Betti number sequences of simplicial complexes, Cohen-Macaulay types and Möbius functions of partially ordered sets, and related topics

Takayuki Hibi

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Betti number sequences of simplicial complexes, Cohen-Macaulay types and Möbius functions of partially ordered sets, and related topics

Takayuki Hibi
Department of Mathematics
Faculty of Science
Hokkaido University
Kita-ku, Sapporo 060, Japan

Abstract

We give a survey of some recent results on $b_i^A(k[\Delta]) = \dim_k \underline{\mathrm{Tor}}_i^A(k[\Delta], k)$ of the face ring (Stanley-Reisner ring) $k[\Delta]$ of a simplicial complex Δ. In particular, we study Cohen-Macaulay types and canonical modules of partially ordered sets.

Introduction

Let $A = k[x_1, x_2, \cdots, x_v]$ be a polynomial ring over a field k, I an ideal of A, and $R = A/I$. A finite free resolution of R as a module over A is an exact sequence

$$0 \to M_h \to M_{h-1} \to \cdots \to M_1 \to M_0 \to R \to 0, \tag{1}$$

where each M_i is a free module over A of finite positive rank. The existence of a finite free resolution of R over A is guaranteed. Moreover, it can easily be shown that these ranks can be simultaneously minimized. The finite free resolution (1) is said to be minimal if each M_i has the smallest possible rank. A minimal (finite) free resolution of R over A is uniquely determined. Define $b_i^A(R)$ to be the rank of M_i appearing in the minimal free resolution of R over A. Thus, in particular, $b_0^A(R) = 1$ and $b_1^A(R)$ is the minimal number of generators of the ideal I. In the language of homological algebra, $b_i^A(R) = \dim_k \underline{\mathrm{Tor}}_i^A(R, k)$. The purpose of this paper is to give an outline of recent developments on the study of $b_i^A(R)$ when I is generated by square-free monomials, i.e., R is equal to the face ring (or Stanley-Reisner ring) $k[\Delta]$ of a simplicial complex Δ, see (1.4).

Here is a brief description of this paper. In Section 1 some fundamental material for algebra, topology and combinatorics on simplicial complexes and partially ordered sets is summarized. In Section 2, first, a result on Betti number sequences $\beta(\Delta; k) = (\beta_0, \beta_1, \cdots, \beta_{d-1})$ (cf. (1.3)) of a Buchsbaum simplicial complex Δ (cf. (1.7)) is given, and, secondly, some relations between $b_i^A(k[\Delta])$'s and $\beta(\Delta; k)$ of a Buchsbaum simplicial complex Δ are studied. On the other hand, Section 3 is devoted to the computation of Cohen-Macaulay types of Cohen-Macaulay complexes. If R is Cohen-Macaulay, then the rank $b_h^A(R)$ of M_h ($\neq 0$) in the minimal free resolution (1) of R over A is called a Cohen-Macaulay type of R, say type(R). Even though the Cohen-Macaulay type is an important invariant of a Cohen-Macaulay ring R, in general, to compute type(R) is quite difficult. We give a combinatorial formula, see Corollary (3.6), for the computation of Cohen-Macaulay types of Stanley-Reisner rings of order complexes of modular lattices by means of Möbius functions. Moreover, in Section 4 we find an explicit expression of the canonical module (cf. (1.12)) of the Stanley-Reisner ring of the order complex of a modular lattice. Finally in Section 5 we conclude this paper with some open questions, which might stimulate further research in the field.

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§1. Background

We here summarize some fundamental material for algebra, topology and combinatorics on simplicial complexes and partially ordered sets. See [12] for an introduction to the topic of algebraic combinatorics on convex polytopes and simplicial complexes. Concerning the detailed and further information, we refer the reader to, e.g., [8], [16] and [22, Chap. II] on commutative algebra and simplicial complexes, [23, Chap. III] on partially ordered sets, and [18] on algebraic topology.

(1.1) Let $V = \{x_1, x_2, \cdots, x_v\}$ be a finite set, called the *vertex set*, and $\triangle$ a *simplicial complex* on V. Thus $\triangle$ is a family of subsets of V such that (i) $\{x_i\} \in \triangle$ for each $1 \leq i \leq v$ and (ii) $\sigma \in \triangle$, $\tau \subset \sigma$ imply $\tau \in \triangle$. Each element of $\triangle$ is called a *face* of $\triangle$. Set $d = \max\{\sharp(\sigma); \sigma \in \triangle\}$. Here $\sharp(\sigma)$ is the cardinality of σ as a finite set. Then the *dimension* of $\triangle$ is defined by $\dim \triangle = d - 1$. We say that a simplicial complex $\triangle$ is *pure* if every maximal face has the same cardinality.

(1.2) When W is a subset of V, we write Δ_W for the simplicial complex $\{\sigma \in \Delta; \sigma \subset W\}$ on the vertex set W. On the other hand, given a face σ of Δ, we define the subcomplex $\mathrm{link}_\Delta(\sigma)$ and $\mathrm{star}_\Delta(\sigma)$ of Δ by

$$\mathrm{link}_\Delta(\sigma) = \{\tau \in \Delta; \sigma \cap \tau = \phi \text{ and } \sigma \cup \tau \in \Delta\}$$

$$\mathrm{star}_\Delta(\sigma) = \{\tau \in \Delta; \sigma \cup \tau \in \Delta\}.$$

Thus, in particular, $\mathrm{link}_\Delta(\phi) = \Delta$. Also, if $\tau \in \Delta' = \mathrm{link}_\Delta(\sigma)$, then $\mathrm{link}_{\Delta'}(\tau) = \mathrm{link}_\Delta(\sigma \cup \tau)$. Moreover, if $\Delta' = \mathrm{star}_\Delta(\sigma)$, then $\mathrm{link}_\Delta(\sigma) = \mathrm{link}_{\Delta'}(\sigma)$.

(1.3) Fix a field k. Let $\tilde{H}_i(\Delta; k)$ be the i-th reduced simplicial homology group of Δ with coefficients k. The i-th (reduced) *Betti number* $\beta_i = \beta_i(\Delta; k)$ of Δ over k is the dimension $\dim_k \tilde{H}_i(\Delta; k)$ of $\tilde{H}_i(\Delta; k)$ as a vector space over k. We say that the sequence $\beta(\Delta; k) = (\beta_0, \beta_1, \cdots, \beta_{d-1})$ is the *Betti number sequence* of Δ over k. The *reduced Euler characteristic* $\tilde{\chi}(\Delta)$ of Δ is defined to be

$$\tilde{\chi}(\Delta) = \sum_{i \geq 1} (-1)^i \dim_k \tilde{H}_i(\Delta; k).$$

Then, the Euler-Poincaré formula says

$$\tilde{\chi}(\Delta) = \sum_{i \geq -1} (-1)^i f_i(\Delta),$$

where $f_i(\Delta)$ is the number of faces σ of Δ with $\sharp(\sigma) = i+1$ (and $f_{-1}(\Delta) = 1$). Thus, in particular, $\tilde{\chi}(\Delta)$ is independent of the characteristic of k. Note that $\tilde{H}_{-1}(\Delta; k) = 0$ if $\Delta \neq \phi$, $\tilde{H}_i(\phi; k) = 0$ for each $i \geq 0$, and $\tilde{H}_{-1}(\phi; k) \cong k$. Hence, $\tilde{\chi}(\phi) = -1$.

