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BLOW-UP OF SOLUTIONS TO SEMILINEAR WAVE EQUATIONS WITH INITIAL DATA OF SLOW DECAY IN LOW SPACE DIMENSIONS

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BLOW-UP OF SOLUTIONS TO SEMILINEAR WAVE EQUATIONS WITH INITIAL DATA OF SLOW DECAY IN LOW SPACE DIMENSIONS

By

Hideo Kubo

§1 Introduction and Statement of Results

This paper is concerned with finite-time blow-up of solutions to the following Cauchy problems:

$$
(1.1) \qquad \begin{cases} u_{tt} - \Delta u = \phi(u_t) & \text{in} \quad \mathbb{R}^n \times [0,\infty), \\ u(x,0) = \varepsilon f(x), \quad u_t(x,0) = \varepsilon g(x) & \text{for} \quad x \in \mathbb{R}^n, \end{cases}
$$

where $u(r,t)$ is a scalar unknown, $n = 1,2,3$, $\phi(\lambda) \in C^2(\mathbb{R})$, $f(x) \in C^3(\mathbb{R}^n)$, $g(x) \in C^2(\mathbb{R}^n)$ and ε is a small parameter. We also assume that there exists a positive number $p > 1$ such that

$$
(1.2) \qquad \phi(\lambda) \geq 2A|\lambda|^p \quad \text{for} \quad \lambda \in \mathbb{R},
$$

where A is a positive constant.

Concerning the problem (1.1), there are many existence and nonexistence results proved in literature [1], [4]-[13], where it is studied that the power of nonlinear term have an effect on a behavior of a classical solution of the problem (1.1). More precisely, there exists a critical value $p(n)$ such that if $1 < p \leq p(n)$, nontrivial classical solutions of the problem (1.1) blow-up provided that the initial data are positive in certain sense and if $p > p(n)$, the problem (1.1) with $\phi(\lambda) = |\lambda|^p$ admits a unique global solution with small initial data. The critical value is $p(n) = (n+1)/(n-1)$, $n = 2,3$. In this present paper, our main interest is effect of decay of the initial data as $|x| \to \infty$. It will be proved that nontrivial classical solutions of the problem (1.1) blow-up for any $p > 1$ under slow decay condition imposed on the initial data, for instance, $f \equiv 0$, $g \equiv 1$ in $\mathbb{R}^n$. (See also Theorem 1.1 below.)

First we consider the case where the initial data are compactly supported. In three space dimensions, the blow-up part for $p = p(3) = 2$ is shown by F. John [5]. T. Sideris [11] proved that this result is also valid for $1 < p < p(3)$ for radial symmetric case. This blow-up result was extended to one and two space dimensions by K. Masuda [8] for the case $p = 2$. In two space dimensions, J. Schaeffer [10] established blow-up result for the critical value $p = p(2) = 3$. R. Agemi [1] also proved for the case $1 < p \leq p(2)$. P. Godin [4] obtained the upper bound for the lifespan for $p = 2, 3$, where lifespan mean the largest T such that classical solutions of the problem (1.1) exist. On the other hand, in three space dimensions, the existence part is established by T. C. Sideris [11] and S. Klainerman [6], [7], independently, for sufficiently small initial data of fast decay. In two space dimensions, P. Godin [4] also proved the similar result. In this paper, we shall present an elementary proof for the blow-up results which is done by simple modification of that of Theorem 1.1. More precisely, we shall prove that for $1 < p \leq p(n)$ ($p(1) = \infty$), nontrivial classical soltions of the problem (1.1) blow-up in finite-time, if

$$f \equiv 0, \quad g(x) \geq 0, \ g(x) \not\equiv 0 \quad \text{for} \quad |x| \geq 1. \tag{1.3}$$

Moreover, we shall obtain upper bound for the lifespan in Theorems 1.2 and 1.3.

