CHARACTERIZATION OF ANTICOMMUTATIVITY SELF-ADJOINT OPERATORS IN CONNECTION WITH CLIFFORD ALGEBRA AND APPLICATIONS

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CHARACTERIZATION OF ANTICOMMUTATIVITY OF SELF-ADJOINT OPERATORS IN CONNECTION WITH CLIFFORD ALGEBRA AND APPLICATIONS*

Asao Arai

A new characterization of anticommutativity of (unbounded) self-adjoint operators is presented in connection with Clifford algebra. Some consequences of the characterization and applications are discussed.

I. INTRODUCTION

In the paper [14] Vasilescu introduced a proper notion of anticommutativity of (unbounded) self-adjoint operators. Further analyses on anticommuting self-adjoint operators have been made by some people [12,9,4]. Samoilenko [12] and Pedersen [9] discussed several equivalent characterizations of anticommutativity and related aspects. In [4] some algebraic properties of anticommuting self-adjoint operators were investigated with special attention to commutation relations of the partial isometries associated with them.

Anticommuting self-adjoint operators may have connections with Clifford algebras. For example, the partial isometries associated with a family of mutually anticommuting self-adjoint operators give a representation of a Clifford algebra [4]. From such a point of view, it is interesting to characterize anticommutativity of self-adjoint operators in connection with Clifford algebra. This is the main purpose of the present paper.

As is well known, in the case of strongly commuting self-adjoint operators, it is possible to characterize their strong commutativity in terms of the one parameter unitary groups generated by them (Lemma 2.1 in the present paper). In the case of anticommuting self-adjoint operators, however, it was shown in [9] that such a characterization is impossible (cf. Proposition 2.3(3) in the present paper). In this paper we show that, if one considers anticommuting self-adjoint operators together with a Clifford algebra, then one can characteize the anticommutativity in terms of the one parameter unitary groups generated by self-adjoint operators formed out of the relevant self-adjoint operators and a representation of the Clifford algebra. This gives a new characterization of anticommutativity of self-adjoint operators.

In Section II we review some results related to both strongly commuting and anticommuting self-adjoint operators. In Section III we present the above mentioned characterization of anticommutativity of self-adjoint operators and discuss some conse-

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quences. The final section is devoted to applications of a result in Section III, where we solve, in abstract forms, the self-adjointness problem for two classes of operators of Dirac type. As examples, we briefly discuss the Dirac-Weyl operator with a strongly singular gauge potential and a Dirac type operator in an abstract Boson-Fermion Fock space.

II. PRELIMINARIES

2.1 Strongly commuting self-adjoint operators

Two *bounded* self-adjoint operators A and B in a Hilbert space are said to commute if $AB = BA$. Two (*not necessarily bounded*) self-adjoint operators A and B in a Hilbert space are said to strongly commute if their spectral projections commute. The following characterization of strong commutativity is well known (e.g., [10, Theorem VIII.13]).

LEMMA 2.1. *Let A and B be self-adjoint operators in a Hilbert space. Then A and B strongly commute if and only if*

$$e^{isA}e^{itB} = e^{itB}e^{isA}$$

for all $s, t \in \mathbb{R}$.

As for products of two strongly commuting self-adjoint operators, we have the following result ($D(T)$ denotes the domain of the operator T).

LEMMA 2.2. *Let A and B be strongly commuting self-adjoint operators in a Hilbert space $\mathcal{H}$ and set $\mathcal{D}_{A,B} = \cap_{n,m\geq 0} D(A^n B^m)$. Let p and q be polynomials in two variables with real coefficients. Then $p(A,B)q(A,B)$ and $q(A,B)p(A,B)$ are essentially self-adjoint on $\mathcal{D}_{A,B}$ and*

$$\overline{p(A,B)q(A,B) \restriction \mathcal{D}_{A,B}} = \overline{q(A,B)p(A,B) \restriction \mathcal{D}_{A,B}},$$

where $\overline{T}$ denotes the closure of the operator T. Moreover, if $q(A,B)$ is bounded with $q(A,B)^2 \geq \delta^2$ for some constant $\delta > 0$, then

$$\overline{p(A,B)q(A,B) \restriction \mathcal{D}_{A,B}} = \overline{p(A,B)}\ \overline{q(A,B)} = \overline{q(A,B)}\ \overline{p(A,B)}.$$

PROOF. Apply Theorems 9.1.2 and 9.1.14 in [13]. ∎

2.2 Anticommuting self-adjoint operators

Two self-adjoint operators A and B in a Hilbert space are said to anticommute if

$$e^{itA}B \subset Be^{-itA}$$

for all $t \in \mathbb{R}$ [9]. This definition of anticommutativity is symmetric in A and B [9].

