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an intermediate subfactor**

Jerzy Wierzbicki

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An estimation of the depth from an intermediate subfactor.

Jerzy Wierzbicki

Abstract. We show that for a triple $K \subset N \subset M$ of type II_1 factors the depth of the inclusion " $K \subset M$ " is not greater than the maximum of depths of the inclusions " $K \subset N$ " and " $N \subset M$ ", provided there is such a factor P , that the diagram

$$\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ K & \subset & N \end{array}$$

is commuting and co-commuting square (or a non-degenerate commuting square) of type II_1 factors. We give also a characterization of the above condition.

In [B] Bisch proved, that if depth of inclusion $K \subset M$ of two type II_1 factors is finite then for any intermediate subfactor N , the depths of " $K \subset N$ " and of " $N \subset M$ " are finite too. In this note we give a converse to the assertion. Similar result as that of ours was obtained recently in the case of the depth two irreducible inclusions in [S] by T. Sano, who used different method.

Preliminaries. We recall here shortly the basic notions we need. We will follow here [GHJ], [P1], [SW] and [PP2].

Let $N \subset M$ be an inclusion of the type II_1 factors with $[M : N] < \infty$. We have the corresponding Jones' tower

$$N \subset M \subset^{e_0} M_1 \subset^{e_1} M_2 \dots$$

with Jones' projections $e_i \in M_{i+1}$. Consider also the tower of relative commutants $\{Y_i\}_{i \geq -1}$, $Y_{-1} = N' \cap N$, $Y_0 = N' \cap M$, $Y_i = N' \cap M_i$. If " $N \subset M$ " is of finite depth then for some $i > 1$, Y_i gets the basic construction for " $Y_{i-2} \subset Y_{i-1}$ ". From [GHJ] or [P3] we know that it happens, if and only if the central support $C(e_{i-1}, Y_i)$ of e_{i-1} in Y_i becomes identity.

Definition 1. The integer $\min\{i \mid C(e_{i-1}, Y_i) = 1\}$ is called the depth of the inclusion " $N \subset M$ ". Let us denote it by $d(N \subset M)$.

In [WW] we defined notion of co-commuting square of type II_1 factors.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\mathcal{T}\mathcal{E}\mathcal{X}$

$$\begin{array}{ccc} M & \subset & L \\ \cup & & \cup \\ K & \subset & N \\ M' & \subset & K' \\ \cup & & \cup \\ L' & \subset & N' \end{array}$$

Definition 2. A diagram \cup of finite factors is a co-commuting square, if their commutants \cup form a commuting square.

It was first analyzed in [SW] in terms of opposite angles between two subfactors. The opposite angles $\text{Op-ang}_L(M, N) = \frac{\pi}{2}$ iff the diagram \cup is a co-commuting square.

From [SW] we see, that the co-commuting square, which is also a commuting square of type II_1 factors coincides with the notion of non-degenerate commuting square of II_1 factors, which was introduced by S.Popa in [P1]. Let us recall a few basic properties, which can be found in [SW] or in [P1].

$$\begin{array}{ccc} & N & \subset & L \\ & \cup & & \cup \\ & K & \subset & M \end{array}$$

Proposition 3. Suppose that the diagram $\mathcal{S} = \cup$ is a commuting square of type II_1 factors. Let also $[L : K] < \infty$. Then the following conditions are equivalent:

- (i) \mathcal{S} is nondegenerate commuting square.
- (ii) $L = \text{Span } N \cdot M$
- (iii) $[L : N] = [M : K]$

We will also use the notion of algebraic basic construction as it was introduced in [PP2].

Definition 4. Suppose that N is a subfactor of a type II_1 factor M with $[M : N] < \infty$. Let M_1 be a von Neumann with a finite, faithful and normal trace τ_1 . Assume that e is a projection in M_1 . If there is a trace preserving *-isomorphism $\phi : \langle M, e_N^M \rangle \rightarrow M_1$ of the basic construction of " $N \subset M$ " onto M_1 such that $\forall x \in M \phi(x) = x$ and $\phi(e_N^M) = e$ then we call M_1 algebraic basic construction for the inclusion " $N \subset M$ ". For convenience we will write $M_1 = \langle M, N, e \rangle$.

