ON MARKOV PROPERTIES OF GAUSSIAN GENERALIZED RANDOM FIELDS

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ON MARKOV PROPERTIES OF GAUSSIAN GENERALIZED RANDOM FIELDS

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1. Introduction.

The paper by Preiss and Kotecky [1] gave us a good occasion to learn delicate points concerning Markov properties of generalized random fields. They clearly pointed out the origin of confusions and misunderstandings so far. Moreover they selected a presumably most suitable definition of the Markov property so that we can avoid unnecessary confusions. In the present note we shall follow their line and observe some interesting phenomenon proper to multi-component Gaussian generalized random fields. Let $\Omega_0^1(\mathbb{R}^d)$ be the space of 1-forms on $\mathbb{R}^d$ with compact supports. We donote by Π the projection operator in the space of square integrable 1-forms onto coclosed 1-forms. Actually the mean 0 Gaussian system $\{X(\varphi)\,;\, \varphi \in \Omega_0^1(\mathbb{R}^d)\}$ with

$$E[\,X(\varphi)^2\,] = (\Pi\varphi\,,\varphi)_{L^2}$$

is Markov and we shall prove this in section 3. Section 2 explains the relation between the Markov property and the locality.

In the rest of this section we shall compare two possible definitions of the Markov property. Let $\{X(\varphi)\,;\, \varphi \in C_0^\infty(\mathbb{R}^d)\}$ be a generalized random field defined on a complete probability space $(\Omega, \mathcal{F}, P)$. If D is an open subset of $\mathbb{R}^d$, then we define

$$\mathcal{F}_D := \sigma\{X(\varphi)\,;\, \text{supp}\ \varphi \subset D\} \vee \mathcal{N},$$

where $\mathcal{N}$ is the trivial sub σ-field of $\mathcal{F}$. On the other hand, for a closed subset C of $\mathbb{R}^d$ we set

$$\mathcal{F}_C := \bigcap\{\mathcal{F}_D;\ D \text{ open}, \supset C\},$$

(1.1) Definition. We say that $\{X(\varphi)\}$ is *MII-Markov*, if $\mathcal{F}_{\overline{D}} \perp \mathcal{F}_{\mathbb{R}^d\setminus D} \mid \mathcal{F}_{\partial D}$ for any open subsets D. (Two sub σ-fields $\mathcal{F}_{\overline{D}}$ and $\mathcal{F}_{\mathbb{R}^d\setminus D}$ are conditionally independent given $\mathcal{F}_{\partial D}$.)

A superstition on the MII-Markov property of Gaussian generalized random fields used to be that the dual bilinear form of the covariance to a MII-Markov Gaussian generalized random field had local property. (This statement is usually described as the spacial independece of the dual generalized random field. And the truth of the statement critically depends on the way how we define the locality.) A counter example, given in [1], to the above assertion is as follows: Let $\{X(\varphi)\}$ be a mean 0 Gaussian system with

$$E[\,X(\varphi)^2\,] = \int_{\mathbb{R}} \left\{ \varphi(x)^2 + \left(\frac{d}{dx}\varphi(x)\right)^2 \right\} dx. \tag{1.2}$$

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The r.h.s. of (1.2) is nothing but the natural inner product of the Sobolev space of order 1. We can check that

$$\mathcal{F}_{\overline{D}} \perp \mathcal{F}_{\mathbb{R}^d \setminus D} \text{ and } \mathcal{F}_{\partial D} = \mathcal{N}$$

holds for any open subset D of $\mathbb{R}$ by examining the corresponding condition in the Sobolev space. Hence $\{X(\varphi)\}$ is MII-Markov. But the dual bilinear form of (1.2) has an integral kernel $\frac{1}{2}e^{-|x-y|}$ and therefore the dual bilinear form is not local.