PROPOSITION 2.3 [9, Proposition 1.2]. *Let A and B be self-adjoint operators in a Hilbert space. Then the following three conditions are equivalent:*

(1) *A and B anticommute.*

(2) *For all $t \in \mathbb{R}$, $\cos tB$ and A strongly commute and $\sin tB$ and A anticommute.*

(3) *For all $s, t \in \mathbb{R}$,*

$$e^{isA}e^{itB} = (\cos tB)e^{isA} + i(\sin tB)e^{-isA}.$$

Proposition 2.3 gives the following result.

LEMMA 2.4. *Let A and B be anticommuting self-adjoint operators in a Hilbert space. Then, for all $s, t \in \mathbb{R}$,*

$$\cos sA \cos tB = \cos tB \cos sA, \tag{2.1}$$

$$\sin sA \cos tB = \cos tB \sin sA, \tag{2.2}$$

$$\sin sA \sin tB = -\sin tB \sin sA. \tag{2.3}$$

PROOF. Equations (2.1) and (2.2) follow from the strong commutativity of $\cos tB$ and A (condition (2) of Proposition 2.3). The anticommutativity of $\sin tB$ and A implies that

$$e^{isA} \sin tB = (\sin tB)e^{-isA}, \quad s, t \in \mathbb{R}. \tag{2.4}$$

Replacing s by $-s$, we have

$$e^{-isA} \sin tB = (\sin tB)e^{isA}, \quad s, t \in \mathbb{R}. \tag{2.5}$$

Subtracting (2.5) from (2.4), we obtain (2.3). ∎

Equation (2.3) is also a sufficient condition for the self-adjoint operators A and B to anticommute:

LEMMA 2.5 [12, Chapt.8, Theorem 1]. *Let A and B be self-adjoint operators in a Hilbert space. Then A and B anticommute if and only if (2.3) holds for all $s, t \in \mathbb{R}$.*

REMARK. In [12], the definition of anticommutativity of self-adjoint operators is given in a way different from ours. But it is shown that they are equivalent (cf. [9]).

III. CLIFFORD ALGEBRA AND CHARACTERIZATION OF ANTICOMMUTATIVITY

Let $\mathfrak{A}_n$ ($n \geq 2$) be the Clifford algebra associated with the n-dimensional Euclidean space, i.e., the algebra generated by n elements e_j, $j = 1, \cdots, n$, and a unit element e_0 satisfying $e_je_k + e_ke_j = 2\delta_{jk}e_0, j, k = 1, \cdots, n$ (e.g., [8]). Let $\mathcal{H}$ be a Hilbert space.

We say that $\{\gamma_j\}_{j=1}^n$ is a self-adjoint representation of $\mathfrak{A}_n$ on $\mathcal{H}$ if each γ_j is a bounded self-adjoint operator on $\mathcal{H}$ satisfying

$$\gamma_j\gamma_k + \gamma_k\gamma_j = 2\delta_{jk}I, \quad j,k = 1,\cdots,n, \tag{3.1}$$

where I denotes the identity operator on $\mathcal{H}$. The first of our main results is the following.

THEOREM 3.1. *Let A and B be self-adjoint operators in a Hilbert space $\mathcal{H}$. Suppose that there exists a self-adjoint representation $\{\gamma_1, \gamma_2\}$ of $\mathfrak{A}_2$ on $\mathcal{H}$ such that each γ_j strongly commutes with A and B. Then A and B anticommute if and only if*

$$e^{is\gamma_1 A}e^{it\gamma_2 B} = e^{it\gamma_2 B}e^{is\gamma_1 A} \tag{3.2}$$

for all $s, t \in \mathbb{R}$.

REMARK. By Lemma 2.2 , $\gamma_1 A(= A\gamma_1)$ and $\gamma_2 B(= B\gamma_2)$ are self-adjoint. Hence $\exp(is\gamma_1 A)$ and $\exp(it\gamma_2 B)$ can be defined via the functional calculus.