The following proposition is a slight modification of [PP2] Proposition 1.2 and it gives more on the algebraic basic construction.

Proposition 5. Let N be a subfactor of a type II_1 factor M with $[M : N] < \infty$. Let M_1 be a von Neumann with a finite, faithful and normal trace τ_1 . Assume that e is a projection in M_1 . $E_M - \tau_1$ preserving conditional expectation of M_1 onto M . The following conditions are equivalent:

- (1) $M_1 = \langle M, N, e \rangle$

- (2) M_1 is a factor, $[e, N] = 0$ and $E_M(e) = [M_1 : M]^{-1} = [M : N]^{-1}$.
 (3) $M_1 = M \vee \{e\}$, $\tau(e) \geq [M : N]^{-1}$ and $exe = E_N(x)e$ for $x \in M$.

The main result.

We construct a system of type II_1 factors from a given commuting and co-commuting square of II_1 factors:

$$\begin{array}{ccc} Q_{1,0} & \subset & Q_{1,1} \\ \cup & & \cup \\ Q & \subset & Q_{0,1} \end{array},$$

where the Jones' index $[Q_{1,1} : Q]$ is finite. At first step we define $Q_{2,1} = \langle Q_{1,1}, e_{Q_{0,1}}^{Q_{1,1}} \rangle$ as the basic construction for the pair $Q_{0,1} \subset Q_{1,1}$ and also $Q_{1,2} = \langle Q_{1,1}, e_{Q_{1,0}}^{Q_{1,1}} \rangle$ and $Q_{2,2} = \langle Q_{1,1}, e_Q^{Q_{1,1}} \rangle$.

This way we obtained the following "bigger" diagram

$$\begin{array}{ccc} & & Q_{2,1} \subset Q_{2,2} \\ & & \cup \quad \cup \\ & & Q_{1,1} \subset Q_{1,2} \\ \text{by [SW]} & \text{is a commuting and co-commuting square too.} & \text{Suppose that the dia-} \\ & & \text{gram} \end{array}$$

$$\begin{array}{ccc} Q_{n,n-1} & \subset & Q_{n,n} \\ \cup & & \cup \end{array}$$

was already defined. Similarly as at the first step,

we put $Q_{n+1,n} = \langle Q_{n,n}, e_{Q_{n-1,n}}^{Q_{n,n}} \rangle$, $Q_{n,n+1} = \langle Q_{n,n}, e_{Q_{n,n-1}}^{Q_{n,n}} \rangle$ and $Q_{n+1,n+1} = \langle Q_{n,n}, e_{Q_{n-1,n-1}}^{Q_{n,n}} \rangle$. Let us use the following notation: $e_n = e_{Q_{n-1,n}}^{Q_{n,n}}$, $f_n = e_{Q_{n,n-1}}^{Q_{n,n}}$, $g_n = e_{Q_{n-1,n-1}}^{Q_{n,n}}$, $\alpha = [Q_{0,1} : Q]$ and $\beta = [Q_{1,0} : Q]$.

The above construction of increasing commuting and co-commuting squares can be found in [SW]. The following properties are noted there too:

$$\text{for } i, j \geq 1 \quad e_i f_j = f_j e_i \text{ and } g_i = e_i f_i.$$

We complete the above system by defining von Neumann subalgebras $Q_{i,j}$ with the following iterative formulas:

$$Q_{i+1,j} = Q_{i,j} \vee \{f_i\} \text{ and } Q_{i,j+1} = Q_{i,j} \vee \{e_j\},$$

where " \vee " denotes generation of a von Neumann algebra. It is obvious, that the definitions agree with the previous construction.