The origin of the superstition in former days was caused by the misunderstanding that the following relation would always hold for any open covering $\{D_+, D_-\}$ of $\mathbb{R}^d$.

$$\text{(1.3)} \qquad \mathcal{F}_{D_+} = \mathcal{F}_{\mathbb{R}^d \setminus D_-} \vee \mathcal{F}_{D_+ \cap D_-}.$$

However the relation (1.3) is not necessarily true unless $\{X(\varphi)\}$ is realized by a continuous random field. In fact the Gausssian system characterized by (1.2) also serves as a counter example to (1.3). To see this we set $D_+ = \mathbb{R}$ and $D_- = (-\infty, 0)$. Then

$$\mathcal{F}_{\mathbb{R}} \neq \mathcal{F}_{[0,\infty)} \vee \mathcal{F}_{(-\infty,0)},$$

because the latter is independent of $\sigma\{X(f)\}$, where $f(x) = e^{-|x|}$.

Based on the discussion so far, Preiss and Kotecky [1] told us that we should include the following postulate in the definition of the Markov property:

$$\text{(1.4)} \qquad \mathcal{F}_{\mathbb{R}^d} = \mathcal{F}_{\partial D} \vee \mathcal{F}_{\mathbb{R}^d \setminus \partial D}, \text{ for all open subsets } D \text{ of } \mathbb{R}^d.$$

This condition seems natural because, as we shall see in the succeeding section, the Markov property with (1.4) and the locality is almost directly related as for Gaussian systems.

(1.5) Definition. We say that $\{X(\varphi)\}$ is *MI-Markov*, if it is MII-Markov and satisfies the condition (1.4).

2. The Markov property and the locality.

We formulate the question in terms of Hilbert space category, since we shall discuss Gaussian systems. Let $\mathbb{E} \to M$ be a real vector bundle over a paracompact C^∞-manifold. We use the notation $\mathcal{D} = \mathcal{D}(\mathbb{E})$ for the totality of C^∞-sections of $\mathbb{E}$ with compact supports. $\mathcal{D}$ is equipped with the Schwartz topology. Our objects to study are continuous linear maps $\gamma : \mathcal{D} \longrightarrow \mathcal{H}$ with real Hilbert spaces $\mathcal{H}$ as targets, in which the images are dense. In the study of a generalized random field $\{X(\varphi)\}$, $\mathcal{H}$ is a subspace of the L^2-space on $(\Omega, \mathcal{F}, P)$ and γ is obtained from the map $\varphi \longmapsto X(\varphi)$. Given $\gamma : \mathcal{D} \longrightarrow \mathcal{H}$, we introduce a family of closed subspaces $\{\mathcal{H}(D)\}$ (resp. $\{\mathcal{H}(C)\}$) of $\mathcal{H}$ indexed by open (resp. closed) subsets of M as follows:

$$\mathcal{H}(D) = \overline{\{\gamma(\varphi)\,;\, \operatorname{supp} \varphi \subset D\}},$$
$$\mathcal{H}(C) = \bigcap \{\mathcal{H}(D)\,;\, D \text{ open}, \supset C\},$$

where $\mathcal{H}(\emptyset) = \{0\}$ is understood.

(2.1) Definition. The map $\gamma : \mathcal{D} \longrightarrow \mathcal{H}$ is called *Markov* if

$$\mathcal{H}(\overline{D}) = \mathcal{H}(\partial D) \oplus \mathcal{H}(M \setminus D)^{\perp} \quad \text{for all open subsets } D,$$

where $\oplus$ means the orthogonal direct sum and $\perp$ the orthogonal complement.

It is easy to see that the followings are equivallent:

$$\begin{aligned} \mathcal{H}(\overline{D}) &= \mathcal{H}(\partial D) \oplus \mathcal{H}(M \setminus D)^{\perp}, \\ \mathcal{H} &= \mathcal{H}(\overline{D})^{\perp} \oplus \mathcal{H}(\partial D) \oplus \mathcal{H}(M \setminus D)^{\perp}. \end{aligned} \tag{2.2}$$