(1.4) We consider the elements of V to be indeterminates over a field k with each $\deg x_i = 1$. Let $A = k[x_1, x_2, \cdots, x_v]$ be the polynomial ring in v-variables over k and write $\mathfrak{m}$ for the irrelevant maximal ideal $(x_1, x_2, \cdots, x_v)$ of A. Define I_Δ to be the ideal of A which is generated by those square-free monomials $x_{i_1}x_{i_2}\cdots x_{i_r}$, $1 \leq i_1 < i_2 < \cdots < i_r \leq v$, such that $\{x_{i_1}, x_{i_2}, \cdots, x_{i_r}\} \notin \Delta$. Set $k[\Delta] = A/I_\Delta$. The algebra $k[\Delta]$ over k is called the *Stanley-Reisner ring* of Δ over k ([19], [20]). The set of those monomials $\prod_{1\leq i\leq v} x_i^{a_i}$, each $0 \leq a_i \in \mathbb{Z}$, such that $\{x_i; a_i > 0\} \in \Delta$ is a basis of $k[\Delta]$ as a vector space over k. We may regard $k[\Delta]$ as a graded module over A with the "quotient grading." It follows easily that the maximal number of homogeneous polynomials of A whose images in $k[\Delta]$ are algebraically independent over k is equal to d, i.e., $\dim_A k[\Delta] = d$. Recall that the depth of $k[\Delta]$ over A is the greatest integer t for which there exists a regular sequence $\theta_1, \theta_2, \cdots, \theta_t$ on $k[\Delta]$ such that each $\theta_i \in A$ is a homogeneous polynomial with $\deg(\theta_i) \geq 1$. Let $t = t_A(k[\Delta])$ denote the depth of $k[\Delta]$ over A. Note that $t > 0$ since $x_1 + x_2 + \cdots + x_v \in A$ is a non-zero divisor on $k[\Delta]$.

(1.5) The i-th *local cohomology module* $\underline{\mathrm{H}}^i_{\mathfrak{m}}(k[\Delta])$ of $k[\Delta]$ over A is defined to be

$$\underline{\mathrm{H}}^i_{\mathfrak{m}}(k[\Delta]) = \varinjlim_n \underline{\mathrm{Ext}}^i_A(A/\mathfrak{m}^n, k[\Delta]).$$

Then $\underline{\mathrm{H}}^i_{\mathfrak{m}}(k[\Delta]) = 0$ unless $t \leq i \leq d$ and $\underline{\mathrm{H}}^d_{\mathfrak{m}}(k[\Delta]) \neq 0$, $\underline{\mathrm{H}}^t_{\mathfrak{m}}(k[\Delta]) \neq 0$. If $\dim_k(\underline{\mathrm{H}}^i_{\mathfrak{m}}(k[\Delta])) < \infty$, then $\dim_k(\underline{\mathrm{H}}^i_{\mathfrak{m}}(k[\Delta])) = \beta_{i-1}(\Delta; k)$. Note that $\dim_k(\underline{\mathrm{H}}^d_{\mathfrak{m}}(k[\Delta])) = \infty$.

(1.6) Let $b_i^A(k[\triangle])$ be the non-negative integer defined by

$$b_i^A(k[\triangle]) = \dim_k \underline{\mathrm{Tor}}_i^A(k[\triangle], k).$$

Thus $b_i^A(k[\triangle] = 0$ unless $0 \le i \le v$. Let $h = h_A(k[\triangle])$ be the greatest integer h for which $b_h^A(k[\triangle]) \neq 0$. Then $h = v - t$ (Auslander-Buchsbaum [1]). Moreover, Hochster [16, Theorem (5.1)] says that the non-negative integer $b_i^A(k[\triangle])$ is given by

$$b_i^A(k[\triangle]) = \sum_{W \subset V} \dim_k(\tilde{H}_{V - \sharp(W) - i - 1}(\triangle_{V-W}; k)). \tag{2}$$

We refer the reader to Munkres [17] and Stanley [21] for topological and combinatorial applications of Eq.(2).

(1.7) We say that a simplicial complex $\triangle$ is *Cohen-Macaulay* over k if $\underline{\mathrm{H}}_{\mathfrak{m}}^i(k[\triangle]) = 0$ for every $0 \le i < d$, i.e., $t = d$ (or $h = v - d$), and that $\triangle$ is *Buchsbaum* if $\dim_k(\underline{\mathrm{H}}_{\mathfrak{m}}^i(k[\triangle])) < \infty$ for every $0 \le i < d$. Every Buchsbaum complex is pure. On the other hand, Reisner's criterion [19] guarantees that a simplicial complex $\triangle$ is Cohen-Macaulay if and only if, for every face σ of $\triangle$ (possibly, $\sigma = \phi$) and for each $i \neq \dim(\mathrm{link}_\triangle(\sigma))$, we have $\tilde{H}_i(\mathrm{link}_\triangle(\sigma); k) = 0$. Moreover, a simplicial complex $\triangle$ is Buchsbaum if and only if $\triangle$ is pure and $\mathrm{link}_\triangle(\sigma)$ is Cohen-Macaulay for every non-empty face σ of $\triangle$.

(1.8) If a simplicial complex $\triangle$ is Cohen-Macaulay over k, then the positive integer $b_{v-d}^A(k[\triangle])$ is called the *Cohen-Macaulay type* of $\triangle$ over k, and is written as type($k[\Delta]$). We say that a Cohen-Macaulay complex Δ over k is *Gorenstein* over k if type($k[\Delta]$) $= 1$. Every subcomplex $\mathrm{link}_\Delta(\sigma)$ of a Gorenstein complex Δ is again Gorenstein.

(1.9) A Cohen-Macaulay complex Δ is called *doubly Cohen-Macaulay* (Baclawski [3]) if the subcomplex $\Delta_{V-\{x\}}$ is Cohen-Macaulay of the same dimension as Δ for every $x \in V$. If Δ is a doubly Cohen-Macaulay complex, then $\mathrm{link}_\Delta(\sigma)$ is also doubly Cohen-Macaulay for every face σ of Δ. Moreover, a Gorenstein complex Δ with $\tilde{\chi}(\Delta) \neq 0$ is doubly Cohen-Macaulay. On the other hand, a Cohen-Macaulay complex Δ is doubly Cohen-Macaulay if and only if type($k[\Delta]$) $= (-1)^{d-1}\tilde{\chi}(\Delta)$, see [3, Corollary (4.7)].

