VANISHING THEOREM FOR THE TEMPERED DISTRIBUTIONS

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0. Introduction

The functor $\mathrm{RH}(\cdot)$ which gives the Riemann-Hilbert correspondence was constructed and deeply studied by Kashiwara [K 3]. This functor plays an important role in the construction of the tempered micro-localization functor due to Andronikof ([A 1, A 2]). The tempered micro-localization functor in Gevrey classes is also constructed by the author of this paper [H 2] using $\mathrm{RH}^{(s)}(\cdot)$. It is an important problem to study the vanishing of the higher cohomology groups of $\mathrm{RH}(\mathbb{C}_\Omega)$ ($\mathrm{RH}^{(s)}(\mathbb{C}_\Omega)$) with Ω a pseudo-convex open set, because it is deeply connected with the concentration of cohomology groups of the tempered micro-localization functor (see section 3). We also need the vanishing of cohomology groups to study the tempered (Gevrey class) micro-localization functor with *Čech* representation.

Let $\Omega \subset X = \mathbb{C}^n$ be an open set and $s \in (1, \infty)$ or ϕ. We denote by $\mathbb{C}_\Omega$ a constructible sheaf whose stalk is $\mathbb{C}$ in Ω and 0 outside of Ω.

Theorem 0.1. *Assume Ω is a relative compact pseudo-convex open set. Then the complex of sheaves $\mathrm{RH}^{(s)}(\mathbb{C}_\Omega)$ is concentrated degree 0, and we have*

$$\mathrm{H}^k(\mathbb{R}\Gamma(X, \mathrm{RH}^{(s)}(\mathbb{C}_\Omega))) = 0 \qquad k \neq 0.$$

We give the definitions and notations of Theorem 0.1 in the next section, and some applications in section 3 . The proof of the main theorem is given in section 2.

1. Notations

Let $M = \mathbb{R}^n$ with a coordinate (x_i) and $s \in (1,\infty)$ or ϕ. We denote by $\mathcal{E}$ $(\mathcal{D})$ the sheaf of infinitesimal differentiable functions (with compact support). We set

$$\| f \|_K^{l,s} = \sup_{x\in K,\alpha} \frac{|D^\alpha f|}{|\alpha|!^{(s)} l^{|\alpha|}} \qquad (s \neq \phi)$$

for a compact set $K \subset M$ and $l > 0$.

We define the sheaf $\mathcal{E}^{(s)}$ $(\mathcal{D}^{(s)})$ of ultradifferentiable functions of Beurling class (s) as follows. For an open set U and $s \neq \phi$,

$$\mathcal{E}^{l,s}(U) := \{ f \in \mathcal{E} : \| f \|_K^{l,s} < \infty \qquad \text{for any} \quad K \subset\subset U \},$$

$$\mathcal{D}^{l,s}(U) := \mathcal{E}^{l,s} \cap \mathcal{D}(U),$$

and

$$\mathcal{E}^{(s)} := \varprojlim_{l} \mathcal{E}^{l,s},$$

$$\mathcal{D}^{(s)} := \mathcal{E}^{(s)} \cap \mathcal{D}.$$

We set $\mathcal{E}^{(\phi)} := \mathcal{E}$ and so on.

The sheaf of ultradistributions $\mathcal{D}_b^{(s)}$ of Beurling class (s) is the topological dual of $\mathcal{D}^{(s)}$. When $s = \phi$, this is nothing but the sheaf of distributions by definition. For the details, refer to Komatsu [Ko 2, Ko 3].

Let U be an open set of M and $\mathbb{C}_U$ a constructible sheaf whose stalk is $\mathbb{C}$ in U and 0 outside of U. We define the sheaf $\mathrm{TH}^{(s)}(\mathbb{C}_U)$ of tempered ultradistributions as follows. For any open set $V \subset M$

$$\mathrm{TH}^{(s)}(\mathbb{C}_U)(V) := \{ u \in \mathcal{D}_b^{(s)}(U \cap V) : \quad \text{there exists a} \quad u' \in \mathcal{D}_b^{(s)}(V) \quad \text{satisfying} \quad u = u'|_{U\cap V} \}.$$

We remark that $\mathrm{TH}(\cdot) := \mathrm{TH}^{(\phi)}(\cdot)$ was constructed by Kashiwara [K 3] for general constructible sheaves.

Proposition 1.1 (cf. [K 3; Lemma 3.3]). *Let $U \subset M$ be a relative compact open set. Then we have the following equivalent conditions (1), (2) and (3) for any $s \in (1,\infty)$ and $u \in \Gamma(U, \mathcal{D}_b^{(s)})$.*

(1) There exists a $v \in \Gamma(M, \mathcal{D}_b^{(s)})$ satisfying $u = v$ in U. (i.e. $u \in \Gamma(M, \mathrm{TH}^{(s)}(\mathbb{C}_U))$).