Unless $\mathcal{H}$ is so embedded in $\mathcal{D}'$ that the composition $\mathcal{D} \xrightarrow{\gamma} \mathcal{H} \hookrightarrow \mathcal{D}'$ yields a natural inclusion $\mathcal{D} \hookrightarrow \mathcal{D}'$, there is no natural way to see how the shape of the index sets is reflected in the filtration $\{\mathcal{H}(D)\}$ (resp. $\{\mathcal{H}(C)\}$). However the situation turns out to be favorable, if we once recall the notion of duality. Since the γ has dense image in its target, the transposed map $\gamma*$ realizes $\mathcal{H}'(\equiv$ the topological dual of $\mathcal{H})$ as a subspace of $\mathcal{D}'$. We shall detect the shape of the index sets in the following way. Let $\kappa : \mathcal{H} \longrightarrow \mathcal{H}'$ be the Riesz isomorphism given by

$${}_{\mathcal{H}'}\langle \kappa u \,, v \rangle_{\mathcal{H}} = (u \,, v)_{\mathcal{H}} \quad u \,, v \in \mathcal{H}.$$

Then we obtain the following

(2.3) Lemma. *Let D be a open subset and C be a closed subset. Then*

$$\begin{aligned} \mathcal{H}(D)^{\perp} &= \{u \in \mathcal{H};\ \mathit{supp}\ \gamma^* \circ \kappa(u) \subset M \setminus D\}, \\ \mathcal{H}(C)^{\perp} &= \overline{\{u \in \mathcal{H};\ \mathit{supp}\ \gamma^* \circ \kappa(u) \subset M \setminus C\}}, \end{aligned}$$

where $\gamma* : \mathcal{H}' \longrightarrow \mathcal{D}'$ stands for the transposed map of γ.

Proof. Let F be a closed subset of M and $T \in \mathcal{D}'$. Then, by definition, supp $T \subset F$ if and only if $\langle T, \varphi \rangle = 0$ for all $\varphi \in \mathcal{D}$ with supp $\varphi \subset M \setminus F$. The first statement immediately follows from this characterization. Considering the orthogonal complement of the defining relation of $\mathcal{H}(C)$, we see that

$$\begin{aligned} \mathcal{H}(C)^{\perp} &= \overline{\bigcup \{\mathcal{H}(D)^{\perp};\ D \text{ open}, \supset C\}}, \\ &= \overline{\{u \in \mathcal{H};\ \text{supp } \gamma^* \circ \kappa(u) \subset M \setminus D \text{ for some } D \text{ open}, \supset C\}}, \end{aligned}$$

and hence the second statement follows. $\square$

Once we know the characterization of the orthogonal complements of the subspaces $\mathcal{H}(C)$ in terms of the duality, our main theorem is an immediate consequence of it (see (2.2)).

(2.4) Theorem. *The map $\gamma : \mathcal{D} \longrightarrow \mathcal{H}$ is Markov if and only if*

$$\begin{aligned} &\overline{\{v \in \mathcal{H}';\ \mathit{supp}\ \gamma^*(v) \subset M \setminus \partial D\}} \\ =&\overline{\{v \in \mathcal{H}';\ \mathit{supp}\ \gamma^*(v) \subset M \setminus \overline{D}\}} \oplus \overline{\{v \in \mathcal{H}';\ \mathit{supp}\ \gamma^*(v) \subset D\}} \end{aligned} \tag{2.5}$$

for all open subsets D.

Now we are at the right position to speak of the locality. Our objects are continuous linear injections $\gamma* : \mathcal{H}' \longrightarrow \mathcal{D}'$ with real Hilbert spaces $\mathcal{H}'$ as sources. (Here we have used the notations $\gamma*$ and $\mathcal{H}'$, in order to keep the continuity of the arguments from the preceeding paragraphs.)

(2.6) Definition. The map $\gamma* : \mathcal{H}' \longrightarrow \mathcal{D}'$ is called *semi-local,* if $v_1, v_2 \in \mathcal{H}'$ are mutually orthogonal whenever $\operatorname{supp} \gamma^*(v_1) \cap \operatorname{supp} \gamma^*(v_2) = \emptyset$.