(1.10) A *finite free resolution* of $k[\Delta]$ as a graded module over A is an exact sequence

$$0 \rightarrow A^{\delta_s} \xrightarrow{\eta_s} A^{\delta_{s-1}} \xrightarrow{\eta_{s-1}} \cdots \xrightarrow{\eta_2} A^{\delta_1} \xrightarrow{\eta_1} A^{\delta_0} \xrightarrow{\eta_0} k[\Delta] \rightarrow 0, \qquad (3)$$

where each A^{δ_i} is a free module over A of rank $\delta_i > 0$ with the basis $e(i;j) = (0,\cdots,0,1,0,\cdots,0) \in A^{\delta_i}$ the "1" in the j-th component, $\deg e(i;j) \in \mathbb{Z}$, $1 \leq j \leq \delta_i$, and where the maps η_i's are degree-preserving. Then $s \geq h_A(k[\Delta])$ and each $\delta_i \geq b_i^A(k[\Delta])$. We say that the finite free resolution (3) is *minimal* if $s = h_A(k[\Delta])$ and $\delta_i = b_i^A(k[\Delta])$ for every $0 \leq i \leq h_A(k[\Delta])$. It is known that there exists a "unique" minimal free resolution of $k[\Delta]$ over A. The non-negative integer $h_A(k[\Delta])$ is called the *homological dimension* of $k[\Delta]$ over A.

(1.11) Let $*$ denote the functor $\underline{\mathrm{Hom}}_A(-, A)$ and $\vee$ the Matlis dual $\underline{\mathrm{Hom}}_A(-, E_A(k))$. Here $E_A(k) = k[x_1^{-1}, \cdots, x_v^{-1}]$, the injective hull of k as a module over A. Then the local duality theorem is $\underline{\mathrm{Ext}}_A^{v-i}(k[\Delta], A)^\vee \cong \underline{\mathrm{H}}_{\mathfrak{m}}^i(k[\Delta])$.

(1.12) Suppose that Δ is a Cohen-Macaulay complex over k and that

$$0 \longrightarrow A^{b_{v-d}} \xrightarrow{\eta_{v-d}} A^{b_{v-d-1}} \xrightarrow{\eta_{v-d-1}} \cdots \xrightarrow{\eta_1} A^{b_0} \xrightarrow{\eta_0} k[\Delta] \longrightarrow 0, \tag{4}$$

is a minimal free resolution of $k[\Delta]$ over A with each $b_i = b_i^A(k[\Delta])$. Note that $\underline{\mathrm{Ext}}_A^{v-i}(k[\Delta], A) \neq 0$ only for $i = d$ since $\underline{\mathrm{H}}_{\mathfrak{m}}^i(k[\Delta]) = 0$ if $i \neq d$. Define the *canonical module* $\Omega(k[\Delta])$ of $k[\Delta]$ to be the graded module

$$\Omega(k[\Delta]) = \underline{\mathrm{Ext}}_A^{v-d}(k[\Delta], A)$$

over $k[\Delta]$. Thus, if we apply $*$ to (4), then we obtain the exact sequence of graded modules over A as follows:

$$0 \longrightarrow (A^{b_0})^* \xrightarrow{\eta_1^*} (A^{b_1})^* \xrightarrow{\eta_2^*} \cdots \xrightarrow{\eta_{v-d}^*} (A^{b_{v-d}})^* \longrightarrow \Omega(k[\Delta]) \longrightarrow 0.$$

The Cohen-Macaulay type $\mathrm{type}(k[\Delta]) = b_{v-d}^A(k[\Delta])$ of Δ over k coincides with the minimal number of generators of the canonical module $\Omega(k[\Delta])$ as a module over $k[\Delta]$. On the other hand, if Δ is Gorenstein over k, then $b_{v-d-i}^A(k[\Delta]) = b_i^A(k[\Delta])$ for every $0 \leq i \leq v - d$.

(1.13) Every partially ordered set ("poset" for short) to be studied is finite. A *chain* is a totally ordered set. The *length* of a chain C is $\ell(C) := \sharp(C) - 1$. A totally ordered subset in a poset P is also called a chain of P. The *rank* of a poset

P is defined by $\text{rank}(P) = \max\{\ell(C); C \text{ is a chain of } P\}$. A poset P is called *pure* if every maximal chain has the same length. When $x, y \in P$, we say that y *covers* x if $x < y$ and $x < z < y$ for no $z \in P$. A chain $x_1 < x_2 < \cdots x_s$ of P is called *saturated* if x_{i+1} covers x_i for each $1 \leq i < s$. If $x \leq y$ in P, then the *open interval* (x, y) (resp. *closed interval* $[x, y]$) of P is the induced subposet $\{z \in P; x < z < y)$ (resp. $\{z \in P; x \leq z \leq y\}$) of P. In particular, $(x, x) = \phi$ and $[x, x] = \{x\}$ for every $x \in P$. Given an arbitrary poset P, we write $\Delta(P)$ for the set of chains of P. Then $\Delta(P)$ is a simplicial complex on the vertex set P. We say that $\Delta(P)$ is the *order complex* of P. Note that $\dim \Delta(P) = \text{rank}(P)$, and that $\Delta(P)$ is pure if and only if P is pure. On the other hand, we define the poset $P\wedge$ by $P\wedge = P \cup \{0\wedge, 1\wedge\}$ such that $0\wedge < x < 1\wedge$ for every $x \in P$. If P is pure of rank $d-1$, then there exists a unique function $\rho : P\wedge \to \{0, 1, \cdots, d+1\}$, called the *rank function* of $P\wedge$, such that $\rho(0\wedge) = 0$, $\rho(1\wedge) = d+1$, and $\rho(\beta) = \rho(\alpha) + 1$ if β covers α in $P\wedge$.

(1.14) The *Möbius function* μ_P of a partially ordered set P is the map μ_P : $\{(x, y) \in P \times P; x \leq y\} \to \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers, defined as follows:

(i) $\mu_P(x, x) = 1$ for each $x \in P$, and

(ii) $\mu_P(x, y) = -\sum_{x \leq z < y} \mu_P(x, z)$ for every $x < y$ in P.

