ASYMPTOTIC BEHAVIORS OF SOLUTIONS TO SEMILINEAR WAVE EQUATIONS WITH INITIAL DATA OF SLOW DECAY

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ASYMPTOTIC BEHAVIOUR OF SOLUTIONS TO SEMILINEAR WAVE EQUATIONS WITH INITIAL DATA OF SLOW DECAY

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Some useful and remarkable property are derived from a representation formula of a radially symmetric solution to the Cauchy problem for a homogeneous wave equation in odd space dimensions. These property provide us with enough information to consider the semilinear case, namely, the associated integral equation with the problem will be considered on a weighted L^∞-space. This formulation enable us to deal with the problem for slowly decaying initial data.

1. Introduction.

This paper studies the asymptotic behaviour of solutions to the following Cauchy problem:

$$u_{tt} - u_{rr} - \frac{n-1}{r}u_r = G(u_t, u_r) \quad \text{in} \quad \mathbb{R} \times [0, \infty), \tag{1.1}$$

$$u(r,0) = f(r), \quad u_t(r,0) = g(r) \qquad \text{for} \quad r \in \mathbb{R}, \tag{1.2}$$

where $u(r,t)$ is a scalar unknown, $n = 2m+3$ with m a nonnegative integer and $G(\lambda)$ is of class C^{m+2} over a neighborhood of $\lambda = 0 \in \mathbb{R}^2$. Moreover, $f(r)$, $g(r) \in C^0(\mathbb{R})$ are even functions such that $r^{m+1}f(r) \in C^{m+3}(\mathbb{R})$, $r^{m+1}g(r) \in C^{m+2}(\mathbb{R})$. As is well known, the equation (1.1) is the radially symmetric version of a special case of the nonlinear wave equation:

$$u_{tt} - \Delta_x u = F(u', u'') \quad \text{in} \quad \mathbb{R}^n \times [0, \infty), \tag{1.3}$$

where u' (resp. u'') represents the vector of first (resp. second) order derivatives of u with respect to time and spatial variables.

We start our consideration from the following remarkable result established by F. John [5]. In [5], he introduced a positive parameter η, which describes the magnitude of the initial data, defined by

$$\eta = \sum_{j=0}^{3} \sup_{r \in \mathbb{R}} \{ |(\frac{d}{dr})^{(j)}(rg(r))| + |(\frac{d}{dr})^{(j+1)}(rf(r))| \}.$$

Under the assumption that η is sufficiently small, he construct a function $u(r,t) \in C^2(\mathbb{R} \times [0, T_\eta))$ satisfying (1.1) with $n = 3$, $G = u_t^2$ and (1.2). Moreover, T_η is estimated as $T_\eta \geq \frac{1}{2}e^{C/\eta}$, where C is a numerical constant. We call such $u(r,t)$ an almost global solution. T. C. Sideris [10] showed global existence of a classical solution to the problem (1.1) with $n = 3$ and $G = (|u_t| + |u_r|)^p$, (1.2), provided $p > 2$ and the support of the initial data are compact. Under the same assumption on the initial data as in [5], H. Takamura [11] proved $t_\eta = \infty$, provided $n = 3$ and $G = u_t^p$ with p a positive integer with $p > 2$. On the other hand, the author recently obtained a blow-up result for more slowly decaying initial data even if $p > 2$ in three space dimensions. In five space dimensions, J. Schaeffer [10] showed there exists a global solution to (1.1), which is classical at points (x,t) with $|x| > 0$, provided $G = (|u_t| + |u_r|)^p$ with $p > 3/2$. Moreover, he obtained blow-up result if $1 < p \leq 3/2$ in [10]. When the initial data are slowly decaying, F. Asakura [2] showed there exists a global solution to (1.1) with $n = 5$, $G = au_t^2 + bu_r^2$, where a, b are real constants.

As far as I know, in higher space dimensions, there are few existence results were established for slowly decaying initial data, such as [2], [5] and [11]. We shall investigate to show global existence theorem for such data in higher space dimensions. (See also (1.4) below). We mention here S. Klainerman' remarkable work which have dealt with the Cauchy problem for (1.3) with rapidly decaying initial data. In [6], [7], he showed that there exists a global classical solution for $n \geq 4$ and an almost global solution for $n = 3$ with small initial data via the global Sobolev inequalities in the Minkowski space $\mathbb{R}^{n+1}$,provided that F is a smooth function vanishing together with its first derivatives for

$(u', u'') = 0$. However, the method requires that the initial data belong at least to $L^2(\mathbb{R}^n_x)$. Hence we can not apply Klainerman's result to our problem.

In this paper, we shall consider the problem (1.1), (1.2) in general odd space dimensions for slowly decreasing initial data such that

$$(1.4) \qquad |r^{m+1+\kappa} g(r)| + \sum_{j=1}^{m+2} |(\frac{d}{dr})^{(j)}(r^{m+1} g(r))| + \sum_{j=0}^{m+3} |(\frac{d}{dr})^{(j)}(r^{m+1} f(r))| \leq \varepsilon,$$

where ε is a small parameter and κ is a sufficiently small positive constant for $n \geq 5$, $\kappa = 0$ if $n = 3$. We also assume that $G(\lambda_1, \lambda_2)$ is even in λ_2 and there exists a positive number p with $2 \leq p \leq 3$ such that

$$(1.5)_1 \qquad |D^\alpha G(\lambda)| \leq A|\lambda|^{p-|\alpha|}, \quad |\alpha| < p, \ |\lambda| \leq 1,$$

where A is a positive constant, $\alpha \in \mathbb{Z}^2_+$ and $\mathbb{Z}_+$ is the set of nonnegative integers. Furthermore, for $|\alpha| = m + 2$, there exist positive constants B, ν with $\nu < 1$ such that

$$(1.5)_2 \qquad |D^\alpha(G(\lambda) - G(\lambda'))| \leq B|\lambda - \lambda'|^\mu, \quad |\lambda|, \ |\lambda'| \leq 1,$$

where we have set $\mu = p - 2$ if $m = 0$ and $\mu = \nu$ if $m \geq 1$. A typical example is $G = au_t^2 + bu_r^2$ for $m \geq 1$.

We are now in a position to state the main result of this paper.

Main Theorem. *Let n be an odd integer with $n \geq 3$. Suppose that (1.4), (1.5) hold and that $p > 2$ for $n = 3$. Then there exists a sufficiently small positive number ε_0, depending on n κ and G, such that for any ε with $0 < \varepsilon \leq \varepsilon_0$, the problem (1.1), (1.2) admits unique global C^2-solutions $u(r,t)$ which are even in r. Furthermore, there exist numerical constants C_i, $1 \leq i \leq 4$ which depend only on n, κ and G such that for any $t \geq 0$,*

$$(1.6) \qquad |u(r,t)| \leq C_1\varepsilon < r >^{-m}, \quad |D^\alpha u(r,t)| \leq C_2\varepsilon < r >^{-m-1}, \quad |\alpha| = 1, 2,$$

$$(1.7) \qquad \begin{aligned} |u(r,t) - u^0(r,t)| &\leq C_3\varepsilon^p < r >^{-m}, \\ |D^\alpha(u(r,t) - u^0(r,t))| &\leq C_4\varepsilon^p < r >^{-m-1}, \quad |\alpha| = 1, 2, \end{aligned}$$

where $u^0(r,t)$ *is a* C^2*-solution for the problem (1.1) with* $G = 0$, *(1.2) and we have set* $< r >= \sqrt{1+r^2}$.

The proof of the above theorem is essentially based on the the following idea, given in [5], which enable us to deal with singularity of a solution $u(r,t)$ at $r = 0$. considering a new unknown $v(r,t) = ru_t(r,t)$ When $n = 3$, it is easy to deal with singularity of a solution $u(r,t)$ at $r = 0$, by considering a new unknown $v(r,t) = ru_t(r,t)$ and imposing higher regularity on it so that $u(r,t)$ has desired regularity. In higher space dimensions cases, however, the procedure is more complicated, because there arise not only the singularity but also the loss of derivatives. (See also [2]). We shall point out remarkable facts, which will be stated in Section 2, to find that the equation (1.1) do have nice structure to overcome such difficulty. In Section 3, the problem (1.1), (1.2) will be deduced to an associated integral equation (3.1) below. The procedure is closely related to the argument of Section 2. Most part of Section 4 is devoted to estimate the kernel of the fundamental solution for the problem (1.1) with $G = 0$, (1.2) with $f = 0$. (See Lemmas 4.3 and 4.4 below). At the end of Section 5, employing iteration method as in F.John [4], we shall prove the above theorem.