Obviously the semi-locality of $\gamma* : \mathcal{H}' \longrightarrow \mathcal{D}'$ is a part of the condition (2.5). In particular, if the semi-locality has been established, we have only to show the following inclusion:

$$\text{(2.7)}\qquad \frac{\{v \in \mathcal{H}'; \operatorname{supp} \gamma^*(v) \subset M \setminus \partial D\}}{\subset \{v \in \mathcal{H}'; \operatorname{supp} \gamma^*(v) \subset M \setminus \overline{D}\} + \{v \in \mathcal{H}'; \operatorname{supp} \gamma^*(v) \subset D\}.}$$

Presumably the easiest method to prove the relation (2.7) is to cut off v by multiplying a suitable C^∞-function. We shall make up the following situation:

(2.8) Definition. The map $\gamma* : \mathcal{H}' \longrightarrow \mathcal{D}'$ is called *compatible with C^∞-cut off*, if the image is stable under the following manipulation:
Let $v \in \mathcal{H}'$. Suppose two disjoint open sets D_1, D_2 and $\chi \in C^\infty(M)$ satisfy that

$$D_1 \cup D_2 \supset \operatorname{supp} \gamma^*(v), \chi = 1 \text{ on } D_1 \text{ and } \chi = 0 \text{ on } D_2.$$

Then $\chi\gamma*(v) \in \gamma*(\mathcal{H}')$.

We remark that the compatibility with C^∞-cut off is an aspect of the locality. Let $\mathbb{F} \to M$ be another real vector bundle over M and $L : \Gamma(\mathbb{F}) \to \Gamma(\mathbb{E})$ be a linear differential operator. $L' : \mathcal{D}'(\mathbb{E}) \to \mathcal{D}'(\mathbb{F})$ stands for the transposed map of L. If we consider the situation in (2.8) with $\gamma*(v)$ replaced by $T \in \mathcal{D}'(\mathbb{E})$, then $L'\chi T = \chi L'T$. Moreover $\{v \in \mathcal{H}'; L'\gamma*(v) = 0\}$ is a closed subspace of $\mathcal{H}'$.

(2.9)Theorem. *If the transposed map of $\gamma : \mathcal{D} \longrightarrow \mathcal{H}$ is semi-local and compatible with C^∞-cut off, then γ is Markov.*

Proof. We show (2.7). Suppose $\operatorname{supp} \gamma^*(v) \subset M \setminus \partial D$. We note that $M \setminus \partial D$ consists of two disjoint open sets $D, M \setminus \overline{D}$. Therefore we can choose two disjoint open sets D_1 and D_2 with

$$\overline{D_1} \subset D, \overline{D_2} \subset M \setminus \overline{D} \text{ and } D_1 \cup D_2 \supset \operatorname{supp} \gamma^*(v).$$

Since M is paracompact, there are C^∞-functions χ_1, χ_2 such that

$$\chi_1 = 1 \text{ on } D_1, \operatorname{supp} \chi_1 \subset D \text{ and } \chi_2 = 1 \text{ on } D_2, \operatorname{supp} \chi_2 \subset M \setminus \overline{D}.$$

Therefore, thanks to the compatibility with C^∞-cut off, there exists $v_1, v_2 \in \mathcal{H}'$ with $\operatorname{supp} \gamma^*(v_1) \subset D, \operatorname{supp} \gamma^*(v_2) \subset M \setminus \overline{D}$ and $v_1 + v_2 = v$. $\square$

The filtration of closed subspaces $\{\mathcal{H}(D)\}$ (resp. $\{\mathcal{H}(C)\}$) of $\mathcal{H}$ behaves quite well, if the map $\gamma : \mathcal{D} \longrightarrow \mathcal{H}$ is Markov. To see this we note that

(2.10) Lemma. *Suppose that $\mathcal{H}(M \setminus D)^\perp \subset \mathcal{H}(\overline{D})$ for all open subsets D. Then $\mathcal{H}(M \setminus D)^\perp \subset \mathcal{H}(D)$ for all open subsets D.*