One of the most important formula for us on Möbius functions is

$$\mu_{P\wedge}(0\wedge, 1\wedge) = \tilde{\chi}(\Delta(P)). \tag{5}$$

Here $\tilde{\chi}(\Delta(P))$ is the reduced Euler characteristic of the order complex $\Delta(P)$ of P. Moreover, if both $0^\wedge = x_0 < x_1 < \cdots < x_s = x$ and $y = y_0 < y_1 < \cdots < y_t = 1^\wedge$ are saturated chains of $P^\wedge$ with $x < y$ in $P^\wedge$ and $\sigma = \{x_0, \cdots, x_s, y_0, \cdots, y_t\} \in \Delta(P)$, then $\mathrm{link}_{\Delta(P)}(\sigma)$ is just the order complex of the open interval (x, y) of $P^\wedge$. Hence

$$\mu_{P^\wedge}(x, y) = \tilde{\chi}(\mathrm{link}_{\Delta(P)}(\sigma)). \tag{6}$$

(1.15) A *lattice* is a poset L for which every pair of elements α and β has a least upper bound (or "join") denoted by $\alpha \vee \beta$, and a greatest lower bound (or "meet") denoted by $\alpha \wedge \beta$. Thus, in particular, every (finite) lattice has a unique minimal element $0^\wedge$ and a unique maximal element $1^\wedge$. Every closed interval of a lattice is again a lattice. An *atom* of a lattice L is an element which covers $0^\wedge$ in L. A lattice L is called *atomic* if every element is the join of atoms of L. Also, a lattice L is called *complemented* if, for every $x \in L$, there exists $y \in L$ such that $x \wedge y = 0^\wedge$ and $x \vee y = 1^\wedge$. Moreover, a lattice L is called *relatively complemented* if every closed interval of L is complemented. On the other hand, we say that a lattice L is *modular* if, for all elements x, y and z in L with $x \leq z$, we have $x \vee (y \wedge z) = (x \vee y) \wedge z$. A lattice L is called *semimodular* if the following condition is satisfied: If $x, y \in L$ both cover $x \wedge y$, then $x \vee y$ covers both x and y. Every modular lattice is semimodular. A semimodular lattice is atomic if and only if it is relatively complemented. A *geometric lattice* is a lattice which is both relatively complemented and semimodular. If L is a geometric lattice, then $\mu_L(0^\wedge, 1^\wedge) \neq 0$. Moreover, a modular lattice L is geometric if and only if $\mu_L(0^\wedge, 1^\wedge) \neq 0$. A *boolean* *lattice* $\mathcal{B}_n$, $0 < n \in \mathbb{Z}$, is a lattice which consists of all subset of $\{1, 2, \cdots, n\}$, "ordered by inclusion." Every boolean lattice is geometric.

(1.16) We say that a poset P is *Cohen-Macaulay* (resp. *Gorenstein*, *Buchsbaum*, *doubly Cohen-Macaulay*) over a field k if the order complex $\Delta(P)$ of P is Cohen-Macaulay (resp. Gorenstein, Buchsbaum, doubly Cohen-Macaulay) over k. For example, if $L = P^\wedge$ is a semimodular lattice, then P is Cohen-Macaulay (see, e.g., [2], [5] and [6]). If $L = P^\wedge$ is a boolean lattice, then P is Gorenstein. Moreover, if $L = P^\wedge$ is a geometric lattice, then P is doubly Cohen-Macaulay [3].

§2. Betti number sequences

First, recall from (1.7) that if a simplicial complex Δ of dimension $d-1$ is Cohen-Macaulay over a field k, then $\beta_i = \beta_i(\Delta; k)$ vanishes for every $0 \le i < d-1$ since $\Delta = \mathrm{link}_\Delta(\phi)$. Thus, in particular, the Betti number sequence of Δ over k is $\beta(\Delta; k) = (0, 0, \cdots, 0, (-1)^{d-1}\tilde{\chi}(\Delta))$. We now ask what can be said about the Betti number sequences of Buchsbaum complexes?

(2.1) Theorem ([7]). *Given a finite sequence $(\beta_0, \beta_1, \cdots, \beta_{d-1}) \in \mathbb{Z}^d$ with each $\beta_i \ge 0$, there exists a simplicial complex Δ of dimension $d-1$ such that, for an arbitrary field k, the simplicial complex Δ is Buchsbaum over k with $\beta(\Delta, k) = (\beta_0, \beta_1, \cdots, \beta_{d-1})$.*

Let Δ be a simplicial complex on the vertex set $V = \{x_1, x_2, \cdots, x_v\}$ of dimension $d-1$, $A = k[x_1, x_2, \cdots, x_v]$ the polynomial ring in v-variables over a field k and

$k[\Delta] = A/I_\Delta$. By (1.5), if Δ is Buchsbaum (but not Cohen-Macaulay) over k with $\beta(\Delta, k) = (\beta_0, \beta_1, \cdots, \beta_{d-1})$, then $\beta_i = 0$ for every $0 \le i \le t-2$ and $\beta_{t-1} \ne 0$. Here $t = t_A([k[\Delta])$ is the depth of $k[\Delta]$ over A. The following result says that each $b_i^A(k[\Delta])$, $v - d < i \le h$ $(= v - t)$, is a linear combination of β_j's with non-negative integer coefficients, where $h = h_A(k[\Delta])$ is the homological dimension of $K[\Delta]$ over A.

(2.2) Theorem ([15]). *We inherit the notation as above. Suppose that Δ is Buchsbaum, but not Cohen-Macaulay. Then, for each integer i with $v-d < i \le h$ $(= v-t)$,*

$$b_i^A(k[\Delta]) = \sum_{j=0}^{v-t-i} \binom{v}{j} \beta_{v-i-1-j}.$$

The i-th *skeleton* $\Delta^{(i)}$, $1 \le i \le d$, of a simplicial complex Δ is defined to be the subcomplex $\{\sigma \in \Delta; \sharp(\sigma) \le i\}$ of Δ. Thus, $\Delta^{(i)}$ is a simplicial complex of dimension $i-1$. If Δ is Cohen-Macaulay(resp. Buchsbaum), then $\Delta^{(i)}$ is Cohen-Macaulay (resp. Buchsbaum) for every $1 \le i \le d$. Moreover, if Δ is Cohen-Macaulay, then $\Delta^{(i)}$ is doubly Cohen-Macaulay for every $1 \le i < d$, see [10, Proposition (2.1)]. It is known, e.g., [11, Corollary (2.6)] that $h_A(k[\Delta^{(i)}]) = h_A(k[\Delta])$, i.e., $t_A(k[\Delta^{(i)}]) = t_A(k[\Delta])$ for every $t \le i \le d$. Thus, in particular, $\Delta^{(t)}$ is Cohen-Macaulay. Let $f_i(\Delta)$ denote the number of faces σ of Δ with $\sharp(\sigma) = i+1$. Since ${\Delta^{(t)}}_{V-W} = {\Delta_{V-W}}^{(t)}$, by the definition of reduced homology groups, if $W \ne \phi$, then $\tilde{H}_{t-\sharp(W)-1}(\Delta_{V-W}; k) = \tilde{H}_{t-\sharp(W)-1}({\Delta^{(t)}}_{V-W}; k)$. It follows from the well-known technique on the long exact sequence of local cohomology modules, e.g., in [11], [13] and [15] that