PROOF. Let $\{E_A\}$ be the spectral family of A and set

$$\mathcal{D}_A = \cup_{n=1}^{\infty} R(E_A([-n,n])).$$

Since γ_1 strongly commutes with A, it leaves $\mathcal{D}_A$ invariant. Hence every $f \in \mathcal{D}_A$ is in $C^\infty(\gamma_1 A)$ and

$$||(s\gamma_1 A)^n f|| = |s|^n ||A^n f||, \quad n = 1, 2, \cdots,$$

which allow one to expand $\exp(is\gamma_1 A)f$ as

$$e^{is\gamma_1 A}f = \sum_{n=0}^{\infty} \frac{(is\gamma_1 A)^n f}{n!}$$

in the sense of strong convergence. Using (3.1) and the strong commutativity of γ_1 and A, we see that

$$(\gamma_1 A)^{2n} f = A^{2n} f, \quad (\gamma_1 A)^{2n+1} f = \gamma_1 A^{2n+1} f.$$

Hence

$$e^{is\gamma_1 A} f = (\cos sA + i\gamma_1 \sin sA) f.$$

Since $\mathcal{D}_A$ is dense in $\mathcal{H}$, we obtain the operator equality

$$e^{is\gamma_1 A} = \cos sA + i\gamma_1 \sin sA, \quad s \in \mathbb{R}. \tag{3.3}$$

Similarly we have

$$e^{it\gamma_2 B} = \cos tB + i\gamma_2 \sin tB, \quad t \in \mathbb{R}. \tag{3.4}$$

The strong commutativity of γ_j with A and B implies that γ_j commutes with $\sin sA$, $\cos sA, \sin tB$, and $\cos tB$. Hence we obtain from (3.3) and (3.4)

$$
\begin{aligned}
e^{is\gamma_1 A}e^{it\gamma_2 B} &= \cos sA\cos tB + i\gamma_2\cos sA\sin tB + i\gamma_1\sin sA\cos tB \\
&\qquad - \gamma_1\gamma_2\sin sA\sin tB, \qquad (3.5)\\
e^{it\gamma_2 B}e^{is\gamma_1 A} &= \cos tB\cos sA + i\gamma_2\sin tB\cos sA + i\gamma_1\cos tB\sin sA \\
&\qquad - \gamma_2\gamma_1\sin tB\sin sA. \qquad (3.6)
\end{aligned}
$$

Now suppose that A and B anticommute. Then, using Lemma 2.4 and (3.1), we see that the RHS of (3.5) is equal to that of (3.6). Thus (3.2) follows.

Conversely, suppose that (3.2) holds. Then, by (3.5) and (3.6), we have

$$W + \gamma_1 X + \gamma_2 Y + \gamma_1\gamma_2 Z = 0 \qquad (3.7)$$

with

$$
\begin{aligned}
W &= \cos sA\cos tB - \cos tB\cos sA, && (3.8)\\
X &= i(\sin sA\cos tB - \cos tB\sin sA), && (3.9)\\
Y &= i(\cos sA\sin tB - \sin tB\cos sA), && (3.10)\\
Z &= -\sin sA\sin tB - \sin tB\sin sA.
\end{aligned}
$$

Note that

$$W^* = -W,\ X^* = X,\ Y = Y^*,\ Z = Z^*,$$

and each γ_j commutes with W, X, Y, and Z. Hence, taking the adjoint of (3.7), we obtain

$$-W + \gamma_1 X + \gamma_2 Y - \gamma_1\gamma_2 Z = 0. \qquad (3.11)$$

Adding (3.7) and (3.11), we have

$$\gamma_1 X + \gamma_2 Y = 0, \qquad (3.12)$$

which, together with (3.7), gives

$$W + \gamma_1\gamma_2 Z = 0. \qquad (3.13)$$

Multiplying (3.13) by γ_1 from the left, we have

$$\gamma_1 W + \gamma_2 Z = 0.$$

Taking the adjoint of this equation, we obtain

$$-\gamma_1 W + \gamma_2 Z = 0.$$

Hence $\gamma_1 W = \gamma_2 Z = 0$, which imply that $W = Z = 0$. Thus we obtain

$$\cos sA \cos tB = \cos tB \cos sA,$$
$$\sin sA \sin tB = -\sin tB \sin sA.$$

The last equation and Lemma 2.5 imply that A and B anticommute. ∎

Using Lemma 2.1, we can rephrase Theorem 3.1 as follows.