(2) There exist $C, l > 0$ satisfying

$$|< u, \psi >| \leq C \| \psi \|_{\bar{U}}^{l,s}$$

for any $\psi \in \mathcal{D}_b^{l,(s)}(U)$.

(3) There exist positive constants C, l and L satisfying

$$|< u, \psi >| \leq C \sup_{x\in U,\alpha} \frac{\exp(L\, \mathrm{dist}(CU, x)^{\frac{-1}{s-1}})|D^\alpha f|}{|\alpha|!^s l^{|\alpha|}}$$

for any $\psi \in \mathcal{D}_b^{l,(s)}(U)$.

In the case of $s = \phi$, we have the same proposition for the tempered distributions [K 3; Lemma 3.3].

Finally we introduce the functor $\mathrm{RH}^{(s)}(\cdot)$. Let X be a complex manifold. We denote by $\bar{X}$ the complex conjugate space of X.

$$\mathrm{RH}^{(s)}(\mathbb{C}_U) := \mathbb{R}\mathcal{H}om_{\mathcal{D}_{\bar{X}}}(\mathcal{O}_{\bar{X}}, \mathrm{TH}^{(s)}(\mathbb{C}_U))$$

where $\mathcal{D}_{\bar{X}}$ is the sheaf of linear differential operators with holomorphic coefficients in $\bar{X}$ and $\mathcal{O}_{\bar{X}}$ is the sheaf of holomorphic functions in $\bar{X}$ (i.e. Cauchy-Riemann system of $X^{\mathbb{R}}$ as $\mathcal{D}_{X\times\bar{X}}$ module).

2. Proof of the main theorem

In this section, we give the proof of Theorem 0.1 in the case of $s \neq \phi$. Since the proof of the vanishing of $\mathrm{RH}^{\phi}(\mathbb{C}_\Omega)$ goes in the similar way, we omit it.

We make a several preparation. Let $s > 1$ and $Q_l(\zeta)$ $(l > 0)$ entire functions in $\mathbb{C} = (\zeta)$ satisfying the following conditions. There exists a function $l' : \mathbb{R}^+ \to \mathbb{R}^+$ with $l'(l) \to \infty$ $(l \to \infty)$, and there exist positive constants B_l and C_l, we have estimates

$$|Q_l(\zeta)| \leq B_l \exp(l|\zeta|^{\frac{1}{2s}}) \qquad \zeta \in \mathbb{C},$$

and

$$|Q_l(\zeta)| \geq C_l \exp(l'(l)|\zeta|^{\frac{1}{2s}}) \qquad \mathrm{Re}\,\zeta \geq 0.$$

We can concretely construct a family of entire functions satisfying the above conditions (see [K 1; Theorem 10.1]). Our first step is to construct a differential operator of infinite order which possesses an elliptic property. Let $X = \mathbb{C}^n = (\zeta_1, \cdots, \zeta_n)$ with a coordinate $(\xi_1 + i\eta_1, \cdots, \xi_n + i\eta_n)$. We set

$$P_l(\zeta) := Q_l(\zeta_1^2 + \cdots + \zeta_n^2).$$

Then $P_l(\zeta)$ satisfies the following estimations.

$$|P_l(\zeta)| \leq B_l \exp(l|\zeta|^{\frac{1}{s}}) \qquad \zeta \in \mathbb{C}^n,$$

and

$$|P_l(\zeta)| \geq C_l \exp(l'(l)|\zeta|^{\frac{1}{s}}) \qquad \frac{1}{\sqrt{2}}|\xi| \geq |\eta|.$$

We consider a kernel function in $X = \mathbb{C}^n = (z_1 = x_1 + iy_1, \cdots, z_n = x_n + iy_n)$ as

$$K_l(z) := \int_{\mathbb{R}^n} \frac{e^{-iz\zeta}}{P_l(\zeta)} d\zeta \qquad z \in \mathbb{R}^n.$$

It is easy to see $K_l(x)$ is an ultradifferentiable function of Gevery order (s) in $\mathbb{R}^n$. Moreover we have the following lemma.