Lemma 6. With the above notations we have:

- (i) $\forall i, j \geq 1$, $Q_{i,j}$ are type II_1 factors.
 (ii) $\forall i, j \geq 0$ $Q_{i+2,j} = \langle Q_{i+1,j}, Q_{i,j}, f_{i+1} \rangle$ and $Q_{i,j+2} = \langle Q_{i,j+1}, Q_{i,j}, e_{j+1} \rangle$.

Proof.

First we show that for any $i \geq 0$ $Q_{i,i+2} = \langle Q_{i,i+1}, Q_{i,i}, e_{i+1} \rangle$. We just check the condition (3) of the Proposition 5. Let us write $E_{i,j}$ for the trace preserving conditional expectation onto subalgebra $Q_{i,j}$. For $x \in Q_{i,i+1} \subset Q_{i+1,i+1}$,

$$e_{i+1} x e_{i+1} = E_{i+1,i}(x) e_{i+1} = E_{i+1,i} E_{i,i+1}(x) e_{i+1} = E_{i,i}(x) e_{i+1}.$$

The trace $\tau(e_{i+1}) = \alpha^{-1}$ and $Q_{i,i+2}$ is generated by e_{i+1} and $Q_{i,i+1}$ right from the definition. In particular, we see that $Q_{i,i+2}$ must be a type II₁ factor.

From the Proposition 5.(2) we see also that $Q_{i+2,i+2} = \langle Q_{i+1,i+2}, Q_{i,i+2}, f_{i+1} \rangle$. Indeed, since $[f_{i+1}, Q_{i,i+1}] = 0$ and $[f_{i+1}, e_{i+1}] = 0$, then $[f_{i+1}, Q_{i,i+2}] = 0$. Also,

$$E_{i+1,i+2}(f_{i+1}) = E_{i+1,i+2}E_{i+2,i+1}(f_{i+1}) = E_{i+1,i+1}(f_{i+1}) = \beta^{-1}.$$

We show now that all the diagrams
$$\begin{array}{ccc} Q_{i+1,i+1} & \subset & Q_{i+1,i+2} \\ \cup & & \cup \\ Q_{i,i+1} & \subset & Q_{i,i+2} \end{array}$$
 are nondegenerate commuting squares. By the Proposition 3.(iii) and [GHJ]Proposition 4.2.1 it is sufficient to check if $E_{i+1,i+1}(Q_{i,i+2}) \subset Q_{i,i+1}$. Take $x \in Q_{i,i+2}$. Then ([PP1]) $x = \sum_k x_k e_{i+1} y_k$, with $x_k, y_k \in Q_{i,i+1}$.

$$E_{i+1,i+1}(x) = \sum_k x_k E_{i+1,i+1}(e_{i+1}) y_k = \alpha^{-1} \sum_k x_k y_k \in Q_{i,i+1}.$$

Similarly, using the Proposition 5.(2) and [GHJ]4.2.1 we obtain $Q_{i+2,i+3} = \langle Q_{i+1,i+2}, Q_{i,i+1}, f_{i+1} e_{i+2} \rangle$.

So we see that the above argument can be repeated sequentially to get

$$Q_{i,i+k} = \langle Q_{i,i+k-1}, Q_{i,i+k-2}, e_{i+k-1} \rangle \text{ and } Q_{i+2,i+k} = \langle Q_{i+1,i+k}, Q_{i,i+k}, f_{i+1} \rangle,$$

for $k = 2, 3, 4, \dots$. By the symmetry we obtain analogous statements also for $Q_{i+k,i}$ with $k = 2, 3, 4, \dots$.