THEOREM 3.2. *Let A, B and $\{\gamma_1, \gamma_2\}$ be as in Theorem 3.1. Then A and B anticommute if and only if $\gamma_1 A$ and $\gamma_2 B$ strongly commute.*

We now discuss some consequences of Theorems 3.1 and 3.2. We fix a self-adjoint representation $\{\gamma_1, \gamma_2\}$ of $\mathfrak{A}_2$ on a Hilbert space $\mathcal{K}$. We denote by $\mathcal{K} \otimes \mathcal{H}$ the tensor product of $\mathcal{K}$ and $\mathcal{H}$.

THEOREM 3.3. *Let A and B be self-adjoint operators in a Hilbert space $\mathcal{H}$. Then A and B anticommute if and only if $\gamma_1 \otimes A$ and $\gamma_2 \otimes B$ strongly commute in the Hilbert space $\mathcal{K} \otimes \mathcal{H}$.*

PROOF. Let $\widetilde{\gamma}_j = \gamma_j \otimes I$. Then $\{\widetilde{\gamma}_1, \widetilde{\gamma}_2\}$ is a self-adjoint representation of $\mathfrak{A}_2$ on the Hilbert space $\mathcal{K} \otimes \mathcal{H}$. It follows that $\widetilde{\gamma}_j$ strongly commutes with $I \otimes A$ and $I \otimes B$. Hence, by Theorem 3.2, $I \otimes A$ and $I \otimes B$ anticommute if and only if $\widetilde{\gamma}_1(I \otimes A)(= \gamma_1 \otimes A)$ and $\widetilde{\gamma}_2(I \otimes B)(= \gamma_2 \otimes B)$ strongly commute. On the other hand, it is not so difficult to see that $I \otimes A$ and $I \otimes B$ anticommute if and only if A and B anticommute. Thus the desired result follows. ∎

REMARK. A simple example of $\mathcal{K}$ and $\{\gamma_1, \gamma_2\}$ is given by

$$\mathcal{K} = \mathbb{C}^2,$$
$$\gamma_1 = \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

The matrices σ_1 and σ_2 are the first two of the so-called Pauli matrices.

We have a "dual" version of Theorem 3.3:

THEOREM 3.4. *Let A and B be self-adjoint operators in a Hilbert space $\mathcal{H}$. Then A and B strongly commute if and only if $\gamma_1 \otimes A$ and $\gamma_2 \otimes B$ anticommute in the Hilbert space $\mathcal{K} \otimes \mathcal{H}$.*

PROOF. Let $\tau_j = \gamma_j \otimes \gamma_j$ in $\mathcal{K} \otimes \mathcal{K}$. By Theorem 3.3, $\gamma_1 \otimes A$ and $\gamma_2 \otimes B$ anticommute if and only if $\tau_1 \otimes A$ and $\tau_2 \otimes B$ strongly commute in the Hilbert space $\mathcal{K} \otimes \mathcal{K} \otimes \mathcal{H}$. Note that $\tau_j^2 = I \otimes I$. Hence, in the same way as in the proof of (3.3) and (3.4), we can show that

$$e^{is\tau_1 \otimes A} = I \otimes I \otimes \cos sA + i\tau_1 \otimes \sin sA, \quad s \in \mathbb{R},$$
$$e^{it\tau_2 \otimes B} = I \otimes I \otimes \cos tB + i\tau_2 \otimes \sin tB, \quad t \in \mathbb{R}.$$

Therefore we have

$$
\begin{aligned}
e^{is\tau_1\otimes A}e^{it\tau_2\otimes B} =& I\otimes I\otimes \cos sA\cos tB + i\tau_2\otimes \cos sA \sin tB \\
&+ i\tau_1\otimes \sin sA\cos tB - \tau_1\tau_2\otimes \sin sA\sin tB, \qquad (3.14)\\
e^{it\tau_2\otimes B}e^{is\tau_1\otimes A} =& I\otimes I\otimes \cos tB\cos sA + i\tau_2\otimes \sin tB\cos sA \\
&+ i\tau_1\otimes \cos tB\sin sA - \tau_2\tau_1\otimes \sin tB\sin sA. \qquad (3.15)
\end{aligned}
$$