We hope that the above theorem will be proved for $(n+1)/(n-1) < p < 2$, $n \geq 6$, where the value $(n+1)/(n-1)$ is seems to be optimal. (See also M. Rammaha [8]).

Throughout this paper, constants arising in the estimates will be denoted by C, depending only on n, κ and G, and will change from line to line.

2. Preliminaries.

In this section, we shall review representations of fundamental solutions. Moreover, we shall prove a very remarkable regularity result of the solutions in Proposition 2.5. First we consider the homogeneous wave equation:

$$(2.1) \qquad u_{tt} - u_{rr} - \frac{n-1}{r}u_r = 0 \qquad \text{for} \quad (r,t) \in \mathbb{R} \times [0,\infty).$$

Lemma 2.1. *Suppose that $H(\rho) \in C^2(\mathbb{R})$ and $n \geq 2$. Set*

$$u(r,t) = \int_{-1}^{1} H(t + r\sigma)(1-\sigma^2)^m d\sigma, \quad m = \frac{n-3}{2}. \tag{2.2}$$

Then $u(r,t)$ belongs to $C^2(\mathbb{R} \times [0,\infty))$ and is even in r. Moreover, $u(r,t)$ satisfies (2.1) with $u_r(0,t) = 0$.

Although the lemma is well known (e.g. Courant-Hilbert [3], p.699-p.703), we shall give an elementary proof.

PROOF OF LEMMA 2.1: It immediately follows from the assumptions that $u(r,t) \in C^2(\mathbb{R} \times [0,\infty))$. In addition, changing the variables as $\sigma' = -\sigma$, it is easy to see that $u(-r,t) = u(r,t)$.

We now show that $u(r,t)$ satisfies (2.1).

$$\begin{aligned}(n-1)u_r(r,t) &= -\int_{-1}^{1} H^{(1)}(t+r\sigma)\frac{d}{d\sigma}(1-\sigma^2)^{m+1}d\sigma \\ &= r\int_{-1}^{1} H^{(2)}(t+r\sigma)(1-\sigma^2)^{m+1}d\sigma \\ &= r\ \{u_{tt}(r,t) - u_{rr}(r,t)\},\end{aligned}$$

where $H^{(k)}(\rho) = (\frac{d}{d\rho})^k H(\rho)$, $k \in \mathbb{Z}_+$. Thus we have proved the lemma. □

From now on, let n be an odd integer with $n \geq 3$, that is, $n = 2m+3$ with m a nonnegative integer.

Lemma 2.2. *Assume that $g(r) \in C^0(\mathbb{R})$ is an even function such that $r^{m+1}g(r) \in C^{m+2}$ $(\mathbb{R})$. Then for $0 \leq j \leq m$, $0 \leq k \leq j+2$, $r^{j+1}g^{(k)}(r)$ is continuously extended at $r = 0$. Moreover, there exists a positive constant $C = C(m)$ such that*

$$|r^{j+1}g^{(j+i)}(r)| \leq C \sup_{|\rho|\leq 1} |(\frac{d}{d\rho})^{m+i}(\rho^{m+1}g(\rho))|, \quad |r| \leq 1,\ 0 \leq j \leq m,\ i = 0,1,2. \tag{2.3}$$

In addition, $g(r) \in C^1(\mathbb{R})$ and satisfies

$$|g^{(i)}(r)| \leq \sup_{|\rho|\leq 1} |(\frac{d}{d\rho})^{m+1+i}(\rho^{m+1}g(\rho))|, \quad |r| \leq 1,\ i = 0,1. \tag{2.4}$$

PROOF: To begin with, let $r \neq 0$. We note that

$$(2.5)_1 \qquad r^{j+1}g^{(k)}(r) = r^{j-k}\sum_{l=0}^{k} C_{k,l}(\frac{d}{dr})^l(r^{l+1}g(r)), \quad 0 \leq k \leq j \leq m.$$

These relations are easily proved by the induction together with Leibniz' rule. Moreover,

$$(2.5)_2 \qquad r^{j+1}g^{(j+i)}(r) = \sum_{l=0}^{j} C_{j,l}(\frac{d}{dr})^{l+i}(r^{l+1}g(r)), \quad i = 0, 1, 2.$$

We shall give an integral expression of $r^j g(r)$, $0 \leq j \leq m$. Setting $\tilde{g}(r) = r^{m+1}g(r)$, we obtain

$$\tilde{g}(r) = \tilde{g}(0) + \int_0^1 \frac{d}{d\lambda}\tilde{g}(r\lambda)d\lambda = r\int_0^1 \tilde{g}^{(1)}(r\lambda)d\lambda,$$

hence we have $r^m g(r) = \int_0^1 \tilde{g}^{(1)}(r\lambda)d\lambda$. Letting $r \to 0$, we get $\tilde{g}^{(1)}(0) = 0$. Repeating the analogous argument, we obtain $\tilde{g}^{(l)}(0) = 0$, $0 \leq l \leq m$. Therefore it follows from Taylor's formula that for $0 \leq j \leq m$,

$$(2.6) \qquad r^j g(r) = \frac{1}{(m-j)!}\int_0^1 (1-\lambda)^{m-j}\tilde{g}^{(m-j+1)}(r\lambda)d\lambda,$$

which implies that $r^j g(r) \in C^{j+1}(\mathbb{R})$, $0 \leq j \leq m$. Hence, for $r = 0$, if we define $r^{j+1}g^{(k)}(r)$ by (2.5), it becomes continuous over $\mathbb{R}$. The other assertions are easily derived from $(2.5)_2$, (2.6). □

Lemma 2.3. *Suppose that the assumptions on $g(r)$ in Lemma 2.2 are fulfilled. Let $u(r,t)$ be as in (2.2) with $H(\rho)$ specified by*

$$(2.7) \qquad H_g(\rho) = \frac{1}{2m!}(\frac{1}{2\rho}\frac{d}{d\rho})^m(\rho^{2m+1}g(\rho)).$$

Then $u(r,t)$ belongs to $C^2(\mathbb{R}\times[0,\infty))$ and satisfies (2.1). Moreover, when $r \neq 0$, we have

$$(2.8) \qquad u(r,t) = \int_{t-r}^{t+r} g(\rho)K(\rho,r,t)d\rho,$$

where

$$(2.9) \qquad \begin{aligned} K(\rho,r,t) &= \frac{(-1)^m}{2m!}(\frac{\rho}{r})^{2m+1}(\frac{\partial}{\partial\rho}\frac{1}{2\rho})^m\phi^m(\rho,r,t),\\ \phi(\rho,r,t) &= r^2-(t-\rho)^2. \end{aligned}$$

Furthermore, we find that

(i) $K(\rho, r, t)$ is not only odd in r but also odd in ρ.

(ii) $u(r, t)$ satisfies (1.2) with $f(r) = 0$.

PROOF: For $\rho \neq 0$, we have from (2.7)

$$H_g(\rho) = \sum_{j=0}^{m} C_j \rho^{j+1} g^{(j)}(\rho). \tag{2.10}$$

Applying Lemma 2.2 with $k = j$, we find that $H_g(\rho)$ belongs to $C^2(\mathbb{R})$. So, by Lemma 2.1, we have $u(r, t) \in C^2(\mathbb{R} \times [0, \infty))$ and satisfies (2.1).

Changing the variables as $\rho = t + r\sigma$ in (2.2) and integrating by parts m times with respect to ρ, we have (2.8).

(*i*) It is clear that $K(\rho, -r, t) = -K(\rho, r, t)$. Since $\phi^m(\rho, r, t)$ is a polynomial of degree $2m$, we find that $(\dfrac{\partial}{\partial \rho}\dfrac{1}{2\rho})^m \phi^m(\rho, r, t)$ is even in ρ. Hence, $K(\rho, r, t)$ is odd in ρ.

(*ii*) Note that (2.10) implies that $H_g(\rho)$ is an odd function, so $u(r, 0) = 0$. For $r \neq 0$, it follows from (2.8) that

$$u_t(r, t) = g(t + r)K(t + r, r, t) - g(t - r)K(t - r, r, t) + \int_{t-r}^{t+r} g(\rho) D_t K(\rho, r, t) d\rho.$$

Since $K(\rho, r, t)$ is odd in ρ, we get $u_t(r, 0) = 2g(r)K(r, r, 0)$. Note that

$$\begin{aligned} \rho^{2m+1} (\frac{\partial}{\partial \rho}\frac{1}{2\rho})^m \phi^m(\rho, r, 0)\Big|_{\rho=r} &= \rho^{m+1} (\frac{1}{2})^m D_\rho^m \phi^m(\rho, r, 0)\Big|_{\rho=r} \\ &= r^{m+1} (\frac{1}{2})^m (-1)^m \, m! \, (2r)^m, \end{aligned}$$

hence $K(r, r, 0) = \dfrac{1}{2}$, so that $u_t(r, 0) = g(r)$. By the continuity of $g(r)$, this also holds for $r = 0$. $\square$

The following lemma is easily derived from the standard argument. So we omit the proof.