Next we consider the case where the initial data are slowly decaying. We start our consideration from a remarkable result established by F. Asakura [3]. In [3], he treated a initial value problem for the equation:

$$u_{tt} - \Delta u = \phi(u) \quad \text{in} \quad \mathbb{R}^3 \times [0, \infty) \tag{1.4}$$

and obtained blow-up result for the initial data of slow decay, even if p is sufficiently large. In two space dimensions, similar result was shown by R. Agemi & H. Takamura [2] and K. Tsutaya [14], independently.

We shall establish blow-up result for the problem (1.1) similar to Asakura [3], even for $p > p(n)$. To do this, for the initial data we assume

$$f \equiv 0, \quad g(x) \geq 2|x|^{-\kappa} \quad \text{for} \quad |x| \geq 1, \tag{1.5}$$

where κ is a positive number. In such framework, F. John [5] obtained the the following estimate of the lifespan T_ε from below for radial symmetric case, when $n = 3$, $p = p(3) = 2$ and $\kappa = 1$: There exists a positive number ε_0 such that

$$(1.6) \qquad T_\varepsilon \geq \frac{1}{2} exp(C/\varepsilon) \quad \text{for} \quad 0 < \varepsilon \leq \varepsilon_0,$$

where C is a numerical constant. Moreover, if p is an integer with $p > 2$, H. Takamura [12] proved that $T_\varepsilon = \infty$ for radial symmetric case, provided $\kappa = 1$ and ε is small.

We shall obtain the upper bound for the lifespan by converting the partial differential equation (1.1) into a first order ordinary differential equation along some characteristic ray (2.6) below. The main result of this paper is as follows.

THEOREM 1.1. *Suppose that (1.2), (1.5) hold. If* $\kappa < \dfrac{1}{p-1}$, *then there exists a positive number* $\varepsilon_0 = \varepsilon_0(n, A, p, \kappa)$ *such that*

$$(1.7) \qquad T_\varepsilon \leq C_1 \varepsilon^{-(p-1)/(1-\kappa(p-1))} \quad \text{for} \quad 0 < \varepsilon \leq \varepsilon_0,$$

where $C_1 = C_1(n, A, p, \kappa)$ *is a positive constant.*

REMARK. *The above result is sharp for* $p > p(3)$, *because the author recently obtained a global radial symmetric* C^2*-solution of the problem (1.1) with* $n = 3$ *provided that for* $\kappa > \dfrac{1}{p-1}$

$$|g(r)| < r >^\kappa + \sum_{j=1}^{2} |(\frac{d}{dr})^j g(r)| < r >^{\kappa+1} \leq 1,$$

where $< r > = \sqrt{1+r^2}$. *The proof will be published elsewhere. Moreover, H.Takamura [13] recently obtained the above result in four and five space dimensions. As for the compactly supported initial data with some positivity, M. A. Rammaha [9] had proved that radial symmetric solutions blow-up provided that* $p = p(n)$ *for odd space dimensional case and that* $1 < p \leq p(n)$ *for even space dimensional case.*

When $1 < p < p(n)$, however, we have the following estimate which is better than the estimate (1.7) for the case $\kappa > \dfrac{n-1}{2}$. In fact:

THEOREM 1.2. *Let $1 < p < p(n)$. Suppose that (1.2), (1.3) hold. Then there exists a positive number $\varepsilon_0 = \varepsilon_0(n, A, p, \kappa)$ such that*

$$T_\varepsilon \leq C_2 \varepsilon^{-(p-1)/(1-m(p-1))}, \quad m = \frac{n-1}{2}. \tag{1.8}$$

where $C_2 = C_2(n, A, p)$ is a positive constant.

For the critical case, we have

THEOREM 1.3. *Let $p = p(n)$. Suppose that (1.2), (1.3) hold. Then there exists a positive number $\varepsilon_0 = \varepsilon_0(n, A, p, \kappa)$ such that*

$$T_\varepsilon \leq exp(C_3/\varepsilon^{p-1}), \tag{1.9}$$

where $C_3 = C_3(n, A, p)$ is a positive constant.