Lemma 2.1. *There exist $\sigma > 0$ and a function $l' : \mathbb{R}^+ \to \mathbb{R}^+$ with $l'(l) \to \infty$ $(l \to \infty)$, $K_l(z)$ is holomorphic in $T_\sigma := \{z; \sigma|x| > |y|\}$ and satisfies the estimate*

$$\sup_{z \in T_\sigma} \left| \frac{l'(l)^{|\alpha+\beta|} D_x^\alpha D_y^\beta K_l(z)}{|\alpha+\beta|!^s} \right| < \infty.$$

Proof. The first assertion of the lemma is easily shown by changing the path of the integral as

$$\eta = \frac{-x}{\sqrt{2}|x|}|\xi|.$$

To obtain the estimate of the lemma, we remark the inequality

$$t^n \exp(-lst^{\frac{1}{s}}) \le l^{-sn} n!^s,$$

for any $t > 0$ and $n \in \mathbb{N}$. Since $K(z)$ is holomorphic in T_σ, it is enough to show the estimates for $(\frac{\partial}{\partial z})^\alpha K_l(z)$, and this is a direct calculation.

We remark that $P_l(\frac{\partial}{\partial z})$ is a partial differential operator of infinete order with Gevrey growth (s) and satisfies

$$P_l(\frac{\partial}{\partial x}) K_l(x) = \delta(x).$$

Let $U \subset \mathbb{R}^n$ be a relative compact open set and $u \in \mathcal{D}_b^{(s)}(\mathbb{R}^n)$ with $\mathrm{supp}(u) \subset \partial U$. We define an ultradistribution in $\mathbb{R}^n$ as

$$g(x) := K_l * u \in \mathcal{D}_b^{(s)}(\mathbb{R}^n).$$

Lemma 2.2. *If we take a sufficiently large l, we have*

(1) $g(x)$ is an ultradifferentiable function of Gevrey class (s) in $\mathbb{R}^n$,

(2) and, g is an analytic function outside of ∂U and can be holomorphically extended over $W_\sigma = \{z \in X : \ |y| < \mathrm{dist}(\partial U, x)\}$ for some $\sigma > 0$. Moreover any derivative of $g(z)$ is bounded in W_σ.

Proof. On account of the estimation of K_l in Lemma 2.1, we esaily obtain the first assertion of the lemma. Since for any point $(x, y) \in W_\sigma$

$$\mathrm{supp}(u) - (x, y) \subset\subset T_\sigma,$$

and $K_l(z)$ is holomorphic in T_σ, g can be holomorphically extended over W_σ. Moreover $K_l(z)$ can be extened over $\mathbb{C}^n$ as an ultradifferentiable function preserving Gevrey growth order. It is easy to see g is bounded in W_σ.

Let Ω a relative compact psudo-convex open set and $\phi = |z|^2$. We denote by $L^2_{(p,q)}(\Omega)$ (resp. $C^\infty_{(p,q)}(\Omega)$, $\mathrm{TH}^{(s)}_{(p,q)}(\mathbb{C}_\Omega)(X)$) the space of (p, q) forms with coefficients in $L^2(\Omega)$ (resp. $C^\infty(\Omega)$, $\mathrm{TH}^{(s)}(\mathbb{C}_\Omega)(X)$). $L^2_{(p,q)}(\Omega)$ is Hilbert space equipped with a norm

$$\|f\|^2_{L^2(\Omega)} := \sum_{I,J} \int_{\mathbb{C}^n} |f_{I,J}|^2 dz \wedge d\bar{z}.$$

We denote by $\theta : C^\infty_{(p,q+1)}(\Omega) \to C^\infty_{(p,q)}(\Omega)$ the formal adjoint operator of $\bar{\partial}$ which has the concrete expression as

$$\theta f := \sum_{I,J} \sum_{j \in [1,n]} \frac{\partial}{\partial z_j} f_{I,Jk} dz^I \wedge d\bar{z}^J,$$

for any $f \in C^\infty_{(p,q+1)}(\Omega)$. From now on, we fix $\phi = |z|^2$.

Proposition 2.3. *Let $f \in C^\infty_{(p,q+1)}(\Omega)$ and $u \in C^\infty_{(p,q)}(\Omega)$. We assume u satisfies*

$$\bar{\partial} u = f,$$

and

$$\theta(e^{-\phi}u) = 0.$$

Then we have the following estimate.

$$\|\delta(z)^i \sum_{|\alpha|=i} D^\alpha u\|^2_{L^2(\Omega)} \le C^i(\|u\|^2_{L^2(\Omega)} + \sum_{|\alpha|=i-1} \|\delta(z)^{|\alpha|} D^\alpha f\|^2_{L^2(\Omega)}) \qquad (i \ge 1)$$

where C is a constant which depends only on Ω and $\delta(z) := \mathrm{Min}(\mathrm{dist}(C\Omega, z), 1)$.

