functions Q_1 and Q_2 such that $N_1 = Q_1 w H_1$ and $N_2 = Q_2 z H_2$. Since $N = [N_1 + N_2]$,

$$M = Q_1 H_1 \cap Q_2 H_2.$$

Since $Q_i H_i$ is the orthogonal complement of N_i and $N_i = (\bar{M}_i)^{\perp}$ for $i = 1, 2$,

$$M_i = Q_i H_i = \bigcap_{j=1}^{n} q_j H_i \quad (i = 1, 2).$$

In the case of one variable an intersection of two Beurling subspaces is also a Beurling subspace. This is not true in the case of two variables by [11, Theorem 2 and its proof]. In fact X_1 and X_2 in [11, Theorem 2] are intersections of two Beurling subspaces and $X_j \subsetneq H^2$ for $j = 1, 2$. If $X_1 = q_1 H^2$ and $X_2 = q_2 H^2$ for some inner functions q_1, q_2 then $\bar{q}_1 X_1 = \bar{q}_2 X_2 = H^2$. This contradicts [11, Theorem 2]. Hence our Theorem 2 is not trivial. If an invariant subspace M is determined by vanishing conditions at finitely many points of U then $M^{\times} = M$ because M is a finite co-dimensional subspace of H^2. We are interested in an invariant subspace determined by vanishing conditions at infinitely many points of U. Let s be an analytic function on U such that $s(U) \subset U$. Put

$$M_s = \{f \in H^2 : f(z, s(z)) = 0 \quad \text{for all } z \in U\}$$

then M_s is an invariant subspace of H^2.

Proposition 3. Let s be an analytic function on U and $s(U) \subset U$. Then

$$M_s^{\times} = QH_1 \cap H_2$$

where $Q = \dfrac{w - s}{1 - \bar{s}w}$. (1) If $|s| = 1$ a.e. on T then $M_s^{\times} = H^2$. (2) If $|s| = r < 1$ a.e. on T for some constant r then

$$M_s^{\times} = M_s = Q' H_1 \cap H_2 = Q' H^2 \cap H^2$$

where $Q' = \dfrac{w - s}{sr^{-1} - rw}$.

Proof. If $f \in M_s$ then for a.e. $z \in T$ $\dfrac{1 - sw}{w - s} f(z, w)$ is w-analytic and hence f belongs to QH_1. Therefore $M_{s,1} \subset QH_1$. Since $(1 - rsw)^{-1} \in H_1$ for any constant r with $0 < r < 1$ and $w - s \in M_s$, $(w - s)(1 - rsw)^{-1}$ belongs to $M_{s,1}$. Since

$$\left| \frac{w - s}{1 - r\bar{s}w} - \frac{w - s}{1 - \bar{s}w} \right| \leq \left| \frac{1 - r}{1 - r\bar{s}w} \right| \rightarrow 0 \quad (\text{as } r \rightarrow 1)$$

because $m\{(z, w) \in T^2 : \bar{s}(z)w = 1\} = 0$, Q belongs to $M_{s,1}$ and hence $M_{s,1} = QH_1$. Since $(w - rs)^{-1} \in H_2$ for any r with $0 < r < 1$ and $w - s \in M_s$, $(w - s)(w - rs)^{-1} \in M_{s,2}$. As $r \rightarrow 1$ the constant 1 belongs to $M_{s,2}$ and hence $M_{s,2} = H_2$. This implies $M_s^{\times} = QH_1 \cap H_2$. (1) is clear because $sH_1 = H_1$. (2) Since $\dfrac{w - s}{1 - sw} = \dfrac{sr^{-1}(w - s)}{sr^{-1} - rw}$, $M_s = Q' H_1 \cap H_2$. Put $N = Q' H^2 \cap H^2$. If $Q' f = g$ for some f and g in H^2 then

$$(w - s)f(z, w) = (sr^{-1} - rw)g(z, w) \qquad (z, w) \in U^2.$$

Thus g belongs to M_s and hence $M_s \supset N$. It is clear that $N_1 \subset Q' H_1$ and $N_2 \subset H_2$. The Q' belongs to N_1 because $(sr^{-1} - rw)^{-1} \in H_1$ and $w - s \in N$. The constant 1 belongs to N_2 because $(w - s)^{-1} \in H_2$ and $w - s \in N$. Thus $N = Q' H_1 \cap H_2$. Since $N^{\times} = N$ by the proof of Theorem 2, $M_s^{\times} = M_s = Q' H_1 \cap H_2 = Q' H^2 \cap H^2$.

§3. Multipliers of an invariant subspace

An invariant subspace M of H^2 is said to be podal if every invariant subspace N of H^2 is unitarily equivalent to M which is a subspace of M (cf. [3]). Agrawal, Clark and Douglas [2] showed that if an invariant subspace of H^2 has full range then it is podal. The following is a generalization of that. For if N is full range then $N^{\times} = H^2$.

Proposition 4. Let M and N be invariant subspaces of H^2 with $M \subset N^{\times}$. If M is unitarily equivalent to N, then $M \subset N$.