REMARK. *The above estimate is weaker than the estimate (1.7) for the case $\kappa < \dfrac{n-1}{2}$.*

§2 Proof of Theorem 1.1

We associate a function $h \in C^0(\mathbb{R}^n)$ with its weighted spherical means on the unit sphere S^{n-1} about the origin: For $n = 2, 3$,

$$\overline{h}(r) = \frac{r^{(n-1)/2}}{\omega_n} \int_{S^{n-1}} h(r\omega) dS_\omega,$$

where ω_n denotes the surface area of S^{n-1} and dS_ω its surface element. When $n = 1$, we set $\overline{h}(r) = \dfrac{1}{2}(h(r) + h(-r))$.

We shall use the following representation of a solution to the problem (1.1). It is due to K. Masuda [8].

LEMMA 2.1. *Let $u(x, t)$ be a global C^2-solution to the problem (1.1). Suppose that (1.2), (1.3) holds. Then $\overline{u}_t(r, t) \geq 0$ for $r \geq t + 1$. Furthermore, we have*

$$\overline{u}_t(r, t) \geq \frac{1}{2}\varepsilon\, \overline{g}(r - t) + \frac{1}{2}\int_0^t \overline{\phi(u_t)}(r - t + \tau, \tau) d\tau \quad \text{for} \quad r \geq t + 1. \tag{2.1}$$

From (1.5), we have

$$\overline{g}(r-t) \geq 2(r-t)^{m-\kappa} \quad \text{for} \quad r \geq t+1. \tag{2.2}$$

where we have set $m = (n-1)/2$. On the other hand, it follows from (1.2) and Lemma 2.1 that

$$\overline{\phi(u_t)}(\lambda, \tau) \geq \frac{2A\lambda^m}{\omega_n} \int_{S^{n-1}} u_t^p(\lambda\omega, \tau) dS_\omega \geq 2A\lambda^{m(1-p)} \overline{u}_t^p(\lambda, \tau) \quad \text{for} \quad \lambda \geq \tau + 1, \tag{2.3}$$

where last inequality is deduced by Jensen's inequality. Substituting the estimates (2.2), (2.3) into (2.1), we obtain for $r \geq t+1$,

$$\overline{u}_t(r,t) \geq \varepsilon(r-t)^{m-\kappa} + A\int_0^t (r-t+\tau)^{m(1-p)} \overline{u}_t^p(r-t+\tau, \tau) d\tau. \tag{2.4}$$

We set $\rho = r - t$ with $\rho \geq 1$ and $w(t) = \overline{u}_t(t+\rho, t)$. Then we have from (2.4)

$$w(t) \geq \varepsilon\rho^{m-\kappa} + A\int_0^t (\rho+\tau)^{m(1-p)} w^p(\tau) d\tau. \tag{2.5}$$

In what follows, we fix ρ and will be chosen appropriately.

We now introduce the following ordinary differential equation:

$$\begin{cases} z'(t) = A(t+\rho)^{m(1-p)} z^p(t) \quad \text{for} \quad t > 0, \\ z(0) = \varepsilon\rho^{m-k}. \end{cases} \tag{2.6}$$

The explicit form of $z(t)$ is as follows:

$$z^{1-p}(t) = z^{1-p}(0)F(t), \tag{2.7}$$

with

$$F(t) = 1 - A(p-1)z^{p-1}(0)\int_0^t (\tau+\rho)^{-m(p-1)} d\tau.$$

Note that $w(t) \geq 0$ for $t \geq 0$ and and $z(t) \geq 0$ whenever $z(t)$ exists. Applying usual comparison theorem, it follows from (2.5), (2.6) that $w(t) \geq z(t)$ whenever $z(t)$ exists. Note that $F'(t) < 0$ and $F(0) = 1$.