$$\begin{aligned}
&\dim_k(\tilde{H}_{i-1}(\Delta^{(i)};k)) \\
&\quad = \dim_k(\tilde{H}_{i-1}(\Delta^{(i+1)};k)) + f_i(\Delta) - \dim_k(\tilde{H}_i(\Delta^{(i+1)};k)) \\
&\quad = \dim_k(\tilde{H}_{i-1}(\Delta;k)) + f_i(\Delta) - \dim_k(\tilde{H}_i(\Delta^{(i+1)};k))
\end{aligned}$$

for every $1 \le i \le d$. Hence

$$\begin{aligned}
&\dim_k(\tilde{H}_{t-1}(\Delta^{(t)};k)) \\
&\quad = \dim_k(\tilde{H}_{t-1}(\Delta;k)) + f_t(\Delta) - \dim_k(\tilde{H}_t(\Delta^{(t+1)};k)) \\
&\quad = \sum_{0\le i\le d-t} (-1)^i \beta_{t-1+i}(\Delta;k) + \sum_{0\le i\le d-t-1} (-1)^i f_{t+i}(\Delta).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathrm{type}(k[\Delta^{(t)}])(= b_h^A(k[\Delta^{(t)}])) \\
&\quad = b_h^A(k[\Delta]) + (-1)^{t-1}\left\{ \sum_{t\le i\le d-1} (-1)^i \left(\beta_i(\Delta;k) - f_i(\Delta)\right)\right\}.
\end{aligned}$$

Moreover, by the Euler-Poincaré formula

$$\sum_{t-1\le i\le d-1} (-1)^i \beta_i(\Delta;k)) = \sum_{-1\le i\le d-1} (-1)^i f_i(\Delta)$$

(since $\beta_i(\Delta;k) = 0$ for every $i < t-1$), we obtain

$$\begin{aligned}
&\mathrm{type}(k[\Delta^{(t)}]) \\
&\quad = (-1)^{t-1} \sum_{-1\le i\le t-1} (-1)^i f_i(\Delta) + b_h^A(k[\Delta]) - \beta_{t-1}(\Delta;k) \\
&\quad = (-1)^{t-1}\tilde{\chi}(\Delta^{(t)}) + (b_h^A(k[\Delta]) - \beta_{t-1}(\Delta;k)).
\end{aligned}$$

(2.3) **Corollary.** *Let Δ be a Buchsbaum complex which is not Cohen-Macaulay. Then the t-th skeleton $\Delta^{(t)}$ of Δ, $t = t_A(k[\Delta])$, is doubly Cohen-Macaulay.*

§3. Cohen-Macaulay types

It would, of course, be of great interest to find an effective (combinatorial) formula to compute Cohen-Macaulay types of Cohen-Macaulay complexes. We here give a brief sketch of [13]. Every result, except Theorem (3.2), appears in [13].

Definition. (a) A face σ of a Cohen-Macaulay complex Δ is called *fundamental* if (i) $\tilde{\chi}(\mathrm{link}_\Delta(\sigma)) \neq 0$ and (ii) $\tilde{\chi}(\mathrm{link}_\Delta(\tau)) = 0$ for every face τ of Δ with $\tau \subset \sigma$ and $\tau \neq \sigma$.

(b) We say that a Cohen-Macaulay complex Δ is *superior* if $\mathrm{link}_\Delta(\sigma)$ is doubly Cohen-Macaulay for every fundamental face σ of Δ.

We write $\mathcal{F}(\Delta)$ for the set of fundamental faces of Δ.

Remark. (a) Let Δ be a Cohen-Macaulay complex. Then, the empty face ϕ is a fundamental face of Δ if and only if $\tilde{\chi}(\Delta) \neq 0$. Moreover, since $\tilde{\chi}(\phi) = -1$, the existence of a fundamental face of Δ is guaranteed.

(b) Suppose that a Cohen-Macaulay complex Δ is superior. Then, for every face τ of Δ with $\tilde{\chi}(\mathrm{link}_\Delta(\tau)) \neq 0$, the subcomplex $\mathrm{link}_\Delta(\tau)$ of Δ is doubly Cohen-Macaulay. Moreover, the Cohen-Macaulay complex $\mathrm{link}_\Delta(\sigma)$ is again superior for every face σ of Δ.

(c) Every Gorenstein complex is superior.

(3.1) Proposition. *Suppose that a simplicial complex Δ is Cohen-Macaulay. Then,* *we have the lower bound inequality*

$$\mathrm{type}(k[\Delta]) \geq \sum_{\sigma\in\mathcal{F}(\Delta)} (-1)^{d-1-\sharp(\sigma)}\tilde{\chi}(\mathrm{link}_{\Delta}(\sigma)) \tag{7}$$

for the Cohen-Macaulay type type$(k[\Delta])$ *of* Δ.

Sketch of Proof. Let a simplicial complex Δ on the vertex set V be Cohen-Macaulay of dimension $d-1$. Then, it follows from Eq.(2) that

$$\mathrm{type}(k[\Delta]) = \sum_{W\subset V} \dim_k(\tilde{H}_{d-\sharp(W)-1}(\Delta_{V-W};k)).$$

The required inequality (7) follows from (a) $\tilde{H}_{d-\sharp(W)-1}(\Delta_{V-W};k) = 0$ unless W is a face of Δ [13, Lemma (3.7)], (b) $\tilde{H}_{d-\sharp(\tau)-1}(\Delta_{V-\tau};k)$, $\tau\in\Delta$, vanishes unless $\tau\supset\sigma$ for some $\sigma\in\mathcal{F}(\Delta)$ [13, Corollary (3.5)], and (c) if $\sigma\in\mathcal{F}(\Delta)$ then $\dim_k(\tilde{H}_{d-\sharp(\sigma)-1}(\Delta_{V-\sigma};k)) = (-1)^{d-1-\sharp(\sigma)}\tilde{\chi}(\mathrm{link}_{\Delta}(\sigma))$ [13, Lemma (3.4)]. Q.E.D.

In order that the equality holds in the above inequality (7), what condition is necessary and sufficient?