Lemma 2.4. *Assume that $f(r) \in C^0(\mathbb{R})$ is an even function such that $\rho^{m+1} f(\rho) \in C^{m+3}(\mathbb{R})$. Then $H_f(\rho) \in C^3(\mathbb{R})$. Moreover, setting*

$$u(r,t) = D_t \int_{-1}^{1} H_f(t + r\sigma)(1-\sigma^2)^m d\sigma, \tag{2.11}$$

we have then $u(r,t)$ belongs to $C^2(\mathbb{R} \times [0,\infty))$ and satisfies (2.1), (1.2) with $g(r) = 0$.

At the end of this section, we shall show that $u(r,t)$ defined in (2.2) does gain regularity by multiplying r.

Proposition 2.5. *Let $H(\rho) \in C^2(\mathbb{R})$ and $u(r,t)$ be as in (2.2). Then $r^{m+1}u(r,t) \in C^{m+3}(\mathbb{R} \times [0,\infty))$. Moreover, for any $\alpha = (\alpha_1, \alpha_2) \in Z_+^2$ with $|\alpha| \le m$, we have*

$$D_{r,t}^{\alpha}(r^{m+1}u(r,t)) = r^{m+1-|\alpha|} \int_{-1}^{1} H(t + r\sigma)\psi_\alpha(\sigma) d\sigma, \tag{2.12}$$

where

$$\psi_\alpha(\sigma) = \sum_{i=0}^{\alpha_1} C_{\alpha,i} \, (-\frac{d}{d\sigma}\sigma)^i \, (-\frac{d}{d\sigma})^{\alpha_2} \, (1-\sigma^2)^m, \quad C_{0,0} = 1. \tag{2.13}$$

Furthermore, when $|\alpha| = m$, we have

$$\begin{aligned} D_t D_{r,t}^{\alpha}(r^{m+1}u(r,t)) &= \int_{-1}^{1} H(t + r\sigma)(-\frac{d}{d\sigma})\psi_\alpha(\sigma) d\sigma \\ &\quad + H(t+r)\psi_\alpha(1) - H(t-r)\psi_\alpha(-1), \end{aligned} \tag{2.14$_1$}$$

$$\begin{aligned} D_r D_{r,t}^{\alpha}(r^{m+1}u(r,t)) &= \int_{-1}^{1} H(t + r\sigma)(-\frac{d}{d\sigma}\sigma)\psi_\alpha(\sigma) d\sigma \\ &\quad + H(t+r)\psi_\alpha(1) + H(t-r)\psi_\alpha(-1). \end{aligned} \tag{2.14$_2$}$$

PROOF: We shall use the following relations:

$$rD_r H(t + r\sigma) = \sigma D_\sigma H(t + r\sigma), \quad rD_t H(t + r\sigma) = D_\sigma H(t + r\sigma). \tag{2.15}$$

Let $n \ge 5$. It follows from (2.2) and (2.15) that

$$\begin{aligned} D_t(r^{m+1}u(r,t)) &= r^m \int_{-1}^{1} H(t + r\sigma)(-\frac{d}{d\sigma})(1-\sigma^2)^m d\sigma, \\ D_r(r^{m+1}u(r,t)) &= r^m \int_{-1}^{1} H(t + r\sigma)(1 - \frac{d}{d\sigma}\sigma)(m-\sigma^2)^m d\sigma, \end{aligned}$$

which implies that (2.12) holds for $|\alpha| = 1$.

Suppose that (2.12) holds for some $\alpha \in \mathbb{Z}_+^2$ with $|\alpha| \leq m$. When $|\alpha| \leq m-1$, since $\psi_\alpha(\pm 1) = 0$, it is easy to see that (2.12) holds for $\alpha+\beta$ with $|\beta| = 1$. Moreover, if $|\alpha| = m$, we get (2.14). Therefore we have proved (2.12), (2.14). When $n = 3$, the procedure is more easy. So we omit the proof for the case. $\square$

3. Formulation of the Problem.

The aim of this section is to formulate the problem (1.1), (1.2) so that its solution is furnished by a solution of the associated integral equation:

$$u(r,t) = u^0(r,t) + L(u)(r,t) \quad \text{in} \quad \mathbb{R} \times [0,\infty), \tag{3.1}$$

where $u^0(r,t)$ is a C^2-solution of the Cauchy problem (2.1), (1.2) and

$$L(u)(r,t) = \int_0^t d\tau \int_{-1}^1 H(t-\tau+r\sigma, \tau)\,(1-\sigma^2)^m d\sigma, \tag{3.2}$$

$$H(\rho,\tau) = \frac{1}{2m!}\left(\frac{1}{2\rho}\frac{\partial}{\partial\rho}\right)^m (\rho^{2m+1} G(u_t, u_r)(\rho,\tau)). \tag{3.3}$$

In other words, we shall find appropriate assumptions on $u(r,t)$ which assure that $H(\rho,\tau) \in C^{2,0}(\mathbb{R} \times [0,\infty))$, where $C^{k,0}(\mathbb{R} \times [0,\infty))$ is the set of functions $u(r,t) \in C^0(\mathbb{R} \times [0,\infty))$ whose derivatives up to order k with respect to r are continuous. Throughout this paper, we denote

$$|u|_k = \sum_{j=0}^k \sup_{\mathbb{R}\times[0,\infty)} |D_r^j u(r,t)| \quad \text{for} \quad u \in C^{k,0}(\mathbb{R} \times [0,\infty)).$$

Now we define a function space X on which we shall find a solution of (3.1):

$$X = \{u(r,t) \in C^1(\mathbb{R} \times [0,\infty)) : r^{m+1}u(r,t) \in C^{m+3,0}(\mathbb{R} \times [0,\infty)),$$
$$r^{m+1}u_t(r,t) \in C^{m+2,0}(\mathbb{R} \times [0,\infty)),\ ||u|| < \infty\},$$

where

$$||u|| = |r^m u|_0 + |D_r(r^{m+1}u)|_{m+2} + |r^{m+1}u_t|_{m+2}. \tag{3.4}$$

Here we shall present an useful lemma which is an analogue to lemma 2.2.

Lemma 3.1. *Let $u \in X$ and $0 \le j \le m+1$. Then we have $r^j D_r^{j+i} u_t(r,t)$, $r^j D_r^{j+k} u(r,t) \in C^0(\mathbb{R}\times[0,\infty))$ for $i = 0,1$, $k = 0,1,2$. Moreover, there exists a positive constant $C = C(m)$ such that*

$$|r^j D_r^{j+i} u_t(r,t)|, \quad |r^j D_r^{j+k} u(r,t)| \le C\|u\|, \quad |r| \le 1. \tag{3.5}$$

PROOF: By applying Lemma 2.2 to $u_t(r,t)$, it follows from (2.3), (2.4) that

$$|r^j D_r^{j+i} u_t(r,t)| \le C \sup_{|\rho|\le 1} |D_\rho^{m+1+i}(\rho^{m+1} u_t(\rho,\tau))|, \quad |r| \le 1, \ i = 0,1.$$

Similarly we have $|r^j D_r^{j+k} u(r,t)| \le C \sup_{|\rho|\le 1} |D_\rho^{m+1+k}(\rho^{m+1} u(\rho,\tau))|$ for $|r| \le 1$, $k = 0,1,2$. So it is easy to see that (3.5) holds. □

It follows from (3.5) and the definition of $\|\cdot\|$ that there exists a positive constant C_0, depending only on n, such that $|u_t(r,t)| + |u_r(r,t)| \le C_0\|u\|$ for any $(r,t) \in \mathbb{R}\times[0,\infty)$. In what follows, we fix a small positive number δ satisfying $C_0\delta \le 1$ and set

$$X_\delta = \{u \in X; \ \|u\| \le \delta\}.$$

Lemma 3.2. *Suppose that $G \in C^{m+2}(B_1(0))$ and satisfy (1.5), where $B_1(0)$ denotes a ball of radius 1 about the center $0 \in \mathbb{R}^2$. Set $F(\rho,\tau) = G(u_t,u_r)(\rho,\tau)$ for $u \in X_\delta$. Then we have*

$$\rho^{k+1} F(\rho,\tau) \in C^{k+2,0}(\mathbb{R}\times[0,\infty)), \quad 0 \le k \le m. \tag{3.6}$$

Moreover, there exist constants $C_i = C_i(m,G) > 0$, $i = 1,2$, such that

$$|D_\rho^j(\rho^{k+1}F(\rho,\tau))| \le C_1\|u\|^p, \quad \text{for} \quad |\rho| \le 1, \ 0 \le j \le k+2, \tag{3.7}$$

$$|D_\rho^j F(\rho,\tau)| \le C_2\|u\|^p|\rho|^{-p(m+1)} \quad \text{for} \quad |\rho| \ge 1, \ 0 \le j \le m+2. \tag{3.8}$$

PROOF: First we shall show (3.6). It is clear that $F(\rho,\tau) \in C^0(\mathbb{R}\times[0,\infty))$, hence it suffices to show

$$\rho^{m+1} F(\rho,\tau) \in C^{m+2,0}(\mathbb{R}\times[0,\infty)), \tag{3.9}$$

by virtue of the argument in the proof of Lemma 2.2.