It is easy to check that τ_1 and τ_2 commute. Using this fact, (3.14), (3.15) and Lemma 2.1, we see that $\tau_1\otimes A$ and $\tau_2\otimes B$ strongly commute if and only if

$$I\otimes I\otimes W + \tau_1\otimes X + \tau_2\otimes Y + \tau_1\tau_2\otimes \Omega = 0, \qquad (3.16)$$

where W, X, Y are given by (3.8), (3.9), (3.10), respectively, and

$$\Omega = \sin tB\sin sA - \sin sA\sin tB.$$

Note that $\Omega^* = -\Omega$. In the same way as in the proof of Theorem 3.1, we can show that (3.16) gives

$$\tau_1\otimes X + \tau_2\otimes Y = 0, \qquad (3.17)$$

$$I\otimes I\otimes W + \tau_1\tau_2\otimes\Omega = 0. \qquad (3.18)$$

Multiplying (3.17) by $\tau_j\otimes I$ from the left, we obtain

$$I\otimes I\otimes X + \tau_1\tau_2\otimes Y = 0. \qquad (3.19)$$

and

$$\tau_1\tau_2\otimes X + I\otimes I\otimes Y = 0 \qquad (3.20)$$

Adding (3.19) and (3.20), we obtain

$$(I+\tau_1\tau_2)\otimes(X+Y) = 0.$$

Subtracting (3.20) from (3.19), we have

$$(I-\tau_1\tau_2)\otimes(X-Y) = 0.$$

Using (3.1), we can show that $\tau_1\tau_2 = \gamma_1\gamma_2\otimes\gamma_1\gamma_2 \neq \pm I$. Thus we conclude that

$$X \pm Y = 0,$$

which give $X = Y = 0$. Similarly it follows from (3.18) that $W = \Omega = 0$. These results imply (2.1). Thus, $\tau_1\otimes A$ and $\tau_2\otimes B$ strongly commute if and only if A and B strongly commute. ∎

REMARK. In the case where $\mathcal{K} = \mathbb{C}^2$ and $\gamma_j = \sigma_j, j = 1, 2$, the necessary condition in Theorem 3.4 has been proven in [6] by a different and more direct method.

IV. APPLICATIONS

In this section we discuss applications of Theorem 3.4 to the self-adjointness problem of operators of Dirac type. We first recall a basic result due to Vasilescu [14]:

LEMMA 4.1 [14, Theorem 2.1, Corollary 2.2]. *Let $\{A_j\}_{j=1}^n$ be a family of mutually anticommuting self-adjoint operators in a Hilbert space ($n < \infty$). Then $\sum_{j=1}^n A_j$ is self-adjoint and*

$$\left(\sum_{j=1}^n A_j\right)^2 = \sum_{j=1}^n A_j^2.$$

Lemma 4.1 can be extended to the case of a countable family of mutually anticommuting self-adjoint operators. Let $\{A_n\}_{n=1}^\infty$ be a family of self-adjoint operators. Then we can define the operator $A := \sum_{n=1}^\infty A_n$ by the relation

$$D(A) = \left\{ f \in \bigcap_{n=1}^\infty D(A_n) \;\middle|\; \mathrm{w} - \lim_{N\to\infty} \sum_{n=1}^N A_n f \text{ exists} \right\},$$

$$Af = \mathrm{w} - \lim_{N\to\infty} \sum_{n=1}^N A_n f, \quad f \in D(A).$$

LEMMA 4.2 [14, Theorem 2.3]. *Let $\{A_n\}_{n=1}^\infty$ be a family of mutually anticommuting self-adjoint operators in a Hilbert space $\mathcal{H}$ such that $D(\sum_{n=1}^\infty A_n)$ is dense in $\mathcal{H}$. Then $\sum_{n=1}^\infty A_n$ is self-adjoint and*

$$\left(\sum_{n=1}^\infty A_n\right)^2 = \sum_{n=1}^\infty A_n^2.$$

We now apply Lemmas 4.1 and 4.2 together with Theorem 3.4 to operators of Dirac type..