Proof. We assume $q \ge 1$. In the case of $q = 0$, we can prove the proposition in the similar way as the following proof. Set

$$U_k = \{z \in \Omega; \quad \mathrm{dist}(C\Omega, z) \ge \frac{1}{2^k}\} \qquad k \ge 0.$$

We choose C^1 functions $\{\chi_k(z)\}_{k\ge 0}$ satisfying

$$0 \le \chi_k \le 1,$$

$$\chi_k(z) = 1 \qquad z \in U_k,$$

$$\mathrm{supp}(\chi_k) \subset \overline{U_{k+1}}$$

and, there exists a positive constant C

$$\sum_{i=0}^n (|\frac{\partial}{\partial z_i}\chi_k| + |\frac{\partial}{\partial \bar{z}_i}\chi_k|)\delta(z) \le C$$

for any k. By the assumption, we have

$$\bar{\partial} u = f,$$

$$\theta u = au$$

where a is a differential operator of order 0 whose coefficients are polynomials of degree 1. We have for any multi index α

$$\bar{\partial}(\chi_k D^\alpha u) = \bar{\partial}\chi_k \wedge D^\alpha u + \chi_k D^\alpha f,$$

$$\theta(\chi_k D^\alpha u) = bD^\alpha u + \chi_k D^\alpha(au)$$

where b is a differential operator of degree 0. Remark that

$$\delta(z)b : L^2_{(p,q+1)}(\Omega) \to L^2_{(p,q)}(\Omega)$$

is a bounded operator. We have for any multi index α ($|\alpha| \geq 1$),

$$\begin{aligned}\|\delta^{|\alpha|+1}\bar{\partial}\chi_k D^{\alpha+1}u\|^2_{L^2(\Omega)} &\leq \|\delta^{|\alpha|+1}\bar{\partial}\chi_k \wedge D^{\alpha}u\|^2_{L^2(\Omega)} + \|\delta^{|\alpha|+1}\chi_k D^{\alpha}f\|^2_{L^2(\Omega)} \\ &\leq C_1(\|\delta^{|\alpha|}D^{\alpha}u\|^2_{L^2(\Omega)} + \|\delta^{|\alpha|}D^{\alpha}f\|^2_{L^2(\Omega)}).\end{aligned}$$

Similary we have

$$\begin{aligned}\|\delta^{|\alpha|+1}\theta(\chi_k D^{\alpha}u)\|^2_{L^2(\Omega)} &\leq C_2(\|\delta^{|\alpha|}D^{\alpha}u\|^2_{L^2(\Omega)} \\ &+ \sum_{|\beta|=|\alpha|-1,\beta\prec\alpha}\|\delta^{|\beta|}D^{\beta}u\|^2_{L^2(\Omega)} + \|\delta^{|\alpha|}D^{\alpha}u\|^2_{L^2(\Omega)}).\end{aligned}$$

Therefore we obtain the following estimation.

$$\begin{aligned}&\sum_{1\leq i\leq n}(\|\delta^{|\alpha|+1}\frac{\partial}{\partial z_i}\chi_k D^{\alpha+1}u\|^2_{L^2(\Omega)} + \|\delta^{|\alpha|+1}\frac{\partial}{\partial \bar{z}_i}\chi_k D^{\alpha+1}U\|^2_{L^2(\Omega)}) \\ &\leq 4(\|\delta^{|\alpha|+1}\partial\chi_k D^{\alpha+1}u\|^2_{L^2(\Omega)} + \|\delta^{|\alpha|+1}\bar{\partial}\chi_k D^{\alpha+1}u\|^2_{L^2(\Omega)}) \\ &\leq C_3(\|\delta^{|\alpha|}D^{\alpha}u\|^2_{L^2(\Omega)} + \|\delta^{|\alpha|}D^{\alpha}f\|^2_{L^2(\Omega)} \\ &+ \sum_{|\beta|=|\alpha|-1,\beta\prec\alpha}\|\delta^{|\beta|}D^{\beta}u\|^2_{L^2(\Omega)})\end{aligned}$$

where C_3 is a positive constat depending only on Ω. Taking limit $k \to \infty$, we have for any multi index α

$$\begin{aligned}&\sum_{1\leq i\leq n}(\|\delta^{|\alpha|+1}\frac{\partial}{\partial x_i}D^{\alpha+1}u\|^2_{L^2(\Omega)} + \|\delta^{|\alpha|+1}\frac{\partial}{\partial \bar{x}_i}D^{\alpha+1}u\|^2_{L^2(\Omega)}) \\ &\leq C_4(\|\delta^{|\alpha|}D^{\alpha}u\|^2_{L^2(\Omega)} + \|\delta^{|\alpha|}D^{\alpha}f\|^2_{L^2(\Omega)} \\ &+ \sum_{|\beta|=|\alpha|-1,\beta\prec\alpha}\|\delta^{|\beta|}D^{\beta}u\|^2_{L^2(\Omega)}).\end{aligned}$$

By induction and the above inequality, we obtain the disired estimation.