Since $F(\rho,\tau) \in C^{m+2,0}((\mathbb{R}\setminus\{0\}) \times [0,\infty))$, we have for $\rho \neq 0$

$$D_\rho^j(\rho^{m+1}F(\rho,\tau)) = \rho^{m+1-j}\sum_{k=0}^{j} C_k\rho^k D_\rho^k F(\rho,\tau), \quad 0 \le j \le m+1. \tag{3.10}$$

By the chain rule, we get for $k \ge 1$

$$D_\rho^k F(\rho,\tau) = \overline{\sum_k} C_{\alpha,\beta}(D^\alpha F)(\rho,\tau)(D_\rho^{\beta_1}u_t(\rho,\tau))^{\alpha_1}(D_\rho^{\beta_2}u_r(\rho,\tau))^{\alpha_2}, \tag{3.11}$$

where $\overline{\sum}_k$ stands for the sum over $\alpha = (\alpha_1,\alpha_2)$, $\beta = (\beta_1,\beta_2) \in \mathbb{Z}_+^2$ such that $1 \le |\alpha| \le k$, $\alpha_1\beta_1 + \alpha_2\beta_2 = k$ and $(D^\alpha F)(\rho,\tau) = (D^\alpha G)(u_t,u_r)(\rho,\tau)$. Substituting (3.11) into (3.10), we have

$$\begin{aligned} D_\rho^j(\rho^{m+1}F(\rho,\tau)) &= C_0\rho^{m+1-j}F(\rho,\tau) \\ &+ \sum_{k=1}^{j} C_k \overline{\sum_k} C_{\alpha,\beta}(D^\alpha F)(\rho,\tau)(\rho^{\beta_1}D_\rho^{\beta_1}u_t(\rho,\tau))^{\alpha_1}(\rho^{\beta_2}D_\rho^{\beta_2}u_r(\rho,\tau))^{\alpha_2}. \end{aligned} \tag{3.12}$$

Employing Lemma 3.1, we find that $D_\rho^j(\rho^{m+1}F(\rho,\tau)) \in C^{1,0}(\mathbb{R}\times[0,\infty))$, *i.e.*, (3.9) holds.

Next we shall show (3.7). It suffices to show (3.7) for $k = m$, because for $0 \le k \le m-1$, we have

$$\rho^{k+1}F(\rho,\tau) = \frac{1}{(m-k-1)!}\int_0^1 (1-\lambda)^{m-k-1}D_\rho^{m-k}\tilde{F}(\rho\lambda,\tau)d\lambda,$$

where $\tilde{F}(\rho,\tau) = \rho^{m+1}F(\rho,\tau)$. From $(1.5)_1$, we have for $\alpha \in \mathbb{Z}_+^2$ with $|\alpha| \le m+2$,

$$|D^\alpha G(\lambda)| \le C|\lambda|^{q(\alpha)} \quad \text{for} \quad |\lambda| \le 1 \tag{3.13}$$

where $q(\alpha) = \max(p - |\alpha|, 0)$. In addition, noting that $|u_t(r,t)| + |u_r(r,t)| \le 1$, we have

$$|(D^\alpha F)(\rho,\tau)| \le C||u||^{q(\alpha)}.$$

Employing this and (3.5), (3.12) yields (3.7) for $0 \le j \le m+1$. Moreover, differentiating (3.12) with $j = m+1$ with respect to ρ and using (3.5) and

$$\begin{aligned} |D_\rho(D^\alpha F)(\rho,\tau)| &= |(D_1D^\alpha F)(\rho,\tau)\ D_\rho u_t(\rho,\tau) + (D_2D^\alpha F)(\rho,\tau)\ D_\rho u_r(\rho,\tau)| \\ &\le C||u||^{q(\alpha)} \quad \text{for } |\alpha| = m+1, \end{aligned} \tag{3.14}$$

we obtain (3.7) for $j = m+2$.

Finally, we shall show (3.8). Let $|\rho| \geq 1$. Since $u_t(\rho,\tau) = \dfrac{1}{\rho^{m+1}}(\rho^{m+1}u_t(\rho,\tau))$, we get

$$|D_\rho^l u_t(\rho,\tau)| \leq C||u|||\rho|^{-(m+1)}, \quad 0 \leq l \leq m+2.$$

In addition, this estimate is also valid for $u_r(r,t)$. Therefore we obtain (3.8) from (3.11) together with (3.13) and (3.14). Thus we have proved the lemma. □

Corollary 3.3. *Let the assumptions on G in Lemma 3.2 be fulfilled and $u \in X_\delta$. Then we have $H(\rho,\tau) \in C^{2,0}(\mathbb{R} \times [0,\infty))$. Moreover, there exists a constant $C_1 = C_1(m,G) > 0$ such that*

$$\text{(3.15)} \qquad |D_\rho^j H(\rho,\tau)| \leq C_1||u||^p < \rho >^{-(p-1)(m+1)}, \quad j = 0,1,2.$$

Furthermore, there exist constants $C_i = C_i(m,G) > 0$, $2 \leq i \leq 7$, such that for u, $\hat{u} \in X_\delta$, the followings hold: Let $0 \leq j \leq m+1$ and $i = 0, 1$. Then

$$\text{(3.16)}_1 \quad |D_\rho^j(F(\rho,\tau) - \hat{F}(\rho,\tau))| \leq C_2|||u-\hat{u}|||\,(|||u|||^{p-1} + |||\hat{u}|||^{p-1})|\rho|^{-p(m+1)}, \quad |\rho| \geq 1,$$

$$\text{(3.16)}_2 \quad |D_\rho^j(H(\rho,\tau) - \hat{H}(\rho,\tau))| \leq C_3|||u-\hat{u}|||\,(|||u|||^{p-1} + |||\hat{u}|||^{p-1}) < \rho >^{-(p-1)(m+1)},$$

where we have set $\hat{F}(\rho,\tau) = G(\hat{u}_t, \hat{u}_r)(\rho,\tau)$, $\hat{H}(\rho,\tau) = \dfrac{1}{2m!}(\dfrac{1}{2\rho}\dfrac{\partial}{\partial\rho})^m(\rho^{2m+1}\hat{F}(\rho,\tau))$ and

$$|||u||| = |r^m u|_0 + |D_r(r^{m+1}u)|_{m+1} + |r^{m+1}u_t|_{m+1},$$

so that

$$|||u||| = ||u|| - \sum_{|\alpha|=1} \sup_{\mathbb{R}\times[0,\infty)} |D_r^{m+2}D_{r,t}^\alpha(r^{m+1}u(r,t))| \quad \text{for} \quad u \in X_\delta.$$

For the top order derivatives, we have

$$\text{(3.16)}_3 \quad \begin{aligned} &|D_\rho^{m+2}(F(\rho,\tau) - \hat{F}(\rho,\tau))| \leq C_4|||u-\hat{u}|||^\mu\,(||u||^{m+2} + ||\hat{u}||^{m+2})|\rho|^{-p(m+1)} \\ &\qquad + C_5||u-\hat{u}||\,(||u||^{p-1} + ||\hat{u}||^{p-1})|\rho|^{-p(m+1)} \quad \text{for } |\rho| \geq 1, \end{aligned}$$

$$
(3.16)_4 \qquad \begin{aligned} |D_\rho^2(H(\rho,\tau)-\hat{H}(\rho,\tau))| &\le C_6|||u-\hat{u}|||^\mu \,(||u||^{m+2}+||\hat{u}||^{m+2}) <\rho>^{-(p-1)(m+1)} \\ &+ C_7||u-\hat{u}||\,(||u||^{p-1}+||\hat{u}||^{p-1}) <\rho>^{-(p-1)(m+1)}, \end{aligned}
$$

where μ is as given below $(1.5)_2$.