THEOREM 4.3. *Let $\{A_j\}_{j=1}^n$ be a family of mutually strongly commuting self-adjoint operators in a Hilbert space $\mathcal{H}$ ($n < \infty$). Let $\{\gamma_j\}_{j=1}^n$ be a self-adjoint representation of $\mathfrak{A}_n$ on a Hilbert space $\mathcal{K}$. Then the operator*

$$\not{D} := \sum_{j=1}^n \gamma_j \otimes A_j$$

is self-adjoint in $\mathcal{K}\otimes\mathcal{H}$ and

$$\not{D}^2=\sum_{j=1}^{n} I\otimes A_j^2.$$

PROOF. By Theorem 3.4, $\{\gamma_j\otimes A_j\}_{j=1}^n$ is a family of mutually anticommuting self-adjoint operators. Thus we can apply Lemma 4.1 with A_j replaced by $\gamma_j\otimes A_j$ to obtain the desired result. ∎

THEOREM 4.4. *Let $\{A_n\}_{n=1}^\infty$ be a family of mutually strongly commuting self-adjoint operators in a Hilbert space $\mathcal{H}$. Let $\{\gamma_n\}_{n=1}^\infty$ be a self-adjoint representation of $\mathfrak{A}_\infty$ on a Hilbert space $\mathcal{K}$. Suppose that $D(\sum_{n=1}^\infty \gamma_n\otimes A_n)$ is dense in $\mathcal{K}\otimes\mathcal{H}$. Then the operator*

$$\not{D}_\infty:=\sum_{n=1}^{\infty}\gamma_n\otimes A_n$$

is self-adjoint in $\mathcal{K}\otimes\mathcal{H}$ and

$$\not{D}_\infty^2=\sum_{n=1}^{\infty} I\otimes A_n^2.$$

PROOF. Simlar to the proof of Theorem 4.3 (use Lemma 4.2). ∎

A sufficient condition for $D(\sum_{n=1}^\infty \gamma_n\otimes A_n)$ in Theorem 4.4 to be dense in $\mathcal{K}\otimes\mathcal{H}$ is given by the following lemma.

Lemma 4.5. *Let A_n and γ_n be as in Theorem 4.4. Suppose that*

$$D:=\left\{f\in\bigcap_{n=1}^{\infty}D(A_n)\,\middle|\,\sum_{n=1}^{\infty}\|A_nf\|^2<\infty\right\}$$

is dense in $\mathcal{H}$. Then $D(\sum_{n=1}^\infty \gamma_n\otimes A_n)$ is dense in $\mathcal{K}\otimes\mathcal{H}$. In particular, the conclusion of Theorem 4.4 holds.

PROOF. Let

$$\widetilde{D}=\mathcal{K}\hat{\otimes}D\quad(\text{ algebraic tensor product}),$$

which is dense in $\mathcal{K}\otimes\mathcal{H}$. Every $\psi:=u\otimes f\in\widetilde{D}$ is in $\cap_{n=1}^\infty D(\gamma_n\otimes A_n)$ and

$$\sum_{n=1}^{\infty}\|\gamma_n\otimes A_n\psi\|^2=\|u\|^2\sum_{n=1}^{\infty}\|A_nf\|^2<\infty. \tag{4.1}$$

In general, the following holds (see the proof of Theorem 2.3 in [14]): *Let $\{B_n\}_{n=1}^\infty$ be a family of mutually anticommuting self-adjoint operators in a Hilbert space. Then $x\in D(\sum_{n=1}^\infty B_n)$ if and only if $x\in\cap_{n=1}^\infty D(B_n)$, $\sum_{n=1}^\infty\|B_nx\|^2<\infty$.* Applying this fact with $B_n = \gamma_n \otimes A_n$, we obtain from (4.1) that $\psi \in D(\sum_{n=1}^{\infty} \gamma_n \otimes A_n)$. Hence $\widetilde{D} \subset D(\sum_{n=1}^{\infty} \gamma_n \otimes A_n)$. Thus $D(\sum_{n=1}^{\infty} \gamma_n \otimes A_n)$ is dense in $\mathcal{K} \otimes \mathcal{H}$. ∎

The operator $\not{D}$ (resp. $\not{D}_\infty$) in Theorem 4.3 (resp. Theorem 4.4) gives a class of operators of Dirac type in an abstract form. Hence Theorems 4.3 and 4.4 solve the self-adjointness problem for such Dirac operators.