Proof. By virtue of Lemma 2.3, all we have to show is that

$$\text{if } u \in \mathcal{H} \text{ satisfies } \operatorname{supp} \gamma^* \circ \kappa(u) \subset D, \text{ then } u \in \mathcal{H}(D).$$

To this end, we choose an open set G so that

$$\operatorname{supp} \gamma^* \circ \kappa(u) \subset G \subset \overline{G} \subset D.$$

We see by Lemma 2.3 that u is actually contained in $\mathcal{H}(M \setminus G)^{\perp}$. Then the assumption, with D replaced by G, tells us that $u \in \mathcal{H}(\overline{G})$. Since $\overline{G} \subset D$, we conclude that $u \in \mathcal{H}(D)$. □

Therefore the definition (2.1) already includes the postulate corresponding to (1.4). Namely we have

(2.11) Corollary. *Let the map $\gamma : \mathcal{D} \longrightarrow \mathcal{H}$ be Markov. Then*

$$\begin{aligned} \mathcal{H}(\overline{D}) &= \mathcal{H}(\partial D) + \mathcal{H}(D), \quad \text{and} \\ \mathcal{H}(M \setminus D) &= \mathcal{H}(\partial D) + \mathcal{H}(M \setminus \overline{D}) \qquad \text{for all open subsets } D. \end{aligned}$$

Finally we see that Definition 2.1 is actually another formulation of the MI-Markov property for Gaussian generalized random fields.

(2.12) Theorem. *The map $\gamma : \mathcal{D} \longrightarrow \mathcal{H}$ is Markov, if and only if*

$$\begin{aligned} \mathcal{H} &= \overline{\mathcal{H}(\partial D) + \mathcal{H}(M \setminus \partial D)}, \quad \text{and} \\ \mathcal{H}(\overline{D}) &= \mathcal{H}(\partial D) \oplus \left(\mathcal{H}(\overline{D}) \cap \mathcal{H}(M \setminus D)^{\perp}\right) \qquad \text{for all open subsets } D. \end{aligned}$$

Proof. We shall prove the sufficiency. Since open sets D and $M \setminus \overline{D}$ cover $M \setminus \partial D$, we see by the first condition that

$$\mathcal{H} = \overline{\mathcal{H}(D) + \mathcal{H}(\partial D) + \mathcal{H}(M \setminus \overline{D})}.$$

Clearly we have $\mathcal{H}(\overline{D}) \supset \mathcal{H}(D) + \mathcal{H}(\partial D)$ and $\mathcal{H}(M \setminus D) \supset \mathcal{H}(\partial D) + \mathcal{H}(M \setminus \overline{D})$. It then follows that

$$\mathcal{H} = \overline{\mathcal{H}(\overline{D}) + \mathcal{H}(M \setminus D)}.$$

On the other hand, we also assume that

$$\begin{aligned} \mathcal{H}(\overline{D}) &= \mathcal{H}(\partial D) \oplus \left(\mathcal{H}(\overline{D}) \cap \mathcal{H}(M \setminus D)^{\perp}\right), \\ \mathcal{H}(M \setminus D) &= \mathcal{H}(\partial D) \oplus \left(\mathcal{H}(M \setminus D) \cap \mathcal{H}(\overline{D})^{\perp}\right). \end{aligned}$$

Combining these relations, we get

$$\mathcal{H} = \overline{\left(\mathcal{H}(\overline{D}) \cap \mathcal{H}(M \setminus D)^{\perp}\right) + \mathcal{H}(\partial D) + \left(\mathcal{H}(M \setminus D) \cap \mathcal{H}(\overline{D})^{\perp}\right)}.$$

Moreover the summands under the closure sign are mutually orthogonal. Hence in the above relation we do not have to put the closure sign and, therefore, we see that

$$\mathcal{H} = \left(\mathcal{H}(\overline{D}) \cap \mathcal{H}(M \setminus D)^{\perp}\right) \oplus \mathcal{H}(\partial D) \oplus \left(\mathcal{H}(M \setminus D) \cap \mathcal{H}(\overline{D})^{\perp}\right).$$