The next lemma can in fact be read off from the proof of [P3]Proposition 2.1. Let state it explicitly. If a von Neumann algebra A contains a projection e then we will write $C(e, A)$ for the central support of e in A . For a subalgebra B we define the following projection:

$$V(e, B) = \bigvee \{ ueu^* \mid u \text{ is a unitary in } B \}.$$

Lemma 7. Let $N \subset M$ be type II₁ factors and $[M : N] < \infty$. B and A are such von Neumann subalgebras that the diagram
$$\begin{array}{ccc} N & \subset & M \\ \cup & & \cup \\ B & \subset & A \end{array}$$
 is commuting square. Suppose that a projection $e \in A$ satisfies $E_N^M(e) = [M : N]^{-1}$. Then $C(e, A) = V(e, B)$.

With the above preparation the proof of our main result becomes easy.

Theorem 8. Let $K \subset M \subset L$ be type II₁ factors with $[L : K] < \infty$. Suppose that $d(K \subset M) < \infty$ and $d(M \subset L) < \infty$. If there is such a type II₁ factor N , that $K \subset N \subset L$ and the diagram
$$\begin{array}{ccc} N & \subset & L \\ \cup & & \cup \\ K & \subset & M \end{array}$$
 is a nondegenerate commuting square, then

$$d(K \subset L) \leq \max(d(K \subset M), d(M \subset L)).$$

Proof. Let make a system of type II₁ factors $\{Q_{i,j}\}$ like in the Lemma 3. with $L = Q_{1,1}$, $N = Q_{1,0}$, $M = Q_{0,1}$ and $K = Q_{0,0} = Q$. Denote $g_i = e_i f_i$ and $q = \max(d(K \subset M), d(M \subset L))$. From the Lemmas 6, 7 and [P3]3.1 we obtain:

$$\begin{aligned} V(g_q, Q' \cap Q_{q,q}) &\geq V(g_q, Q' \cap Q_{q,q} \cap \{f_q\}') = \\ (*) \quad \bigvee \{ue_q u^* f_q | u \in U(Q' \cap Q_{q-1,q})\} &\geq V(e_q, Q' \cap Q_{0,q}) f_q = \\ C(e_q, Q' \cap Q_{0,q+1}) f_q &= f_q, \end{aligned}$$

so that

$$\begin{aligned} C(g_q, Q' \cap Q_{q+1,q+1}) &= V(g_q, Q' \cap Q_{q,q}) = V(V(g_q, Q' \cap Q_{q,q}), Q' \cap Q_{q,q}) \geq \\ (**) \quad V(f_q, Q' \cap Q_{q,q}) &\geq V(f_q, (Q_{0,1} \vee \{e_2, e_3 \dots e_q\})' \cap Q_{q,q}) \\ &= V(f_q, Q'_{0,1} \cap Q_{q,1}) = C(f_q, Q'_{0,1} \cap Q_{q+1,1}) = 1. \end{aligned}$$

Remark. The example
$$\begin{array}{ccc} L \otimes K & \subset & L \otimes M \\ \cup & & \cup \\ N \otimes K & \subset & N \otimes M \end{array}$$
, where $K \subset M$ and $N \subset L$ are finite index inclusions of type II₁ factors, shows that the above estimation is sharp. However in some cases there is no equality. Let α be an outer action of the symmetric group S_3 on a type II₁ factor K . Consider the following commuting and co-commuting square:

$$\begin{array}{ccc} K \rtimes_{\alpha} \langle(1, 2, 3)\rangle & \subset & K \rtimes_{\alpha} S_3 \\ \cup & & \cup \\ K & \subset & K \rtimes_{\alpha} S_2 \end{array}$$

We see that

$$d(K \subset K \rtimes_{\alpha} S_3) = 2 < \max(d(K \subset K \rtimes_{\alpha} S_2), d(K \rtimes_{\alpha} S_2 \subset K \rtimes_{\alpha} S_3)) = \max(2, 4).$$

Using the above method we can show a little more.