Proof: It follows from (3.3) and $(2.5)_2$ that

$$
(3.17) \qquad H(\rho,\tau)=\sum_{j=0}^{m} C_j\rho^{j+1}D_\rho^j F(\rho,\tau)=\sum_{k=0}^{m} C_k D_\rho^k(\rho^{k+1}F(\rho,\tau)),
$$

hence by (3.6), $H(\rho,\tau)\in C^{2,0}(\mathbb{R}\times[0,\infty))$. Moreover, combining (3.7) and (3.8), we obtain (3.15).

We shall show only $(3.16)_3$, because the others are obtained in a similar fashion. From (3.11), we have

$$
(3.18) \qquad \begin{aligned} D_\rho^{m+2}(F(\rho,\tau)-\hat{F}(\rho,\tau)) = \overline{\sum_k} C_{\alpha,\beta}[&(D^\alpha F-D^\alpha\hat{F})(D_\rho^{\beta_1}u_t)^{\alpha_1}(D_\rho^{\beta_2}u_\tau)^{\alpha_2} \\ &+(D^\alpha F)((D_\rho^{\beta_1}u_t)^{\alpha_1}-(D_\rho^{\beta_1}\hat{u}_t)^{\alpha_1})(D_\rho^{\beta_2}u_\tau)^{\alpha_2} \\ &+(D^\alpha F)(D_\rho^{\beta_1}\hat{u}_t)^{\alpha_1}((D_\rho^{\beta_1}u_\tau)^{\alpha_2}-(D_\rho^{\beta_1}\hat{u}_\tau)^{\alpha_2})] \end{aligned}
$$

For $|\rho|\ge 1$ and $|\alpha|\le m+1$, we have from $(1.5)_1$

$$
|D^\alpha F-D^\alpha\hat{F}|\le C\;|||u-\hat{u}|||\;(||u||^{p-|\alpha|-1}+||\hat{u}||^{p-|\alpha|-1})\;|\rho|^{-(p-|\alpha|)(m+1)}.
$$

In addition, $(1.5)_2$ yields

$$
|D^\alpha F-D^\alpha\hat{F}|\le C\;|||u-\hat{u}|||^\mu\;|\rho|^{-\mu(m+1)}, \quad |\alpha|=m+2.
$$

On the other hand, we have

$$
\begin{aligned} |(D_\rho^{\beta_1}u_t)^{\alpha_1}-(D_\rho^{\beta_1}\hat{u}_t)^{\alpha_1}| &=|(D_\rho^{\beta_1}u_t-D_\rho^{\beta_1}\hat{u}_t)\sum_{k=0}^{\alpha_1-1}C_k(D_\rho^{\beta_1}u_t)^k(D_\rho^{\beta_1}\hat{u}_t)^{\alpha_1-k-1}| \\ &\le C||u-\hat{u}||\;(||u||^{\alpha_1-1}+||\hat{u}||^{\alpha_1-1})\;|\rho|^{\alpha_1(m+1)}, \end{aligned}
$$

if $\alpha_1\ge 1$. Combining these estimates with (3.18), we obtain $(3.16)_3$. □

At the end of this section, we shall show that the problem (1.1), (1.2) is equivalent to (3.1).

Proposition 3.4. *Let $u \in X_\delta$ such that $u(-r,t) = u(r,t)$. If $u(r,t)$ satisfies (3.1), then it is a unique C^2-solution to the Cauchy problem (1.1), (1.2).*

PROOF: As is well known, the uniqueness of a solution to (1.1) is valid. (See [5], Theorem 4, R. Agemi [1], p.156). So, it suffices to show that $L(u)(r,t)$ is a C^2-solution to (1.1) with zero initial data, because $u^0(r,t)$ is a C^2-solution to (2.1) with (1.2).

Since $u(r,t)$ is even in r, by (3.18), $H(\rho,\tau)$ is odd in ρ. Therefore we have

$$D_t L(u)(r,t) = \int_0^t d\tau D_t \int_{-1}^1 H(t-\tau+r\sigma)(1-\sigma^2)^m d\sigma. \tag{3.19}$$

From this and (3.2), we have $L(u)(r,0) = D_t L(u)(r,0) = 0$. Moreover, it follows from Lemmas 2.1 and 2.3 (*ii*) that

$$\begin{aligned} AL(u)(r,t) = \int_0^t d\tau A \int_{-1}^1 H(t-\tau+r\sigma)(1-\sigma^2)^m d\sigma \\ + D_t \int_{-1}^1 H(t-\tau+r\sigma)(1-\sigma^2)^m d\sigma \bigg|_{\tau=t} = G(r,t), \end{aligned}$$

where we have set $A = D_t^2 - D_r^2 - \dfrac{n-1}{r} D_r$. Thus we have proved the proposition. □

4. Homogeneous Wave Equations.

Proposition 4.1. *Assume that $f(r), g(r) \in C^0(\mathbb{R})$ are even functions such that $r^{m+1}f(r) \in C^{m+3}(\mathbb{R})$ and $r^{m+1}g(r) \in C^{m+2}(\mathbb{R})$. Moreover, suppose that (1.4) holds. Let $u^0(r,t)$ be a C^2-solution of the problem (2.1), (1.2). Then $u^0(r,t) \in X$ and there exists a positive constant $C_i = C_i(m)$, $i = 1,2$ such that*

$$|u^0(r,t)| \le C_1 \varepsilon < r >^{-m}, \quad |D^\alpha u^0(r,t)| \le C_2 \varepsilon < r >^{-m-1}, \quad |\alpha| = 1,2, \tag{4.1}$$

where $\alpha \in \mathbb{Z}_+^2$. In addition, we have $||u^0|| \le C_3 \varepsilon$.

PROOF: By Lemmas 2.1 and 2.4, we have

$$u^0(r,t) = \int_{-1}^1 (H_f^{(1)} + H_g)(t+r\sigma)\,(1-\sigma^2)^m d\sigma. \tag{4.2}$$

By (2.10), Lemma 2.2 and (1.4), we have $(H_f^{(1)}+H_g)(\rho) \in C^2(\mathbb{R})$ and there exists a positive constant C such that

$$(4.3) \qquad |D_\rho^i(H_f^{(1)} + H_g)(\rho)| \le C\varepsilon, \quad i = 0, 1, 2.$$

Therefore, by Proposition 2.5, we have $r^{m+1}u^0(r,t) \in C^{m+3}(\mathbb{R} \times [0,\infty))$, $r^{m+1}u_t^0(r,t) \in C^{m+2}(\mathbb{R} \times [0,\infty))$, hence we have only to show (4.1). Moreover, since $u^0(r,t)$ is even in r, in what follows, we assume that $r \ge 0$.

By employing Proposition 2.5 together with (4.3), it is easy to see that (4.1) holds for $0 \le r \le 1$. Before carrying out the the proof of (4.1) for $r \ge 1$, we shall review another representations of $u^0(r,t)$ for such r, which is a direct consequence of Lemma 2.3.

Lemma 4.2. *Let $r \ge 1$. Then we have*

$$(4.4) \qquad r^{m+1}u^0(r,t) = I(g)(r,t) + D_t I(f)(r,t),$$

where

$$(4.5) \qquad I(g)(r,t) = \frac{1}{r^m}\int_{t-r}^{t+r} H_g(\rho)\phi^m(\rho,r,t)d\rho.$$

Moreover, we have

$$(4.6) \qquad I(g)(r,t) = \int_{|t-r|}^{t+r} g(\rho)\tilde{K}(\rho,r,t)d\rho.$$

where $\tilde{K}(\rho,r,t) = r^{m+1}K(\rho,r,t)$ and $K(\rho,r,t)$ and $\phi(\rho,r,t)$ is as in (2.9).

By assuming following two lemmas, we shall complete the proof of (4.1) for $r \ge 1$.

Lemma 4.3. *Suppose that $-1 \le t - r \le \rho \le t + r$ or $0 \le r - t \le \rho \le t + r$. In addition, let $r \ge 1$. Then for any $\alpha \in \mathbb{Z}_+^2$, there exist a positive constant $C_{\alpha,m}$ such that*

$$(4.7) \qquad |D_{r,t}^\alpha \tilde{K}(\rho,r,t)| \le C_{\alpha,m}\ \chi_\alpha(\rho),$$

where

$$(4.8)\qquad \chi_\alpha(\rho)=\begin{cases} <\rho>^{m+1}, & |\alpha|\geq 0,\\ <\rho>^{m}+r^{-1}<\rho>^{m+1}, & |\alpha|\geq 1,\\ <\rho>^{m-1}+r^{-1}<\rho>^{m}+r^{-2}<\rho>^{m+1}, & |\alpha|\geq 2.\end{cases}$$

Lemma 4.4.