Now we show the proof of the main theorem.

proof. Consider the complex

$$\mathrm{TH}^{(s)}_{(p,q)}(\mathbb{C}_{\Omega})(X) \xrightarrow{\bar{\partial}} \mathrm{TH}^{(s)}_{(p,q+1)}(\mathbb{C}_{\Omega})(X) \xrightarrow{\bar{\partial}} \mathrm{TH}^{(s)}_{(p,q+2)}(\mathbb{C}_{\Omega})(X).$$

Given $f \in \mathrm{TH}^{(s)}_{(p,q+1)}(\mathbb{C}_{\Omega})(X)$ with $\bar{\partial}f = 0$. We take $f' \in \mathcal{D}^{(s)}_b(X)$ satisfying

$$f'|_U = f|_U,$$

$$\mathrm{supp}(f') \subset \bar{U},$$

and

$$\mathrm{supp}(\bar{\partial}f') \subset \partial U.$$

For a sufficently large $l > 0$, we set

$$h := K_l * \bar{\partial} f',$$

where $K(z, \bar{z})$ is a kernel function associated with $P_l(\zeta, \bar{\zeta})$ constructed in the beginning of this section. Remark that, on account of Lemma 2.1, $K_l(z, \bar{z})$ is a real analytic function in the underlaying space $X^R = R^{2n}$ except for the origin and can be holomorphically extended over the domain $T_\sigma \subset X \times \bar{X}$ described in the same lemma. Therefore h satisfies the following.

(1) $\bar{\partial} h = k_l * \bar{\partial}\bar{\partial} f' = 0$ in X^R.
(2) h is a $(p, q+2)$ form with coefficients in ultradifferentiable functions of Gevrey order (s) in X^R.
(3) There exists a positive constat C, we have the estimate

$$|\frac{\partial}{\partial z}^{\alpha} \frac{\partial}{\partial \bar{z}}^{\beta} h| \leq C \frac{|\alpha+\beta|!}{\operatorname{dist}(C\Omega, z)^{|\alpha+\beta|}} \qquad (z, \bar{z}) \in X^{\mathbb{R}} \subset X \times \bar{X}$$

on account of Lemma 2.2.

We fix a relative compact convex open set $W \subset X^R$ with $\Omega \subset\subset W$. By the theorem of Hörmander [Hö; chapter 4], we can find $g \in C^{\infty}_{p,q+1}(W)$ satisfying

$$\bar{\partial} g = h \qquad \text{in} \quad W,$$

and

$$\theta(e^{-\phi} u)|_U = 0$$

where $\phi = |z|^2$. Then by proposition 2.3, we have

$$\|\delta(z)^{|\alpha+\beta|} \frac{\partial}{\partial z}^{\alpha} \frac{\partial}{\partial \bar{z}}^{\beta} g\|_{L^2(\Omega)} \leq C^{|\alpha+\beta|} |\alpha+\beta|! \qquad (z \in \Omega)$$

with a positive constant C. Since

$$P_l(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}) = \Sigma C_{\alpha\beta} \frac{\partial}{\partial z}^{\alpha} \frac{\partial}{\partial \bar{z}}^{\beta}$$

is a differential operator of infinite order with Gevrey class (s), there exists a positive constants l satisfying

$$|C_{\alpha\beta}| \leq \frac{l^{|\alpha+\beta|}}{|\alpha+\beta|!^s}.$$

Therefore we obtain

(2.1)

$$\begin{aligned}
\|(P_l(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}) g) \exp(-l' \delta(z)^{\frac{-1}{s-1}})\|_{L^2(\Omega)} &\leq \Sigma \frac{(Cl)^{|\alpha+\beta|} |\alpha+\beta|!}{|\alpha+\beta|!^s} \operatorname{supp}_{z \in \Omega} |\delta(z)^{-|\alpha+\beta|} \exp(-l' \delta(z)^{\frac{-1}{s-1}})| \\
&\leq C_1 \Sigma \frac{(Cl)^{|\alpha+\beta|}}{|\alpha+\beta|!^{(s-1)}} \frac{|\alpha+\beta|!^{(s-1)}}{(l')^{|\alpha+\beta|}} \\
&\leq C_2 \Sigma (\frac{Cl}{l'})^{|\alpha+\beta|} < \infty
\end{aligned}$$

for a sufficiently large l'.