EXAMPLE 1. *The Dirac-Weyl operator with a strongly singular gauge potential* [6]. Let $\{\mathbf{a}_\nu\}_{\nu=1}^n$ be a finite set of isolated points in $\mathbb{R}^2$ and set $M := \mathbb{R}^2 \setminus \{\mathbf{a}_\nu\}_{\nu=1}^n$. Let

$$A_1(x,y) = -\sum_{\nu=1}^{n} \sum_{0 \le \alpha+\beta \le m} \frac{C_{\alpha,\beta}^{(\nu)}}{2\pi} D_x^\alpha D_y^\beta \left(\frac{y - a_{\nu 2}}{|\mathbf{r} - \mathbf{a}_\nu|^2} \right), \quad (x,y) \in M,$$

$$A_2(x,y) = \sum_{\nu=1}^{n} \sum_{0 \le \alpha+\beta \le m} \frac{C_{\alpha,\beta}^{(\nu)}}{2\pi} D_x^\alpha D_y^\beta \left(\frac{x - a_{\nu 1}}{|\mathbf{r} - \mathbf{a}_\nu|^2} \right), \quad (x,y) \in M,$$

where $C_{\alpha,\beta}^{(\nu)}$ are real constants, and D_x and D_y are the distributional partial differential operators in x and y, respectively. Let

$$P_1 = -iD_x - A_1$$

and

$$P_2 = -iD_y - A_2$$

acting in $L^2(\mathbb{R}^2)$. It was shown in [5] that (i) each P_j is essentially self-adjoint ; (ii) if the constant $C_{0,0}^{(\nu)}$ is an integer multiple of 2π for all $\nu = 1, \cdots, n$ (in this case we say that the magnetic flux is "locally quantized"), then $\bar{P}_1$ and $\bar{P}_2$ strongly commute. Thus, applying Theorem 4.3 to the *Dirac-Weyl operator*

$$\not{D} = \sigma_1 \otimes \bar{P}_1 + \sigma_2 \otimes \bar{P}_2$$

in $\mathbb{C}^2 \otimes L^2(\mathbb{R}^2)$, we see that, if the magnetic flux is locally quantized, then $\not{D}$ is self-adjoint and

$$\not{D}^2 = I \otimes \bar{P}_1^2 + I \otimes \bar{P}_2^2.$$

EXAMPLE 2. *A class of operators of Dirac type in an abstract Boson-Fermion Fock space.* Let $\mathcal{H}$ be a separable Hilbert space with $\dim \mathcal{H} = \infty$ and consider the the Boson Fock space over $\mathcal{H}$:

$$\mathcal{F}_{\mathrm{b}}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} S_n(\mathcal{H}),$$

where $S_n(\mathcal{H})$ denotes the n-fold symmetric tensor product of $\mathcal{H}$ ($S_0(\mathcal{H}) := \mathbb{C}$) (e.g., [7, §5.2], [10, §II.4]). We denote by $a(f), f \in \mathcal{H}$, the annihilation operators in $\mathcal{F}_{\mathrm{b}}(\mathcal{H})$ and by

Ω_b the vacuum in $\mathcal{F}_\mathrm{b}(\mathcal{H}) : \Omega_\mathrm{b} = \{1, 0, 0, \cdots,\}$ (e.g., [7, §5.2], [11, §X.7]). Let D_b be the subspace algebraically spanned by vectors of the form

$$a(f_1)^* \cdots a(f_n)^* \Omega_\mathrm{b}, \quad f_j \in \mathcal{H}, j = 1, \cdots, n, n \geq 0. \tag{4.1}$$

Then D_b is dense in $\mathcal{F}_\mathrm{b}(\mathcal{H})$. Moreover, the operators $a(f)$ and $a(f)^*$ leave D_b invariant, satisfying the canonical commutation relations

$$[a(f), a(g)^*] = (f, g)_\mathcal{H},$$
$$[a(f), a(g)] = 0 = [a(f)^*, a(g)^*], \quad f, g \in \mathcal{H},$$

on D_b, where $[A, B] := AB - BA$ and $(\ \cdot\ ,\ \cdot\ )_\mathcal{H}$ denotes the inner product of $\mathcal{H}$. Let

$$\Phi(f) = \frac{1}{\sqrt{2}} \left(a(f)^* + a(f) \right), \quad f \in \mathcal{H},$$

which is called the Segal field operator [11, §X.7]. The following facts are known.