(*i*) *Assume that $r\geq 1$. Then for any β, $\gamma\in\mathbb{Z}_+^2$, there exists a positive constant $C_{\beta,\gamma}$ such that*

$$(4.9)\qquad |D^{\beta}_{r,t}(D^{\gamma}_{r,t}\tilde{K}(\rho,r,t)\Big|_{\rho=t\pm r})|\leq C_{\beta,\gamma}\ \chi_{\beta+\gamma}(t\pm r);$$

(*ii*) *Assume that $r\geq t$. Then for any β, $\gamma\in\mathbb{Z}_+^2$, there exists a positive constant $\tilde{C}_{\beta,\gamma}$ such that*

$$(4.10)\qquad |D^{\beta}_{r,t}(D^{\gamma}_{r,t}\tilde{K}(\rho,r,t)\Big|_{\rho=r-t})|\leq \tilde{C}_{\beta,\gamma}\ \chi_{\beta+\gamma}(r-t),$$

End of the proof of Proposition 4.1: We first assume $f(r)=0$ hence by (4.4), $r^{m+1}u^0(r,t)=I(g)(r,t)$. By (1.4) and (4.7) with $\alpha=0$, (4.6) yields $|I(g)(r,t)|\leq C\varepsilon r$, so that $|r^m u^0(r,t)|\leq C\varepsilon$.

Let $t\geq r$ and $m\geq 1$. It follows from (4.6) that for any $\theta\in\mathbb{Z}_+^2$ with $1\leq|\theta|\leq m+2$,

$$\begin{aligned} D_r^{\theta_1}D_t^{\theta_2}D_t(r^{m+1}u^0(r,t)) &= \int_{t-r}^{t+r} g(\rho)D_r^{\theta_1}D_t^{\theta_2+1}\tilde{K}(\rho,r,t)d\rho\\ &+\sum_{k=1}^{\theta_2+1} D_r^{\theta_1}D_t^{\theta_2+1-k}(\sum_{\sigma=\pm1}\sigma\ g(t+r\sigma)D_t^{k-1}\tilde{K}(t+r\sigma,r,t))\\ &+\sum_{l=1}^{\theta_1} D_r^{\theta_1-l}(\sum_{\sigma=\pm1} g(t+r\sigma)D_r^{l-1}D_t^{\alpha_2+1}\tilde{K}(t+r\sigma,r,t)),\end{aligned}$$

which is easily proved by the induction.

By (1.4), (4.7) with $|\alpha|=1$ and (4.9) with $\beta+\gamma=0$, we obtain

$$(4.11)\qquad |D^{\theta}_{r,t}(r^{m+1}u^0_t(r,t))|\leq C\varepsilon\int_{t-r}^{t+r}\{<\rho>^{-1-\kappa}+r^{-1}\}d\rho+C\varepsilon\leq C\varepsilon.$$

In a similar fashion, we have $|D^{\theta}_{r,t}D_r(r^{m+1}u^0(r,t))|\leq C\varepsilon$, hence we get $||u||\leq C\varepsilon$.

When $g(r) = 0$, we have $r^{m+1}u^0(r,t) = D_t I(f)(r,t)$. Proceeding as before, we obtain the same results. The other cases are also similar, so we omit the proof. Thus we prove the Proposition 4.1. □

In the rest of this section, we shall prove Lemmas 4.3 and 4.4. We now set

$$\overline{K}(\rho,r,t) = r^m \tilde{K}(\rho,r,t) = \frac{(-1)^m}{2m!}\rho^{2m+1}\left(\frac{\partial}{\partial\rho}\frac{1}{2\rho}\right)^m \phi^m(\rho,r,t). \tag{4.12}$$

Then Lemma 4.3 is equivalent to the following lemma.

Lemma 4.5. *We make the assumptions of Lemma 4.3. Then for any $\alpha \in \mathbb{Z}^2_+$, there exists a positive constant $C = C(m,\alpha)$ such that*

$$|D^\alpha_{r,t}\overline{K}(\rho,r,t)| \le Cr^m\ \chi_\alpha(\rho).$$

PROOF: It follows from (4.12) that

$$D^\alpha_{r,t}\overline{K}(\rho,r,t) = \sum_{j=0}^{m} C_j K_{j,\alpha}(\rho,r,t), \tag{4.13}$$

where we have set

$$K_{j,\alpha}(\rho,r,t) = \rho^{j+1}D_r^{\alpha_1}D_t^{\alpha_2+j}\phi^m(\rho,r,t).$$

We set $\beta = (\alpha_1, \alpha_2 + j)$. When $0 \le |\alpha| + j \le m$, we have

$$\begin{aligned} D^\beta_{r,t}\phi^m(\rho,r,t) &= D^\beta_{r,t}(r+t-r)^m\ (r-t+r)^m \\ &= \sum_{\sigma\le\beta} C_\sigma (r+t-r)^{m-|\sigma|}\ (r-t+r)^{m-|\beta-\sigma|}. \end{aligned}$$

Since $r+t-\rho \le 2r$ and $r-t+\rho \le 2r,\ 2<\rho>$, we get

$$|D^\beta_{r,t}\phi^m(\rho,r,t)| \le C_{m,\beta}r^m\ \ <\rho>^{m-|\beta|},$$

hence $|D^\alpha_{r,t}K_{j,\alpha}(\rho,r,t)| \le C_{m,\beta}\ r^m\ \ <\rho>^{m+1-|\alpha|}$.

If $|\beta| = |\alpha| + j = m+1$, we have then $|D^\alpha_{r,t} K_{j,\alpha}(\rho, r, t)| \le C_{m,\beta} < \rho >^{m+2-|\alpha|} r^{2m-|\beta|}$. Moreover, when $|\beta| \ge m+2$, we obtain $D^\alpha_{r,t} K_{j,\alpha}| \le C_{m,\beta} < \rho >^{m+1} r^{2m-|\beta|}$. Combining these estimates, we get the desired estimates. □

We shall prepare the following lemma to prove Lemma 4.4.

Lemma 4.6.

(*i*) *Assume that* $r \ge 1$. *Let* $\beta, \gamma \in \mathbb{Z}^2_+$. *If* $|\beta + \gamma| \le m - 1$ *or* $2m + 1 \le |\beta + \gamma|$, *we have* $D^\beta_{r,t}(D^\gamma_{r,t}\phi^m(\rho, r, t)|_{\rho = t \pm r}) = 0$. *Moreover, when* $m \le |\beta + \gamma| \le 2m$, *there exists a positive constant* $C_1 = C_1(m, \beta, \gamma)$ *such that*

$$|D^\beta_{r,t}(D^\gamma_{r,t}\phi^m(\rho, r, t)|_{\rho = t \pm r})| \le C_1 r^{2m-|\beta+\gamma|}; \tag{4.14}$$

(*ii*) *Assume that* $r \ge t$. *Let* $\beta, \gamma \in \mathbb{Z}^2_+$. *If* $|\beta+\gamma| \ge 2m+1$, $D^\beta_{r,t}(D^\gamma_{r,t}\phi^m(\rho, r, t)|_{\rho = r-t}) = 0$. *Moreover, when* $|\beta + \gamma| \le 2m$, *there exist positive constants* $C_i = C_i(m, \beta, \gamma)$, $i = 2, 3$ *such that*

$$|D^\beta_{r,t}(D^\gamma_{r,t}\phi^m(\rho, r, t)|_{\rho = r-t})| \le C_2\, r^m < r - t >^{m-|\beta+\gamma|}, \quad |\beta + \gamma| \le m, \tag{4.15$_1$}$$

$$|D^\beta_{r,t}(D^\gamma_{r,t}\phi^m(\rho, r, t)|_{\rho = r-t})| \le C_3\, r^{2m-|\beta+\gamma|}, \quad m+1 \le |\beta + \gamma| \le 2m. \tag{4.15$_2$}$$

Proof: We shall show only (4.15), because the others are handled in a similar fashion. It follows from Leibniz' rule that for $|\gamma| \le 2m$

$$D^\gamma_{r,t}\phi^m(\rho, r, t)|_{\rho = r-t} = \sum_{\sigma,\gamma} C_\sigma\ (2t)^{m-|\sigma|}(2(r-t))^{m-|\gamma-\sigma|},$$

where $\sum_{\sigma,\gamma}$ stands for the sum over $\sigma \in \mathbb{Z}^2_+$ such that $\sigma \le \gamma$ and $m - |\sigma|$, $m - |\gamma - \sigma| \ge 0$. Moreover, we get

$$|D^\beta_{r,t}(D^\gamma_{r,t}\phi^m(\rho, r, t)|_{\rho = r-t})| \le \sum_{\sigma+\delta,\gamma+\beta} C_{\sigma,\delta}\ t^{m-|\sigma+\delta|}\ (r-t)^{m-|\gamma+\beta-\sigma-\delta|}. \tag{4.16}$$