Because the second condition claims that the second and the third summands constitute $\mathcal{H}(M \setminus D)$, we see that the first summand coincides with $\mathcal{H}(M \setminus D)^{\perp}$. By exactly same reasoning, we also conclude that the third summand coincides with $\mathcal{H}(\overline{D})^{\perp}$. Thus we have obtained (2.2). □

3. An example.

Let d, n and r be non negative integers with $0 < d, 2n < d$ and $0 < r < d$. We use $\wedge_r\mathbb{R}^d$ to denote the r-fold exterior product of $\mathbb{R}^d$. The natural inner product of $\wedge_r\mathbb{R}^d$ shall be denoted by $(\cdot\,,\cdot)$, which is induced by the universality of the exterior algebra. We obtain a Hilbert space $\mathcal{H}'$ from the following function space:

$$\{v : \mathbb{R}^d \longrightarrow \wedge_r\mathbb{R}^d \otimes \mathbb{C};\ \text{measurable}, v(-k) = \overline{v(k)}\,, v(k)\llcorner k = 0 \text{ a.e.} \\ \text{and} \int_{\mathbb{R}^d} (v(k)\,, v(-k))\,|k|^{2n}\,dk < \infty\},$$

where $\overline{}$ is the natural complex conjugation in $\wedge_r\mathbb{R}^d \otimes \mathbb{C}$ and $\llcorner$ denotes the interior multiplication. Since $2n < d$, it follows that

$$\int_{\mathbb{R}^d} (\widehat{\varphi}(k)\,, \widehat{\varphi}(-k))\,|k|^{-2n}\,dk < \infty \text{ for all } \varphi \in \mathcal{D} := C_0^\infty(\mathbb{R}^d \to \wedge_r\mathbb{R}^d).$$

Here $\widehat{\varphi}$ is the Fourie transform of φ. Therefore a continuous injection $\gamma* : \mathcal{H}' \longrightarrow \mathcal{D}'$ is defined by

$$\langle \gamma*(v)\,, \varphi\rangle = \int_{\mathbb{R}^d} (v(k)\,, \widehat{\varphi}(-k))\,dk, \quad \varphi \in \mathcal{D}.$$

(3.1) Lemma. *The map $\gamma^* : \mathcal{H}' \longrightarrow \mathcal{D}'$ is semi-local and compatible with C^∞-cut off.*

Proof. Let f be a locally integrable function on $\mathbb{R}^d$. We shall denote the distributional derivative of f of orden n by $\nabla^n f$. We define

$$W_n := \{f \in L^{2d/(d-2n)}(\mathbb{R}^d)\,; \nabla^n f \in L^2\}.$$

Fix a real number λ with $0 < \lambda < d$ and set $g(x) := |x|^\lambda, x \in \mathbb{R}^d$. Then $\|g\|_{q,w} < \infty$ for $q = \frac{d}{\lambda}$. The generalized Young inequality provides us the Sobolev inequality

$$\|g * f\|_p \leq \text{ Const}_{q,p'}\|g\|_{q,w}\|f\|_{p'},$$

where $p; p'$ are positive numbers with $1 < p, p'$ and $1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{p'}$. Taking into account that $\widehat{g}(k) \propto |k|^{\lambda-d}$, we set $\lambda - d = -\alpha$. Moreover we specify that $p' = 2$. Then we get, for $p = \frac{2d}{d-2\alpha}$,

$$\|g * f\|_p \leq \text{ Const } \|f\|_2,$$

where α must satisfy that $0 < 2\alpha < d$. Therefore the inverse Fourie transformation $\mathfrak{F}^{-1}$ embeds $\mathcal{H}'$ into $W_n \otimes \wedge_r\mathbb{R}^d$. On the other hand $W_n \otimes \wedge_r\mathbb{R}^d$ and $\mathcal{D}$ are naturally paired, i.e.