First we consider the case $m(p-1) \neq 1$. Direct computation yields

$$F(\rho) = 1 - B_1\varepsilon^{p-1}\rho^{1-\kappa(p-1)} \quad \text{for} \quad m(p-1) \neq 1$$

where we have set $B_1 = A(p-1)(2^{1-m(p-1)}-1))/(1-m(p-1)) > 0$.

Let ε_0 be the least upper bound for ε satisfying

$$\tilde{B}_1\varepsilon^{-(p-1)/(1-\kappa(p-1))} \geq 1,$$

where we have set $\tilde{B}_1 = B_1^{-1/(1-\kappa(p-1))}$. We take

$$\rho_1 = \tilde{B}_1\varepsilon^{-(p-1)/(1-\kappa(p-1))} \quad \text{for} \quad 0 < \varepsilon \leq \varepsilon_0.$$

Then we have $F(\rho_1) = 0$. Therefore $T_\varepsilon \leq \rho_1$, hence (1.7) holds when $m(p-1) \neq 1$.

Next we consider the case $m(p-1) = 1$, *i.e.*, $p = p(n)$. Then we have

$$F(\rho) = 1 - B_2\varepsilon^{p-1}\rho^{(m-\kappa)(p-1)}.$$

where we have set $B_2 = A(p-1)\log 2$.

Let ε_0 be the least upper bound for ε satisfying

$$\tilde{B}_2\varepsilon^{-(p-1)/(1-\kappa(p-1))} \geq 1,$$

where we have set $\tilde{B}_2 = B_2^{-(p-1)/(1-\kappa(p-1))}$. We take

$$\rho_2 = \tilde{B}_2\varepsilon^{-(p-1)/(1-\kappa(p-1))} \quad \text{for} \quad 0 < \varepsilon \leq \varepsilon_0.$$

Then we have $F(\rho_2) = 0$. Therefor we obtain $T_\varepsilon \leq \rho_2$ which implies (1.7) for $p = p(n)$. Thus we have proved Theorem 1.1.

§3 Proof of Theorems 1.2 and 1.3

From (1.3), we may assume that there exists $x_1 \in \mathbb{R}^n$ with $|x_1| = 1$ such that $g(x_1) \neq 0$, hence $\overline{g}(1)$ is a positive constant depending only on g. We denote it $2C(g)$. Applying Lemma 2.2 with $r - t = 1$ and employing the estimate (2.3), we obtain an analogue to (2.5);

$$w(t) \geq C(g)\varepsilon + A\int_0^t (\tau+1)^{m(1-p)}w^p(\tau)d\tau, \tag{3.1}$$

where we have set $w(t) = \overline{u}_t(t+1,t)$. Similarly to the proof of Theorem 1.1, it is enough to study the behavior of a solution to the following ordinary differential equation:

$$(3.2) \qquad \begin{cases} z'(t) = A(t+1)^{m(1-p)} z^p(t) \quad \text{for} \quad t > 0, \\ z(0) = C(g)\,\varepsilon. \end{cases}$$

Note that $z(t)$ represents (2.7) with $\rho = 1$.

PROOF OF THEOREM 1.2: We note that $1 < p < p(n)$ implies $0 \leq m(p-1) < 1$. Set

$$t_1 = (1 + 1/A')^{1/(1-m(p-1))} - 1 \quad \text{with} \quad A' = B\varepsilon^{p-1},\ B = A(p-1)/(1-m(p-1)).$$

Then $F(t_1) = 0$. Therefore we have $T_\varepsilon \leq t_1$, hence (1.8) holds.

PROOF OF THEOREM 1.3: We note that $p = p(n)$ implies $m(p-1) = 1$. Set $t_2 = exp(1/(A(p-1)\varepsilon^{p-1})) - 1$. Then $F(t_2) = 0$. Therefore we have $T_\varepsilon \leq t_2$, hence (1.9) holds.

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