We set $f''(z) := P_l(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}})g$. Since W is convex and

$$\bar{\partial}(f' - f'') = \bar{\partial} f' - P_l \bar{\partial} g = \bar{\partial} f' - P_l h = \bar{\partial} f' - P_l(K_l * \bar{\partial} f') = 0$$

on account of Palamdov [P; p.300 Theorem 1], we can find $u' \in \mathcal{D}_b^{(s)}(W)$ with $\bar{\partial} u' = f' - f''$. On the other hand, since $\bar{\partial} f''|_\Omega = \bar{\partial} f'|_\Omega = 0$ and f'' satisfies the esitmation (2.1), we can find $u'' \in L^{loc}_{(p,q)}(\Omega)$ satisfying

$$f'' = \bar{\partial} u'' \qquad \text{in} \quad \Omega$$

and the estimation

$$\|u'' \frac{\exp(-l'\delta(z)^{\frac{-1}{s-1}})}{(1+|z|^2)}\|_{L^2(\Omega)} < \infty$$

due to [Hö; Theorem 4.4.2] . The above inequality means, in particular, $u'' \in \mathrm{TH}^{(s)}_{(p,q)}(\mathbb{C}_\Omega)(X)$. Thus $u = u' + u'' \in \mathrm{TH}^{(s)}_{(p,q)}(\mathbb{C}_\Omega)(X)$ satisfies

$$\bar{\partial}(u' + u'')|_U = ((f' - f'') + f'')|_U = f|_U.$$

3. The application of the main thorem

One of the application of the main theorem, we give the vanishing of distribution solutions along the submanifolds which are micro-locally weakly pseudo-convex.

Let M be a real analytic submanifold in a complex manifold X, and $\pi : T^*X \to X$ (resp. $\pi : T^*_M X \to X$) a cotangent bundle of X (resp. conormal bundle of M). We set for $p \in T^*X$,

$$\begin{aligned}
&E(p) = T_p(T^*X),\\
&\lambda_M(p) = T_p(T^*_M X),\\
&\lambda_0(p) = T_p(\pi^{-1}\pi(p)),\\
&\rho_M(p) = \lambda_M(p) \cap i\lambda_M(p),\\
&\delta(p) = \dim_{\mathbb{C}}(\lambda_M(p) \cap i\lambda_M(p) \cap \lambda_0(p)).
\end{aligned}$$

The vector space $(E(p), \sigma)$ has a natural complex symplectic structure. For any real Lagrangian vector space $\lambda \in E(p)^{\mathbb{R}}$, we denote by λ^{ρ_M} the quotient space

$$((\lambda \cap \rho(p)^{\perp}_M) + \rho_M(p))/\rho_M(p).$$

We also define the bilinear form γ_{λ_M} on $\lambda_0^{\rho_M}(p)$ by setting for $(u, v) \in \lambda_0^{\rho_M}(p) \times \lambda_0^{\rho_M}(p)$

$$\gamma_{\lambda_M}(u, v) := \sigma_{\rho_M}(u, \bar{v})$$

where $\bar{v}$ is the complex conjugate of v with respect to $\lambda_M^{\rho_M}(p)$. We denote by $S_+(M, p)$ (resp $S_-(M, p)$) the number of positive (resp. negative) eigen values of the hermitian form γ_{λ_M}, and define $S(M, p) = S_+(M, p) - S_-(M, p)$.

Theorem 3.1 (cf. [K-S 1; Theorem 11.3.5]). *Let $p_0 \in T^*_M X \backslash T^*_X X$. We assume the following conditions.*

(1) $\dim_{\mathbb{R}}(\lambda_M(p_0) \cap \nu(p_0)) = 1$ *where $\nu(p_0)$ is a complex line generated by the Euler vector field.*

(2) $S_-(M, p) - \delta(p)$ *is locally constant in a neighborhood of p_0.*

Then the complex $\mathrm{T}\text{-}\,\mu_M(\mathcal{O}_X)_{p_0}$ *is concentrated degree* $k = \operatorname{codim} M + S_-(M, p_0) - \delta(p_0)$, *and the canonical morphism*

$$\mathrm{H}^k\,\mathrm{T}\text{-}\,\mu_M(\mathcal{O}_X)_{p_0} \to \mathrm{H}^k\,\mu_M(\mathcal{O}_X)_{p_0}$$

is injective.