(3.2) Theorem. *Let* Δ *be a simplicial complex on* $V=\{x_1,x_2,\cdots,x_v\}$, $\dim\Delta = d-1$, $A=k[x_1,x_2,\cdots,x_v]$, $k[\Delta]=A/I_{\Delta}$, *and* $h_A(k[\Delta])$ *the homological dimension of* $k[\Delta]$ *over* A. *If* Δ *is Cohen-Macaulay, then the following conditions are equivalent:*

(i) type$(k[\Delta])$ *coincides with the right-hand side of the inequality (7);*

(ii) $\tilde{H}_{d-\sharp(\tau)-1}(\Delta_{V-\tau};k)=0$ *for every face* τ *of* Δ *with* $\tau\notin\mathcal{F}(\Delta)$;

(iii) $\tilde{H}_{d-\sharp(\tau)-1}(\Delta_{V-\tau}; k) = 0$ *if* τ *is face of* Δ *such that* $\sigma \subset \tau$ *and* $\sigma \neq \tau$ *for some* $\sigma \in \mathcal{F}(\Delta)$;

(iv) $h_A(k[\Delta_{V-\sigma}]) = h_A(k[\Delta_{V-\tau}])$ *for all faces* σ *and* τ *of* Δ *with* $\sigma \in \mathcal{F}(\Delta)$, $\sigma \subset \tau$ *and* $\sharp(\tau) = \sharp(\sigma) + 1$;

(v) $h_A(k[\Delta_{V-\tau}]) < v - (d - \sharp(\tau))$ *if* τ *is a face of* Δ *such that* $\sigma \subset \tau$ *and* $\sigma \neq \tau$ *for some* $\sigma \in \mathcal{F}(\Delta)$;

(vi) *the* $(d - \sharp(\sigma))$*-th skeleton* ${\Delta_{V-\sigma}}^{(d-\sharp(\sigma))}$ *of* $\Delta_{V-\sigma}$ *is doubly Cohen-Macaulay for every face* $\sigma \in \mathcal{F}(\Delta)$.

Proof. If $\tilde{H}_{d-\sharp(\tau)-1}(\Delta_{V-\tau}; k) \neq 0$, $\tau \in \Delta$, then $h_A(k[\Delta_{V-\tau}]) = v - (d - \sharp(\tau))$, i.e., $t_A(k[\Delta_{V-\tau}]) = d - \sharp(\tau)$. Here $t_A(k[\Delta_{V-\tau}])$ is the depth of $k[\Delta_{V-\tau}]$. In fact, thanks to Eq. (2), $t_A(k[\Delta_{V-\tau}]) \leq d - \sharp(\tau)$, while $t_A(k[\Delta_{V-\tau}]) \geq d - \sharp(\tau)$ by [13, Lemma (3.1)]. Hence, (v)$\Rightarrow$(iii). Moreover, since $h_A(k[\Delta_{V-\sigma}]) = v - (d - \sharp(\sigma))$ if $\sigma \in \mathcal{F}(\Delta)$, again by [13, Lemma (3.1)], we see (iv)$\Leftrightarrow$(v). Let $\mathcal{G}(\Delta)$ denote the set of those faces τ of Δ such that $\sigma \subset \tau$ and $\sigma \neq \tau$ for some $\sigma \in \mathcal{F}(\Delta)$. Since

$$b^A_{v-(d-\sharp(\tau))}(k[\Delta_{V-\tau}]) = \sum_{\tau \subset \tau' \in \Delta} \dim_k(\tilde{H}_{d-\sharp(\tau')-1}(\Delta_{V-\tau'}; k)),$$

by Eq. (2) and [13, Lemma (3.7)], if $\tilde{H}_{d-\sharp(\tau)-1}(\Delta_{V-\tau}; k) = 0$ for every face $\tau \in \mathcal{G}(\Delta)$, then $b^A_{v-(d-\sharp(\tau))}(k[\Delta_{V-\tau}]) = 0$, i.e., $h_A(k[\Delta_{V-\tau}]) < v - (d - \sharp(\tau))$, for every face $\tau \in \mathcal{G}(\Delta)$. Thus, (iii)$\Rightarrow$(v), while (ii)$\Leftrightarrow$(iii) by [13, Corollary (3.5)]. On the other hand, (iv)$\Leftrightarrow$(vi) follows from, e.g., [11, Corollary (2.6)], see also the proof of

[13, Lemma (3.7)]. Moreover, we see (i)$\Leftrightarrow$(iii) from the above Sketch of Proof of Proposition (3.1). Q.E.D.

It is desirable to find a combinatorial (sufficient) condition for a Cohen-Macaulay complex to satisfy one of (ii), (iii), (iv), (v) and (iv) of Theorem (3.2). The following Lemma (3.3), which plays an important role in our research on the computation of Cohen-Macaulay types, is essentially [13, Lemma (3.6)].

(3.3) Lemma. *If a Cohen-Macaulay complex $\triangle$ is superior, then $\tilde{H}_{d-\sharp(\tau)-1}(\triangle_{V-\tau};k) = 0$ for every face τ of $\triangle$ with $\tau \notin \mathcal{F}(\triangle)$.*

(3.4) Corollary. *Suppose that a Cohen-Macaulay complex $\triangle$ is superior. Then the Cohen-Macaulay type* type$(k[\triangle])$ *of $\triangle$ is equal to the right-hand side of the inequality* (7).

The above Corollary (3.4) is a powerful tool for the explicit computation of Cohen-Macaulay types of certain Cohen-Macaulay partially ordered sets. Let P be a Cohen-Macaulay poset. If $C : 0^\wedge = x_0 < x_1 < \cdots < x_s < x_{s+1} = 1^\wedge$ is a chain of $P^\wedge$, then we set

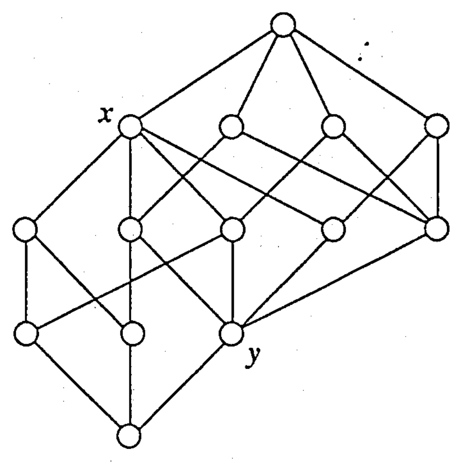
$$\mu(C) = \mu(x_0, x_1)\mu(x_1, x_2)\cdots\mu(x_s, x_{s+1}).$$

Here $\mu = \mu_{P^\wedge}$ is the Möbius function of $P^\wedge$. Thus, by Eq. (6), $|\mu(C)| = |\tilde{\chi}(\mathrm{link}_{\triangle(P)}(\sigma))|$, where $\sigma = \{x_1, x_2, \cdots, x_s\} \in \triangle(P)$. A chain $C : 0^\wedge = x_0 < x_1 < \cdots < x_s < x_{s+1} = 1^\wedge$ of $P^\wedge$ is called *fundamental* if the face $\{x_1, \cdots, x_s\}$ of $\triangle(P)$ is fundamental. Let $\mathcal{F}(P^\wedge)$ denote the set of fundamental chains of $P^\wedge$. We say that a Cohen-Macaulay poset P is *superior* if the Cohen- Macaulay complex $\triangle(P)$ is superior. For example, if $L = P^\wedge$ is a modular lattice, then the Cohen-Macaulay poset P is superior, since every closed interval $[x, y]$ of $L = P^\wedge$ with $\mu_{P^\wedge}(0^\wedge, 1^\wedge) \neq 0$ is geometric, see (1.15).