Proof. By [2, Lemma 1] $M = qN$ for some unimodular function q. Then $M^{\times} = qN^{\times}$ and $M^{\times} \subset N^{\times}$. By [2, Proposition 3] the q is an inner function.

Proposition 5. Let M and N be invariant subspaces of H^2 with $M^{\times} = N^{\times}$. Then M is unitarily equivalent to N if and only if $M = N$.

Proof. If M is unitarily equivalent to N then $M = qN$ for some unimodular function q. Then $M^{\times} = qN^{\times}$ and hence $M^{\times} = qM^{\times}$. Thus q is an inner function. By the same argument $N^{\times} = \bar{q}N^{\times}$ and hence q is also an inner function. Thus q is constant and $M = N$.

If M and N are invariant subspaces of finite codimension in H^2, then we can show $M^{\times} = N^{\times}$ and hence by Proposition 5 $M = N$. This is Corollary 3 in [2] (see [5, Corollary 6]). Douglas and Yan [3] gave the following question : Can one characterize podal invariant subspaces ?

Proposition 6. Let M be a nonzero invariant subspace of H^2. M is podal if and only if any unimodular functions in $\mathfrak{M}(M)$ belong to H^{∞}.

Proof. If ϕ is a unimodular function in $\mathfrak{M}(M)$ then $\phi M \subset H^2$ and ϕM is unitarily equivalent to M. If M is podal then $\phi M \subset M$ and hence $\phi \in H^{\infty}$ by [2, Proposition 3]. Suppose any unimodular functions in $\mathfrak{M}(M)$ belong to H^{∞}. If M is unitarily equivalent to N which is an invariant subspace in H^2 then $N = \phi M$ for some unimodular function ϕ. Since $\phi \in \mathfrak{M}(M)$, by the hypothesis ϕ belongs to H^{∞} and so M is podal.

By the proposition above, if $\mathfrak{M}(M) = H^{\infty}$ then M is podal. We will characterize an invariant subspace M with $\mathfrak{M}(M) = H^{\infty}$. In general $\mathfrak{M}(M)$ is an invariant subspace of L^{∞} which contains H^{∞}. The structure of $\mathfrak{M}(M)$ is simpler than that of M.

Theorem 7. If M is a nonzero invariant subspace of H^2 then

$$\mathcal{M}(M) = \bar{Q}_1 H_1 \cap \bar{Q}_2 H_2 \cap L^\infty$$

where $M^\times = Q_1 H_1 \cap Q_2 H_2$ and Q_i is a unimodular function in H_i for $i = 1, 2$. Hence $\mathcal{M}(M) = H^\infty$ if and only if

$$\bar{Q}_1 H_1 \cap \bar{Q}_2 H_2 \cap L^\infty = H^\infty.$$

Proof. If $\phi \in \mathcal{M}(M)$ then $\phi M \subset H^2$. Hence for $i = 1, 2$

$$\phi M_i \subset H_i \quad \text{and} \quad M_i = Q_i H_i$$

where Q_i is a unimodular function in H_i. Thus $\phi \in \bar{Q}_1 H_1 \cap \bar{Q}_2 H_2 \cap L^\infty$ and hence $\mathcal{M}(M) \subset \bar{Q}_1 H_1 \cap \bar{Q}_2 H_2 \cap L^\infty$. Conversely let ϕ be in $\bar{Q}_1 H_1 \cap \bar{Q}_2 H_2 \cap L^\infty$ then

$$\phi = \bar{Q}_1 \phi_1 = \bar{Q}_2 \phi_2$$

where ϕ_i is in H_i for $i = 1, 2$. If $f \in M$ then $f \in M^\times$ and hence

$$f = Q_1 f_1 = Q_2 f_2$$

where f_i is in H_i for $i = 1, 2$. Therefore $\phi f = \phi_1 f_1 = \phi_2 f_2$. This implies that $\phi f \in H_1 \cap H_2 = H^2$ and hence $\phi \in \mathcal{M}(M)$. Thus $\bar{Q}_1 H_1 \cap \bar{Q}_2 H_2 \cap L^\infty \subset \mathcal{M}(M)$. This implies the theorem.

For $\phi \in L^\infty$, the Hankel operator determined by ϕ is

$$H_\phi = (I - P) M_\phi | H^2$$

where P is the orthogonal projection from L^2 to H^2.

Proposition 8. The kernel of a Hankel operator is an invariant subspace M with $M^{\times} = M$.

Proof. Suppose H_{ϕ} is the Hankel operator defined by $\phi \in L^{\infty}$ and let M be its kernel. It is clear that M is an invariant subspace. Since M is the kernel of H_{ϕ}, $\phi M \subset H^2$. By Theorem 7, $\mathfrak{m}(M) = \mathfrak{m}(M^{\times})$ and $\phi M^{\times} \subset H^2$ This implies $M = M^{\times}$ because $M = \{\phi f \in H^2 : f \in H^2\}$.

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