When X is the complexification of a real analytic manifold M, the above theorem has been shown by Maltineau [Ma 1]. Andronikof, in his papers [A 1, A 2], has obtained the edge of the wedge with tempered growth order, and proved the same results in the complex case (i.e. M is a complex manifold). The first assertion replaced the functor $\mathrm{T}\text{-}\,\mu_M$ with a micro-localization functor μ_M was established by Kashiwara-Schapira [K-S 1]. They proved this to construct a quantized contact transformation cohomologically and reduce the problem to the case M is a weakly pseudo-convex hypersurface. Therefore the important point is to construct a quantized contact transformation cohomologically for the tempered micro-localization functor. We first study how to construct the quantized contact transformation. Let X,Y be complex manifolds and $N \subset Y$, $M \subset X$ real analytic submaifolds, $\Phi : T^*X \to T^*Y$ a complex contact transformation which interchange $(T^*_M X, p_0)$ and $(T^*_N Y, \Phi(p_0))$ in a neighborhood of $p_0 \in T^*_M X \backslash T^*_X X$. We denote by Q_1, Q_2 the first and the second projection from $X \times Y$ respectively and by P_1, P_2 the first and the second projection from $T^*(X \times Y)$. We set $P_1^a = P_1 \cdot a$ where a is the anti-podal map on T^*X. Let $\Lambda \subset T^*(X \times Y)$ be a complex conic Lagrangian submanifold in a neighborhood of $(p_0, -\Phi(p_0))$ obtained by taking the image of graph of Φ by the anti-podal map on T^*X. Moreover we may assume $Z = \pi(\Lambda) \subset X \times Y$ is a smooth complex hypersurface. We denote by K a pure sheaf $\mathbb{C}_Z[-1]$. And if necessarily, we cut the support of K in a neighborhood of $\pi(p, -\Phi(p))$ satisfying that $Q_2^a|_{\mathrm{supp}(K)}$ is proper over $\mathrm{supp}(\mathbb{C}_N)$. This does not affect on the following arguments because we consider the problem in the micro-local point of view. To construct the quantized contact transformation, we use direct image and restriction of the tempered micro-localization functor due to Andronikof [A 1, A 2]. We consider the following diagram.

$$\begin{array}{ccccc}
X & \xleftarrow{Q_1} & X \times (X \times Y) & & \\
 & & \Big\uparrow \iota & & \\
 & & \Delta_X \times Y \simeq X \times Y & \xrightarrow{Q_2} & Y.
\end{array}$$

$$(\text{T-}\mu hom(K, \mathcal{O}_{X\times Y}) \underset{\mathcal{O}_{X\times Y}}{\otimes} \Omega_{X\times Y/Y})_{(-p_0,\Phi(p_0))} \boxtimes \text{T-}\mu hom(\mathbb{C}_M, \mathcal{O}_X)_{p_0}$$
$$\to \text{T-}\mu hom(\mathbb{C}_M \boxtimes K, \mathcal{O}_{X\times X\times Y})_{(p_0,-p_0,\Phi(p_0))} \underset{\mathcal{O}_{X\times Y}}{\otimes} \Omega_{X\times Y/Y}$$
$$\to \mathcal{D}_{X\times Y\to X\times X\times Y} \underset{\mathcal{D}_{X\times X\times Y}}{\overset{\mathbb{L}}{\otimes}} \text{T-}\mu hom(\mathbb{C}_M \boxtimes K, \mathcal{O}_{X\times X\times Y})_{(p_0,-p_0,\Phi(p_0))} \underset{\mathcal{O}_{X\times Y}}{\otimes} \Omega_{X\times Y/Y}$$
$$\xrightarrow{i^*} \text{T-}\mu hom(i^{-1}(\mathbb{C}_M \boxtimes K), \mathcal{O}_{X\times Y})_{(-p_0,\Phi(p_0))} \underset{\mathcal{O}_{X\times Y}}{\otimes} \Omega_{X\times Y/Y}$$
$$\to \mathcal{D}_{Y\to X\times Y} \underset{\mathcal{D}_{X\times Y}}{\overset{\mathbb{L}}{\otimes}} \text{T-}\mu hom(i^{-1}(\mathbb{C}_M \boxtimes K), \mathcal{O}_{X\times Y})_{(-p_0,\Phi(p_0))}$$
$$\xrightarrow{\int_!} \text{T-}\mu hom(\mathbb{R}Q_{2*}i^{-1}(\mathbb{C}_M \boxtimes K), \mathcal{O}_Y)_{\Phi(p_0)}[-\dim_{\mathbb{C}} X].$$