$$\langle f\,, \varphi\rangle = \int_{\mathbb{R}^d} (f(x)\,, \varphi(x))\,dx, \quad f \in W_n \otimes \wedge_r\mathbb{R}^d\,, \varphi \in \mathcal{D}.$$

We obtain a commutative diagram:

$$\begin{array}{ccc} \mathcal{H}' & \xrightarrow{\gamma^*} & \mathcal{D}' \\ {\scriptstyle \mathfrak{F}^{-1}}\big\downarrow & & \big\| \\ W_n \otimes \wedge_r\mathbb{R}^d & \xrightarrow{\subset} & \mathcal{D}' \end{array}$$

In this diagram the elements of $\mathcal{H}'$ are characterized by the following condition:

$$\delta f := \sum_{i=1}^{d} \frac{\partial}{\partial x^i} f \llcorner e_i = 0,$$

where $\{e_1, e_2, \ldots, e_d\}$ is the canonical base of $\mathbb{R}^d$. In $\gamma(\mathcal{H}')$ the induced norm reads as $\|(d+\delta)^n f\|_{L^2}$. (In the Euclidean frame the exterior derivative d is given by

$$df := \sum_{i=1}^{d} e_i \wedge \frac{\partial}{\partial x^i} f.\Big)$$

Hence the semi-locality and the compatibility with C^∞-cut off are clearly satisfied (see the remarks after the definition (2.8)). □

We consider the transposed map $\gamma : \mathcal{D} \longrightarrow \mathcal{H} \equiv (\mathcal{H}')'$ of $\gamma *$. We note that $\gamma(\mathcal{D})$ is dense in $\mathcal{H}$. As the dual space of $\mathcal{H}'$, $\mathcal{H}$ carries a natural Hilbert space structure given by

$$\|\gamma(\varphi)\|_{\mathcal{H}} = \sup\{|\langle \gamma*(v), \varphi\rangle| ; v \in \mathcal{H}', \|v\|_{\mathcal{H}'} \le 1\}, \quad \varphi \in \mathcal{D}.$$

We can easily see that if $\varphi_1, \varphi_2 \in \mathcal{D}$, then

$$\begin{aligned}
&(\gamma(\varphi_1), \gamma(\varphi_2))_{\mathcal{H}} \\
&= \int_{\mathbb{R}^d} (|k|^{-2}(k \wedge \widehat{\varphi_1}(k))\llcorner k, |k|^{-2}(k \wedge \widehat{\varphi_2}(-k))\llcorner k)\, |k|^{-2n}\, dk \qquad (3.2)\\
&= \int_{\mathbb{R}^d} (k \wedge \widehat{\varphi_1}(k), k \wedge \widehat{\varphi_2}(-k))\, |k|^{-2(n+1)}\, dk.
\end{aligned}$$

Combining Theorem 2.9 and Lemma 3.1, we get

(3.3) Theorem. *The map $\gamma : \mathcal{D} \longrightarrow \mathcal{H}$ is Markov.*

Let us return to the discussion of generalized random fields. We consider a mean 0 Gaussian system $\{X(\varphi)\}$ indexed by $\mathcal{D}$ with

$$\begin{aligned}
E[X(\varphi)^2] = \int_{\mathbb{R}^d} \{ & c_1(k \wedge \widehat{\varphi_1}(k)), k \wedge \widehat{\varphi_2}(-k)) \qquad (3.4)\\
& + c_2(\widehat{\varphi_1}(k)\llcorner k, \widehat{\varphi_2}(-k)\llcorner k) \}\, |k|^{-2}\, dk,
\end{aligned}$$

where c_1 and c_2 are nonnegative real numbers with $c_1 + c_2 > 0$. Combining (3.3) and known results (cf. [2]) on Gaussian generalized random fields with non-degenerate covariance, we conclude that

(3.5) Theorem. *$\{X(\varphi)\}$ with (3.4) is MI-Markov if and only if $c_1 = c_2 > 0$ or $c_1 c_2 = 0$.*

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