Let $|\beta + \gamma| \le m$. If $t \le r/2$, *i.e.*, $t \le r - t$, we have

$$t^{m-|\sigma+\delta|}(r-t)^{m-|\gamma+\beta-\sigma-\delta|} \le (r-t)^{2m-|\beta+\gamma|} \le r^m < r - t >^{m-|\beta+\gamma|}.$$

If $t \geq r/2$, *i.e.*, $t \geq r - t$, we have

$$t^{m-|\sigma+\delta|}(r-t)^{m-|\gamma+\beta-\sigma-\delta|} \leq t^{m-|\sigma+\delta|}(r-t)^{m-|\beta+\gamma|}\, t^{|\sigma+\delta|}$$
$$\leq r^m < r-t >^{m-|\beta+\gamma|}\,.$$

Therefore we have $(4.15)_1$ from (4.16). Furthermore, when $m+1 \leq |\beta+\gamma| \leq 2m$, we have then

$$t^{m-|\sigma+\delta|}(r-t)^{m-|\gamma+\beta-\sigma-\delta|} \leq r^{2m-|\beta+\gamma|},$$

hence we obtain (4.15). □

The following corollary is equivalent to Lemma 4.4.

Corollary 4.7.

(*i*) *Assume that* $r \geq 1$. *Then for any* β, $\gamma \in \mathbb{Z}_+^2$, *there exists a positive constant* $C_1 = C_1(m,\beta,\gamma)$ *such that*

$$(4.17) \qquad |D^\beta_{r,t}(\,D^\gamma_{r,t}\overline{K}(\rho,r,t)\big|_{\rho=t\pm r})| \leq C_1\; r^m\; \chi_{\beta+\gamma}(t\pm r);$$

(*ii*) *Assume that* $r \geq t$. *Then for any* β, $\gamma \in \mathbb{Z}_+^2$, *there exists a positive constant* $C_2 = C_2(m,\beta,\gamma)$ *such that*

$$(4.18) \qquad |D^\beta_{r,t}(\,D^\gamma_{r,t}\overline{K}(\rho,r,t)\big|_{\rho=r-t})| \leq C_2\; r^m\; \chi_{\beta+\gamma}(r-t).$$

PROOF: Analogously to (4.13), we have

$$D^\gamma_{r,t}\overline{K}(\rho,r,t)\big|_{\rho=t\pm r} = \sum_{j=0}^{m} C_j \check{K}_j(r,t),$$

where we have set $\check{K}_j(r,t) = (t\pm r)^{1+j}\;(D^\gamma_{r,t}D^j_\rho\phi^m)(\rho,r,t)\big|_{t\pm r}$. Proceeding as in the proof of Lemma 4.5 with employing Lemma 4.6 (*i*), we obtain (4.17). Moreover, we get (4.18) in a similar fashion. □

5. Proof of the Main Theorem.

Proposition 5.1. *Let $u \in X_\delta$. Suppose that $G(\lambda_1, \lambda_2) \in C^{m+2}(B_1(0))$ is even in λ_2 and satisfy (1.5) and that $p > 2$ for $n = 3$. Then we have $L(u) \in X$ and there exists a positive constant $C_i = C_i(m, \kappa, G)$, $i = 1, 2$ such that*

$$|L(u)(r,t)| \leq C_1 < r >^{-m} ||u||^p. \tag{5.1}$$

Moreover, we have $||L(u)|| \leq C_2 ||u||^p$.

PROOF: We set

$$E(r,t;\tau) = \int_{-1}^{1} H(t - \tau + r\sigma, \tau)\,(1-\sigma^2)^m d\sigma. \tag{5.2}$$

By Corollary 3.3, $H(\rho,\tau) \in C^{2,0}(\mathbb{R} \times [0,\infty)$, hence we have $E(r,t;\tau) \in C^{2,2,0}(\Omega)$, where we have set $\Omega = \mathbb{R} \times [0,\infty) \times [0,\infty)$. Moreover by Proposition 2.5, we have $r^{m+1} D_t E(r,t;\tau) \in C^{m+2,m+2,0}(\Omega)$, $r^{m+1} E(r,t;\tau) \in C^{m+3,m+3,0}(\Omega)$. Note that we have from (3.2) and (3.19)

$$L(u)(r,t) = \int_0^t E(r,t;\tau) d\tau, \quad D_t L(u)(r,t) = \int_0^t D_t E(r,t;\tau) d\tau. \tag{5.3}$$

Therefore we get $L(u)(r,t) \in C^2(\mathbb{R} \times [0,\infty))$, $r^{m+1} D_t L(u)(r,t) \in C^{m+2,0}(\mathbb{R} \times [0,\infty))$ and $r^{m+1} L(u)(r,t) \in C^{m+3,0}(\mathbb{R} \times [0,\infty))$. Hence we have only to show (5.1) for $r \geq 0$, because $L(u)(r,t)$ is even in r.

Analogously to the proof of (4.1) for $0 \leq r \leq 1$, we obtain for such r

$$|r^m E(r,t;\tau)| \leq C < t - \tau >^{-(p-1)(m+1)} ||u||^p, \tag{5.4}$$

if we employ (3.15). Hence we obtain (5.1) for $0 \leq r \leq 1$, because of the assumptions on p.

In what follows, let $r \geq 1$. To begin with, we note an analogue to Lemma 4.2.

Lemma 5.2. *Let $r \geq 1$ and set $\gamma_\pm = t - \tau \pm r$ and*

$$I(r,t,\tau) = r^{m+1} E(r,t;\tau). \tag{5.5}$$

Then we have

$$I(r,t;\tau) = \frac{1}{r^m} \int_{\gamma_-}^{\gamma_+} H(\rho,\tau)\, \phi^m(\rho, r, t-\tau) d\rho. \tag{5.6}$$

Moreover, we get

$$I(r,t;\tau) = \int_{|\gamma_-|}^{\gamma_+} F(\rho,\tau)\ \tilde{K}(\rho,r,t-\tau)d\rho. \tag{5.7}$$

We shall now return to the proof of (5.1) for $r \geq 1$. By (5.3) and (5.5), we have

$$r^{m+1}L(u)(r,t) = \int_0^t I(r,t;\tau)d\tau. \tag{5.8}$$

We divide the τ-integral into $K_i(r,t)$, $i=1,2$, that is,

$$K_2(r,t) = \int_{\tau_-}^{\tau_+} I(r,t;\tau)d\tau, \quad K_1(r,t) = \int_0^t I(r,t;\tau)d\tau - K_2(r,t), \tag{5.9}$$

where we have set $\tau_\pm = \max(t-r\pm 1, 0)$.

We first deal with the case $0 \leq \tau \leq t_-$ or $t_+ \leq \tau \leq t$ where $|\gamma_-| \geq 1$. Therefore (5.7) yields

$$|I(r,t;\tau)| \leq C||u||^p \int_{|\gamma_-|}^{\gamma_+} < \rho >^{-(p-1)(m+1)} d\rho, \tag{5.10}$$

if we employ (3.8) and (4.7) with $\alpha = 0$. Therefore we have

$$|K_1(r,t)| \leq C||u||^p \int_0^t d\tau \int_{|\gamma_-|}^{\gamma_+} < \rho >^{-(p-1)(m+1)} d\rho. \tag{5.11}$$

Applying the following lemma to (5.11), we easily have

$$|K_1(r,t)| \leq C||u||^p r. \tag{5.12}$$

Lemma 5.3. *For any $\mu > 0$, there exist positive constant $C_i = C_i(\mu)$, $i=1,2$ such that*

(*i*) $\quad |\int_0^t d\tau \int_{\gamma_1}^{\gamma_2} < \rho >^{-1-\mu} d\rho| \leq C_1 r;$

(*ii*) $\quad |\int_0^t d\tau \int_{|\gamma_1|}^{\gamma_2} < \rho >^{-2-\mu} d\rho| \leq C_2.$

Proof: (*i*) Changing the variables as $\xi = \tau + \rho$, $\eta = \tau - \rho$, we have

$$\begin{aligned} |\int_0^t d\tau \int_{\gamma_1}^{\gamma_2} < \rho >^{-1-\mu} d\rho| \leq & C\int_{t-r}^{t+r} d\xi \int_{-\xi}^{-\xi+2t} < \frac{\xi-\eta}{2} >^{-1-\mu} d\eta \\ \leq & C\int_{t-r}^{t+r} d\xi \int_{\mathbb{R}} < \eta >^{-1-\mu} d\eta. \end{aligned}$$

(ii) As above we have

$$\int_0^t d\tau \int_{|\gamma_1|}^{\gamma_2} <\rho>^{-2-\mu} d\rho = \frac{1}{2}\int_{t-r}^{t+r} d\xi \int_{-\xi}^{t-r} <\frac{\xi-\eta}{2}>^{-2-\mu} d\eta.$$

Since $<\frac{\xi-\eta}{2}>^{-2-\mu} \leq C(1+\xi-\eta)^{-1-\mu}$, ξ-integral is dominated by $C(1+\xi-t+r)^{-1-\mu}$. Therefore we have the desired estimate. □

Next we consider the case $\tau_1 \leq \tau \leq \tau_2$ where we shall use (5.6). The following lemma is proved similarly to Lemma 4.5, so we omit the proof.