(3.5) Corollary. *If a Cohen-Macaulay poset P is superior, then* $\text{type}(k[\Delta(P)]) = \sum_{C \in \mathcal{F}(P^\wedge)} |\mu(C)|$.

(3.6) Corollary. *We have the equality* $\text{type}(k[\Delta(P)]) = \sum_{C \in \mathcal{F}(P^\wedge)} |\mu(C)|$ *if $L = P^\wedge$ is a modular lattice.*

(3.7) Example. Let $L = P^\wedge$ be the semimodular lattice drawn below. Then $\mathcal{F}(\Delta(P)) = \{\{x\}, \{y\}\}$. We easily see that the Cohen-Macaulay complex $\Delta(P)$ satisfies the condition (iv) of Theorem (3.2). Hence $\text{type}(k[\Delta(P)]) = \sum_{C \in \mathcal{F}(P^\wedge)} |\mu(C)| = 3$. Note that the Cohen-Macaulay poset P is not superior since the open interval $(0^\wedge, x)$ is not doubly Cohen-Macaulay.



§4. Canonical modules

The topic of this section is the canonical module of the Stanley-Reisner ring of the order complex of a Cohen-Macaulay partially ordered sets. Let P be a Cohen-Macaulay poset and, since P is pure, let ρ denote the rank function of $P^\wedge$. Let us first recall that the set of those monomials $\prod_{\alpha \in P} \alpha^{e(\alpha)}$, each $0 \leq e(\alpha) \in \mathbb{Z}$, such that $\{\alpha; e(\alpha) > 0\} \in \Delta(P)$ is a basis of $k[\Delta(P)]$ as a vector space over k. Hence, if $\alpha < \beta$ in $P^\wedge$ with $\rho(\beta) - \rho(\alpha) = r + 1$, then the $(r-1)$-th reduced homology group $\tilde{H}_{r-1}(\Delta((\alpha, \beta)); k)$ of the order complex $\Delta((\alpha, \beta))$ of the open interval (α, β) of $P^\wedge$ can be imbedded in $k[\Delta(P)]$. Given a chain $C : 0^\wedge = x_0 < x_1 < \cdots < x_s < x_{s+1} = 1^\wedge$ of $P^\wedge$ with each $\rho(x_{i+1}) - \rho(x_i) = r(i) + 1$, we write $\mathcal{R}(C)$ for the subspace of $k[\Delta(P)]$ spanned by those polynomials $f_0 {x_1}^2 f_1 {x_2}^2 \cdots f_{s-1} {x_s}^2 fs$ such that $f_i \in \tilde{H}_{r(i)-1}(\Delta((x_i, x_{i+1})); k)$ for every $0 \leq i \leq s$, where each monomial of f_i is of the form $\alpha_1 \alpha_2 \cdots \alpha_{r(i)}$ with $x_i < \alpha_1 < \cdots < \alpha_{r(i)} < x_{i+1}$. Thus, in particular, $\dim_k(\mathcal{R}(C)) = |\mu(C)|$. Let $\mathcal{F}(P^\wedge)$ denote the set of fundamental chains of $P^\wedge$ and define $\mathcal{I}^*(k[\Delta(P)])$ to be the ideal of $k[\Delta(P)]$ generated by all $\mathcal{R}(C)$ with $C \in \mathcal{F}(P^\wedge)$.

(4.1) Theorem ([14]). *Suppose that P is a Cohen-Macaulay partially ordered set. Then, the ideal $\mathcal{I}^*(k[\Delta(P)])$ is isomorphic to the canonical module $\Omega(k[\Delta(P)])$ of $k[\Delta(P)]$, as graded modules over $k[\Delta(P)]$ up to shift in grading, if and only if the Cohen-Macaulay type of $k[\Delta(P)]$ is equal to $\sum_{C \in \mathcal{F}(P^\wedge)} |\mu(C)|$.*

Remark. If $\mu(0^\wedge, 1^\wedge) \neq 0$, then the above Theorem (4.1) essentially coincides with Baclawski [4, Theorem 2].

Since, by Corollary (3.6), type$(k[\Delta(P)]) = \sum_{C \in \mathcal{F}(P^\wedge)} |\mu(C)|$ if $L = P^\wedge$ is a modular lattice, we immediately obtain

(4.2) Corollary ([14]). *If $L = P^\wedge$ is a modular lattice, then $\mathcal{I}^*(k[\Delta(P)])$ is iso-*

morphic to the canonical module $\Omega(k[\Delta(P)])$ *of* $k[\Delta(P)]$.

§5. Open questions

We conclude with some open questions, which is unlikely to be terribly difficult.

(5.1) Let $p > 0$ be a prime number, $\lambda = (\lambda, \lambda_2, \cdots, \lambda_\ell) \in \mathbb{Z}^\ell$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$, and $G_\lambda(p)$ the finite abelian p-group of type λ, i.e., $(\mathbb{Z}/p^{\lambda_1}\mathbb{Z}) \times (\mathbb{Z}/p^{\lambda_2}\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p^{\lambda_\ell}\mathbb{Z})$. Let $\mathcal{L}(G_\lambda(p))$ denote the lattice of subgroups of $G_\lambda(p)$, ordered by inclusion. It is known, and easy to prove, that $\mathcal{L}(G_\lambda(p))$ is a modular lattice. Let μ be the Möbius function of $\mathcal{L}(G_\lambda(p))$. If $H \subset N$ are subgroups of $G_\lambda(p)$, then $\mu(H, N) \neq 0$ if and only if the quotient group N/H is of type $(1, 1, \cdots, 1)$. Moreover, if N/H is of type $(1, \cdots, 1) \in \mathbb{Z}^n$, then $\mu(H, N) = (-1)^n p^{\binom{n}{2}}$. See, e.g., [23, pp. 126-127]. Hence, if a chain $C : 0\hat{} = H_0 < H_1 < \cdots < H_s < H_{s+1} = 1\hat{}$ of $\mathcal{L}(G_\lambda(p))$ is fundamental, then $|\mu(C)|$ is a power of p. Given a sequence $\delta = (\delta_1, \delta_2, \cdots, \delta_{s+1}) \in \mathbb{Z}^{s+1}$ with each $\delta_i \geq 0$ such that $\sum_{1 \leq i \leq s+1} \delta_i = \sum_{1 \leq i \leq \ell} \lambda_i$, define $f_\delta(p)$ to be the number of those fundamental chains $0\hat{} = H_0 < H_1 < \cdots < H_s < H_{s+1} = 1\hat{}$ of $\mathcal{L}(G_\lambda(p))$ such that H_i/H_{i-1} is of type $(1, \cdots, 1) \in \mathbb{Z}^{\delta_i}$ for every $1 \leq i \leq s+1$.