The second map is given by choosing a canonical section $1_{X\times Y\to X\times X\times Y} \in \mathcal{D}_{X\times Y\to X\times X\times Y}$. Since

$$\text{T-}\mu hom(K, \mathcal{O}_{X\times Y}) = \mathcal{E}^{\mathbb{R},f}_{X\times Y} \underset{\mathcal{D}_{X\times Y}}{\otimes} \mathcal{B}_{Z|X\times Y}$$

and $\mathcal{B}_{Z|X\times Y}$ is a simple holonomic system, we can take a section $s \in \mathcal{E}^{\mathbb{R},f}_{Z|X\times Y}$ which generates $\mathcal{E}^{\mathbb{R},f}_{Z|X\times Y}$ over $\mathcal{E}^{\mathbb{R},f}_{X\times Y}$. Then we have the morphism

$$\text{T-}\mu hom(\mathbb{C}_M, \mathcal{O}_X)_{p_0} \to (\text{T-}\mu hom(K, \mathcal{O}_{X\times Y}) \underset{\mathcal{O}_{X\times Y}}{\otimes} \Omega_{X\times Y/Y})_{(-p_0,\Phi(p_0))} \boxtimes \text{T-}\mu hom(\mathbb{C}_M, \mathcal{O}_X)_{p_0},$$

by $u \to s \boxtimes u$. Thus we have the cohomological quantized contact transformation map

$$\text{T-}\mu hom(\mathbb{C}_M, \mathcal{O}_{X\times Y})_{p_0} \to \text{T-}\mu hom(\mathbb{R}Q_{2*}i^{-1}(\mathbb{C}_M \boxtimes K), \mathcal{O}_Y)_{\Phi(p_0)}[-\dim_{\mathbb{C}} X].$$

Since $\mathbb{R}Q_{2*}i^{-1}(\mathbb{C}_M \boxtimes K)$ is a simple sheaf along $T^*_N Y$ with the shift $-\frac{1}{2}\dim M - \frac{1}{2}(S(N,p) - S(M,\Phi(p))$ in a neighborhood p_0, we have

$$\text{T-}\mu hom(\mathbb{C}_M, \mathcal{O}_{X\times Y})_{p_0} \to \text{T-}\mu hom(\mathbb{C}_N, \mathcal{O}_Y)_{\Phi(p_0)}[d]$$

where $d := \frac{1}{2}(\dim M - \dim N + S(M,p_0) - S(N,\Phi(p_0)))$. We can easily check the map constructed above satisfies the composition law, and is isomorphism. Moreover we have a commutative diagram

$$\begin{array}{ccc}
\text{T-}\mu hom(\mathbb{C}_M, \mathcal{O}_X)_{p_0} & \longrightarrow & \text{T-}\mu hom(\mathbb{C}_N, \mathcal{O}_Y)_{\Phi(p_0)}[d] \\
\downarrow & & \downarrow \\
\mathbb{R}\mu hom(\mathbb{C}_M, \mathcal{O}_X)_{p_0} & \longrightarrow & \mathbb{R}\mu hom(\mathbb{C}_N, \mathcal{O}_Y)_{\Phi(p_0)}[d].
\end{array}$$

proof of Theorem 3.1. In the same way of the proof of [K-S 1, Theorem 11.3.5], we can find a hypersurface N which is the boundary of a pseudo-convex open set $j : \Omega \to X$ (in particular, $S_-(N,q) = 0$) and a contact transformation ϕ interchanging $(T^*_M X, p_0)$ and $(T^*_N Y, q_0)$ in a neighborhood of p_0. We obtain

$$\text{T-}\,\mu hom(\mathbb{C}_M, \mathcal{O}_X)_{p_0}[\operatorname{codim} M + S_-(M,p_0) - \delta(p_0)] = \text{T-}\,\mu hom(\mathbb{C}_N, \mathcal{O}_Y)_{q_0}[1].$$

Since there is a triangle

$$\text{T-}\,\mu hom(\mathbb{C}_N, \mathcal{O}_Y)_{q_0} \to \mathcal{O}_{\pi(q_0)} \to \mathrm{RH}(\mathbb{C}_\Omega)_{\pi(q_0)} \xrightarrow{+1},$$

we obtain the first assertion of Theorem 3.1 due to Theorem 0.1. Moreover

$$\mathrm{H}^0(\mathrm{RH}(\mathbb{C}_\Omega))/\mathcal{O} \to j_*\mathcal{O}/\mathcal{O}$$

is clearly injective, we have injectivity of $\mathrm{H}^k\,\text{T-}\,\mu_M(\mathcal{O}_X) \to \mathrm{H}^k\,\mu_M(\mathcal{O}_X)$.

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