Corollary 9. Let the diagram
$$\begin{array}{ccc} N & \subset & L \\ \cup & & \cup \\ K & \subset & M \end{array}$$
 be as in the Theorem 8. We denote $a = d(N \subset L)$, $b = d(K \subset M)$, $c = d(K \subset N)$ and $d = d(M \subset L)$. Then

$$\begin{aligned} d(K \subset L) &\leq \max(\min(a, b), \min(c, d)) = \\ &\min(\max(a, c), \max(a, d), \max(b, c), \max(b, d)). \end{aligned}$$

Proof. If q is the right-hand side of the above inequality then the inequality (*) in the proof of 8. may be replaced by the following one:

$$(*)' \quad \bigvee \{ue_q u^* f_q | u \in U(Q' \cap Q_{q-1,q})\} \geq V(e_q, Q'_{1,0} \cap Q_{1,q}) f_q,$$

except the case $q = 1$ which we consider later. Also the inequality (**) above may be replaced by:

$$(**)' \quad V(f_q, Q' \cap Q_{q,q}) \geq V(f_q, Q' \cap Q_{q,0}).$$

By the symmetry this ends the proof in all cases except when $a = d = 1$ and $c, d > 1$. It can be solved by Theorem 8. and the next remark.

Remark. Let $N \subset M$ be a pair of type II₁ factors with $[M : N] < \infty$. Then $d(N \subset M) = 1$, iff " $N \subset M$ " \cong " $N \subset N \otimes M_n(\mathbb{C})$ ".

Indeed, if $d(N \subset M) = 1$ then the second relative commutant $N' \cap \langle M, e_N^M \rangle$ is a factor. So, there is a Pimsner-Popa basis of M over N composed of the elements of $N' \cap M$. Therefore, $M = N \vee (N' \cap M) \cong N \otimes (N' \cap M) = N \otimes M_n(\mathbb{C})$, where the last equality comes from the factoriness of M .

We give a characterization of the condition used in Theorem 8. For a given pair $Q \subset N$ of II₁ factors with finite index A.Ocneanu ([O]) and also Bisch ([B]) considered a set of projections corresponding to intermediate subfactors between Q and N denoted in [B] by $IS(Q, N)$. We recall one of the equivalent definitions. If $N_1 = \langle N, e_Q^N \rangle$ is the basic construction for $Q \subset N$ then

$$IS(Q, N) = \{q \in P(Q' \cap N_1) \mid qe_Q^N = e_Q^N, E_{N_1}^{N_1}(q) \in \mathbb{C} \text{ and } qNq \subset Nq\}.$$

Proposition 10. Let $Q \subset K \subset N$ be a triple of type II₁ factors with $[N : Q] < \infty$ and $Q' \cap N = \mathbb{C}$. Then there exists such type II₁ factor B that $\begin{array}{ccc} B & \subset & N \\ \cup & & \cup \\ Q & \subset & K \end{array}$ is a nondegenerate commuting square, if and only if there is in $IS(Q, N)$ a Jones' projection e corresponding to the inclusion $Q \subset K$ i.e. such that $\forall x \in K, exe = E_Q(x)e$ and $E_K(e) = [K : Q]^{-1}$.

Proof. If such a factor B exists then we can put $e = e_B^N \in IS(Q, N)$. Let now $e \in IS(Q, N)$. If $B = \{e\}' \cap N$ then as in [B]4.2 we see that for $x \in N$, $exe = E_B(x)e$, and that $[N : B]^{-1} = \tau(e)$. Since $E_{K'}^{Q'}(e) = [K : Q]^{-1}$ and $\begin{array}{ccc} K' & \subset & Q' \\ \cup & & \cup \\ N' & \subset & B' \end{array}$ (by 5.(2)) $B' = \langle N', \langle N, B, e \rangle', e \rangle$, the diagram of commutants $\begin{array}{ccc} K' & \subset & Q' \\ \cup & & \cup \\ N' & \subset & B' \end{array}$ is a commuting square. By 3.(iii) it is also a co-commuting square, which ends the argument.

Remark. It is obvious that instead of $Q' \cap N = \mathbb{C}$ in the above proposition we could have demanded for $\{e_Q^K\}' \cap N$ to be a factor.

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