Question. (a) Is $f_\delta(p)$ a polynomial in p with non-negative integer coefficients?

(b) (follows from (a)) Let P be the Cohen-Macaulay poset with $P\hat{} = \mathcal{L}(G_\lambda(p))$. Is type$(k[\Delta(P)])$ a polynomial in p with non-negative integer coefficients?

(5.2) Suppose that $L = P^\wedge$ is a semimodular lattice and let $\mu = \mu_{P^\wedge}$ be the Möbius function of $P^\wedge$. Then, the Cohen-Macaulay poset P is superior if and only if the following condition is satisfied: if $0^\wedge \leq x \leq z \leq w \leq y \leq 1^\wedge$ in $P^\wedge$ and $\mu(x,y) \neq 0$, than $\mu(z,w) \neq 0$. See [13, Example (4.13)]. On the other hand, there exists a semimodular lattice $L = P^\wedge$ such that the Cohen-Macaulay poset P is not superior, but $\mathrm{type}(k[\triangle(P)]) = \sum_{C \in \mathcal{F}(P^\wedge)} |\mu(C)|$ (cf. Example (3.7)).

Question. (a) Find a combinatorial formula to compute $\mathrm{type}(k[\triangle(P)])$ for a semimodular lattice $L = P^\wedge$.

(b) Classify the semimodular lattices $L = P^\wedge$ for which the Cohen-Macaulay type of $\triangle(P)$ is equal to $\sum_{C \in \mathcal{F}(P^\wedge)} |\mu(C)|$.

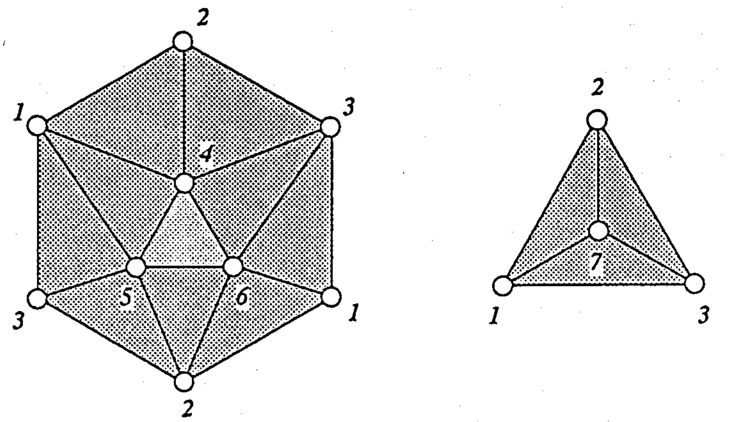
(5.3) Recall that a boolean lattice $\mathcal{B}_n$ is a lattice which consists of all subsets of $\{1, 2, \cdots, n\}$, ordered by inclusion. Let P denote the Cohen-Macaulay poset with $\mathcal{B}_n = P^\wedge$. Since $\sharp(P) = 2^n - 2$ and P is Gorenstein, $h_A(k[\triangle(P)]) = 2^n - n - 1$ and $b^A_{2^n-n-1}(k[\triangle(P)]) = \mathrm{type}(k[\triangle(P)]) = 1$. Moreover, $b^A_{(2^n-n-1)-i}(k[\triangle(P)]) = b^A_i(k[\triangle(P)])$ for every $0 \leq i \leq 2^n - n - 1$. Let $b_i = b_i(n) = b^A_i(k[\triangle(P)])$ and $\ell = 2^n - n - 1$.

Question. (a) Find an explicit formula for $b_i(n)$.

(b) Is the sequence $(b_0, b_1, \cdots, b_\ell)$ unimodal, i.e., $b_0 \leq b_1 \leq \cdots \leq b_{[\ell/2]}$?

(5.4) We inherit the notation in, e.g., (1.6) and (1.10). It follows immediately from (2) that $b^A_0(k[\triangle]) = 1$ and $b^A_1(k[\triangle])$ is the minimal number of generators of

I_Δ, i.e., the number of minimal subsets W of V with $W \notin \Delta$. Thus, in particular, $b_1^A(k[\Delta])$ is independent of the field characteristic of k. On the other hand, let Δ denote the simplicial complex drawn below with $v = 7$ and $d = 3$. Then Δ is Cohen-Macaulay over every field k with $b_0^A(k[\Delta]) = 1$, $b_1^A(k[\Delta]) = 13$ and $b_2^A(k[\Delta]) = 27$. If $\mathrm{char}(k) \neq 2$, then $b_3^A(k[\Delta]) = 22$, $b_4^A(k[\Delta]) = 7$, however, if $\mathrm{char}(k) = 2$, then $b_3^A(k[\Delta]) = 23$, $b_4^A(k[\Delta]) = 8$.



Question. (a) Suppose that Δ is a simplicial complex with $v - d \leq 2$. Then is $\tilde{H}_i(\Delta; k)$ independent of $\mathrm{char}(k)$?

(b) (follows from (a)) Let Δ be an arbitrary simplicial complex. Then is $b_2^A(k[\Delta])$ independent of $\mathrm{char}(k)$?

(5.5) Again, let G be a finite group and $\mathcal{L}(G)$ its lattice of subgroups. Then, (a) $\mathcal{L}(G)$ is Cohen-Macaulay if and only if G is supersolvable ([5]); (b) $\mathcal{L}(G)$ is Gorenstein if and only if G is a cyclic group whose order is either square-free or prime power ([9]); (c) $\mathcal{L}(G)$ is doubly Cohen-Macaulay if and only if G is supersolvable with $\mu_{\mathcal{L}(G)}(0\hat{}, 1\hat{}) \neq 0$ ([24]).

Question. (a) Classify the supersolvable groups G for which the Cohen-Macaulay poset $\mathcal{L}(G)$ is superior.

(b) What can be said about the subgroup lattices $\mathcal{L}(G)$ of supersolvable groups G with $\text{type}(k[\Delta(\mathcal{L}(G))]) = 2$?

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