LEMMA 4.6. *For all $f \in \mathcal{H}$, $\Phi(f)$ is essentially self-adjoint on D_b. If $(f, g)_\mathcal{H}$ is real, then $\overline{\Phi(f)}$ and $\overline{\Phi(g)}$ strongly commute.*

PROOF. See, e.g., [11, §X.7]. ∎

Let $\mathcal{K}$ be another separable Hilbert space with $\dim \mathcal{K} = \infty$ and consider the Fermion Fock space over $\mathcal{K}$

$$\mathcal{F}_\mathrm{f}(\mathcal{K}) = \bigoplus_{p=0}^{\infty} A_p(\mathcal{K}),$$

where $A_p(\mathcal{K})$ is the p-fold antisymmetric tensor product of $\mathcal{K}$ ($A_0(\mathcal{K}) := \mathbb{C}$)(e.g., [7, §5.2], [10, §II.4]). We denote by $b(u), u \in \mathcal{K}$, the annihilation operators in $\mathcal{F}_\mathrm{f}(\mathcal{K})$, which are bounded linear operators satisfying the canonical aniticommutation relations

$$\{b(u), b(v)^*\} = (u, v)_\mathcal{K}, \tag{4.2}$$
$$\{b(u), b(v)\} = 0 = \{b(u)^*, b(v)^*\}, \quad u, v, \in \mathcal{K}, \tag{4.3}$$

where $\{A, B\} := AB + BA$.

Let $\{e_n\}_{n=1}^\infty$ be a complete orthonormal system of $\mathcal{K}$ and define

$$\gamma_{2n-1} = b(e_n) + b(e_n)^*, \quad \gamma_{2n} = i\left(b(e_n)^* - b(e_n) \right), \quad n \geq 1.$$

Obviously each γ_n is a bounded self-adjoint operator. Moreover, by (4.2) and (4.3), we can show that

$$\{\gamma_n, \gamma_m\} = 2\delta_{nm}, \quad n, m \geq 1. \tag{4.4}$$

Hence $\{\gamma_n\}_{n=1}^\infty$ gives a self-adjoint representatoin of $\mathfrak{A}_\infty$ on $\mathcal{F}_\mathrm{f}(\mathcal{K})$.

Let

$$\mathcal{F} = \mathcal{F}_{\mathrm{f}}(\mathcal{K}) \otimes \mathcal{F}_{\mathrm{b}}(\mathcal{H}),$$

which may be called the *Boson-Fermion Fock space* over $(\mathcal{H}, \mathcal{K})$. In this Hilbert space we introduce an operator of Dirac type. Let $\mathbf{h} = \{h_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{H}$ and

$$\not{D}_{\mathbf{h}} = \sum_{n=1}^{\infty} \gamma_n \otimes \overline{\Phi(h_n)}.$$

THEOREM 4.7. *Suppose that*

$$\sum_{n=1}^{\infty} ||h_n||_{\mathcal{H}}^2 < \infty$$

and $(h_n, h_m)_{\mathcal{H}}$ *is real for all* $n, m \geq 1$. *Then* $\not{D}_{\mathbf{h}}$ *is self-adjoint with*

$$\not{D}_{\mathbf{h}}^2 = \sum_{n=1}^{\infty} I \otimes \overline{\Phi(h_n)}^2.$$

PROOF. By Lemma 4.6 and Theorem 4.4, we need only to show that $D(\not{D}_{\mathbf{h}})$ is dense in $\mathcal{F}$. The following estimate is known (e.g.,[11, §X.7]):

$$||\overline{\Phi(f)}\psi|| \leq C_{\psi}||f||_{\mathcal{H}}, \quad f \in \mathcal{H}, \psi \in D_{\mathrm{b}},$$

with a constant $C_{\psi} > 0$. Hence we have for all $\psi \in D_{\mathrm{b}}$

$$\begin{aligned} \sum_{n=1}^{\infty} ||\overline{\Phi(h_n)}\psi||_{\mathcal{F}_{\mathrm{b}}}^2 &\leq C_{\psi}^2 \sum_{n=1}^{\infty} ||h_n||_{\mathcal{H}}^2 \\ &< \infty, \end{aligned}$$

which implies that

$$D_{\mathrm{b}} \subset \mathcal{G} := \left\{ \psi \in \bigcap_{n=1}^{\infty} D\left(\overline{\Phi(h_n)}\right) \,\middle|\, \sum_{n=1}^{\infty} ||\overline{\Phi(h_n)}\psi||_{\mathcal{F}_{\mathrm{b}}}^2 < \infty \right\}.$$

Hence $\mathcal{G}$ is dense in $\mathcal{F}_{\mathrm{b}}(\mathcal{H})$. Thus, by Lemma 4.5, we conclude that $D(\not{D}_{\mathbf{h}})$ is dense in $\mathcal{F}$. ∎

REMARK. Other classes of Dirac type operators in Boson-Fermion Fock spaces are discussed in [1-3].

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