Lemma 5.4: We put $\Phi(\rho,r,t) = \frac{1}{r^m}\phi^m(\rho,r,t)$. Let χ_α be as given in (4.8). Assume that $-1 \leq t-r \leq \rho \leq t+r$ or $0 \leq r-t \leq \rho \leq t+r$. Let $r \geq 1$.

(i) For any $\alpha \in \mathbb{Z}_+^2$, there exists a positive constant $C = C(m,\alpha)$ such that

$$|D_{r,t}^\alpha \Phi(\rho,r,t)| \leq C <\rho>^{-1} \chi_\alpha(\rho); \tag{5.13}$$

(ii) Let $\alpha \in \mathbb{Z}_+^2$. Then we have

$$D_{r,t}^\alpha \Phi(\rho,r,t)\big|_{t\pm r} = 0, \quad \text{if} \quad |\alpha| \leq m-1. \tag{5.14}$$

Combining (3.15) and (5.11) with $\alpha = 0$, (5.6) yields

$$|I(r,t;\tau)| \leq C||u||^p \int_{\gamma_-}^{\gamma_+} <\rho>^{-(p-1)(m+1)+m} d\rho \leq Cr||u||^p. \tag{5.15}$$

Since $t_+ - t_- \leq 2$, we have $|K_2(r,t)| \leq Cr||u||^p$ from (5.9) and (5.15). Hence combining this with (5.9) and (5.12), we obtain $|L(u)(r,t)| \leq C <r>^{-m} ||u||^p$.

Proceeding as before and repeating the similar argument to Proposition 4.1, we also obtain the estimates for the derivatives of $r^{m+1}L(u)$, $r^{m+1}D_tL(u)$. □

Proposition 5.5. *We make the assumptions of Proposition 5.1. Then there exist constants $C_i = C_i(m,\kappa,G) > 0$, $1 \leq i \leq 3$, such that for u, $\hat{u} \in X_\delta$, the followings hold:*

$$|||L(u) - L(\hat{u})||| \leq C_1|||u-\hat{u}||| \, (|||u|||^{p-1} + |||\hat{u}|||^{p-1}), \tag{5.16}$$

$$
(5.17) \qquad \begin{aligned} &||L(u) - L(\hat{u})|| - |||L(u) - L(\hat{u})||| \\ &\leq C_2|||u-\hat{u}|||^{\mu}\ (||u||^{m+2} + ||\hat{u}||^{m+2}) + C_3||u-\hat{u}||\ (||u||^{p-1} + ||\hat{u}||^{p-1}), \end{aligned}
$$

where μ is as given below $(1.5)_2$.

Proof: Note that for $0 \leq r \leq 1$,

$$
L(u) - L(\hat{u}) = \int_0^t d\tau \int_{-1}^{1} \{H(t-\tau+r\sigma,\tau) - \hat{H}(t-\tau+r\sigma,\tau)\}\ (1-\sigma^2)^m d\sigma
$$

and for $r \geq 1$,

$$
\begin{aligned} L(u) - L(\hat{u}) &= \frac{1}{r^{m+1}} \int_0^t d\tau \int_{t-\tau-r}^{t-\tau+r} \{H(\rho,\tau) - \hat{H}(\rho,\tau)\}\ \phi^m(\rho,r,t-\tau) d\rho \\ &= \int_0^t d\tau \int_{t-\tau-r}^{t-\tau+r} \{F(\rho,\tau) - \hat{F}(\rho,\tau)\}\ \tilde{K}(\rho,r,t-\tau) d\rho \end{aligned}
$$

Employing (3.16) and repeating the analogous argument of the proof of Proposition 5.1, we obtain the proposition. □

Now we are in a position to prove the Main theorem.

Proof of the Main Theorem: We shall concentrate on the case $m \geq 1$ or $m = 0$ and $2 < p < 3$, because the procedure is simpler when $m = 0$ and $p = 3$. We define a sequence of functions $\{u_k\}_{k\in\mathbb{Z}_+}$ by

$$
(5.18) \qquad u_k = u_0 + L(u_{k-1}) \quad \text{for} \quad k \geq 1, \quad u_0 = u^0,
$$

where $u^0(r,t)$ is the solution of the problem (2.1), (1.2). By Proposition 4.1, there exists a positive constant C_0, depending only on n, such that

$$
(5.19) \qquad ||u^0|| \leq C_0\varepsilon.
$$

Moreover, (5.16) and (5.17) imply that for $u \in X_\delta$,

$$
(5.20) \qquad ||L(u)|| \leq C_4||u||^p,
$$

where $C_4 = C_1 + C_2 + C_3$ and C_1, C_2, C_3 are as in (5.16) and (5.17).

Let ε_0 be the greatest upper bound of ε satisfying

$$2C_0\varepsilon \le \min(\delta, 2), \quad 2^{p+2}C_4(C_0\varepsilon)^{p-1} \le 1. \tag{5.21}$$

In what follows, we assume $0 < \varepsilon \le \varepsilon_0$.

First we shall show $u_k \in X_\delta$ for any $k \in \mathbb{Z}_+$. To do this, it is enough to show

$$||u_k|| \le 2||u_0||, \quad k \in \mathbb{Z}_+, \tag{5.22}$$

because of (5.19) and (5.21). Indeed, (5.22) follows from the induction with employing (5.18), (5.20) and (5.21).

Next we shall show that $\{u_k\}$ is a Cauchy sequence in X_δ. From (5.18), (5.16), (5.22) and (5.21), we obtain

$$|||u_{k+1} - u_k||| = |||L(u_k) - L(u_{k-1})||| \le \frac{1}{4}|||u_k - u_{k-1}|||, \quad k \ge 1. \tag{5.23}$$

Moreover, employing these relations repeatedly, we have

$$|||u_k - u_{k-1}||| \le (\frac{1}{4})^{k-1} |||u_1 - u_0||| \le (\frac{1}{4})^k 4C_4||u_0||^p, \quad k \ge 1,$$

where we have used (5.18), (5.20). Noting $C_0\varepsilon \le 1$, we have from this and (5.19),

$$|||u_k - u_{k-1}||| \le 4C_4(C_0\varepsilon)^p(\frac{1}{4})^k \le 4C_4C_0\varepsilon(\frac{1}{2})^k, \quad k \ge 1.$$

Combining this with (5.17) and (5.23), we obtain

$$||L(u_k) - L(u_{k-1})|| \le \frac{1}{2}||u_k - u_{k-1}|| + C_5\varepsilon^p(\frac{1}{2})^{k\mu}, \tag{5.24}$$

where we have set $C_5 = 2^{m+3+2\mu}C_3C_4^\mu C_0^p$.

Now we shall show that $\sum_{k=1}^\infty ||u_k - u_{k-1}||$ converges. Indeed, noting $0 < \mu < 1$, we have from (5.18) and (5.24),

$$||u_k - u_{k-1}|| \le (\frac{1}{2})^{k-1}||u_1 - u_0|| + C_5\varepsilon^p(k-1)(\frac{1}{2})^{k\mu},$$

which yields $\sum_{k=1}^\infty ||u_k - u_{k-1}|| \le 2C_4(C_0\varepsilon)^p + C_6\varepsilon^p$. Therefore u_k is a Cauchy sequence in X_δ. Since X_δ is complete, there exists $u \in X_\delta$ such that u_k converges to u. In addition,

u satisfies the integral equation (3.1), hence u is the C^2-solution of the Cauchy problem (1.1), (1.2), because of Proposition 3.4. This completes the proof of the Main Theorem. □

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