# OVERDETERMINED SYSTEMS OF FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS WITH SINGULAR SOLUTION

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# OVERDETERMINED SYSTEMS OF FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS WITH SINGULAR SOLUTION

SHYUICHI IZUMIYA AND BING LI

ABSTRACT. We study overdetermined systems of first order partial differential equations with singular solutions. The main result gives a characterization of such systems and asserts that the singular solution is equal to the contact singular set. We can also give a local normal form of such systems up to contact diffeomorphism.

## 0. INTRODUCTION

In this paper we study overdetermined systems of first order partial differential equations with singular solution. The first example of singular solutions was discovered by Brook Tayler about 290 years ago (cf. [9]). 20 years after that Alex Claude Clairaut studied a class of equations which have singular solutions [4]. This equation is called the Clairaut equation now: $y = x \cdot \frac{dy}{dx} + f(\frac{dy}{dx})$. It has a quite beautiful geometric structure as follows : There exists a "general solution" that consists of lines ; $y = t \cdot x + f(t)$, where $t$ is a parameter and the singular solution is the envelope of such a family. However, there are some examples that the situation is not necessary like as the singular solution of the Clairaut equation (cf., Examples 3.2, 3.3). In classical treatises of equations (Cartan [2], Carathéodory [3] , Courant-Hilbert [5], Forsyth [6] [7], Ince [7], Petrovski [16]) the discussions of equations with singular solutions are informal. In these, a "general solution" of a differential equation is defined to be an one-parameter family of solutions and a "singular solution" is a solution which is not contained in the "general solution". This definition of singular solutions is very confused as usual in classical literature on differential equations. Even in modern articles ([10,13]), they studied under special assumptions. In this paper we try to give a rigorous definition of singular solutions. The vagueness of the definition of singular solutions of equations is caused by that of general solutions, so that we adopt the notion of complete solutions instead of "general solutions".

The main results are Theorems 1.1 and 1.4. We establish the notion of *overdetermined systems of first order differential equations with singular solution* in Theorem 1.1. This result is a generalization of the similar result on single equations in [11]. Furthermore we

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give a local normal form of such a class of differential equations up to contact diffeomorphism in Theorem 1.4. In Section 2 we give the proofs of results. Some examples in Section 3 are illustrating the results.

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All maps considered here are differentiable of class $C^\infty$ unless stated otherwise.

## 1. Main results

A first order differential equation is most naturally interpreted as being a closed subset of the 1-jet space of functions of $n$-variables $J^1(\mathbb{R}^n,\mathbb{R})$. Unless the contrary is specifically stated, we use the following definition. *A first order differential equations* (or briefly, *an equation*) is a relation $F=0$, where $F:(J^1(\mathbb{R}^n,\mathbb{R}),z_0)\to(\mathbb{R}^d,0)$ is a submersion germ and $1\le d\le n$. If $1<d$, then $F=0$ is said to be *an overdetermined system.* Let $\theta$ be the canonical contact form on $J^1(\mathbb{R}^n,\mathbb{R})$ which is given by $\theta=dy-\sum_{i=1}^n p_i dx_i$, where $(x,y,p)$ are canonical coordinates of $J^1(\mathbb{R}^n,\mathbb{R})$. Throughout the remainder of this paper, we shall consider $J^1(\mathbb{R}^n,\mathbb{R})$ as a contact manifold whose contact structure is given by the canonical 1-form. We also have the canonical projection $\pi:J^1(\mathbb{R}^n,\mathbb{R})\to\mathbb{R}^n\times\mathbb{R}$ by $\pi(x,y,p)=(x,y)$. The notion of a solution of an equation is given by the philosophy of Lie. *A Legendrian solution* (*or, a geometric solution*) *of* $F=0$ is defined to be an immersion $i:(L,q_0)\to(J^1(\mathbb{R}^n,\mathbb{R}),z_0)$ from an $n$-dimensional manifold such that $i^*\theta=0$ and $i(L)\subset F^{-1}(0)$ (i.e. a Legendrian submanifold which is contained in $F^{-1}(0)$). Then the jet extension $j^1f:\mathbb{R}^n\to J^1(\mathbb{R}^n,\mathbb{R})$ is a Legendrian embedding. Hence, in our terminology, the smooth solution of $F=0$ is a smooth function $f$ such that $j^1f(\mathbb{R}^n)\subset F^{-1}(0)$. On the other hand, we can show that a Legendrian solution $i$ is given by (at least locally) a jet extension of a smooth function if and only if $\pi\circ i$ is a non-singular map. Thus we can define the notion of singularities of solutions. We say that $z_0\in L$ is *a Legendrian singular point* if $z_0$ is a singular point of $\pi\circ i$. We also define notions of singularities of an equation $F^{-1}(0)$. We say that $z_0$ is *a contact singular point* if $\theta(T_{z_0}F^{-1}(0))=0$. We denote the set of contact singular points by $\Sigma_c(F)$. We call it *a contact singular set of $F$.* A point $z\in(E^r,z_0)$ is called *a $\pi$-singular point* if $\operatorname{rank} d(\pi|F^{-1}(0))_z<n$. We denote the set of $\pi$-singular points by $\Sigma_\pi(F)$. We say that an equation $F=0$ is *involutory at $z\in F^{-1}(0)$* if there is a Legendrian submanifold $L$ tangent to $F^{-1}(0)$ at $z$. We also say that an equation $F=0$ is *involutory* if it is involutory at any point of $(F^{-1}(0),z_0)$. Since single equations are automatically involutory, the notion of involutory equations is essential for overdetermined systems of first order partial differential equations (i.e. $d\ge 2$) (cf. [14],[15]). An equation $F=0$ is said to be *completely integrable* if there exists a foliation by Legendrian solutions

on $F^{-1}(0)$. In this case such a foliation is called *a complete solution* of $F = 0$. Then we can define the notion of singular solutions. A Legendrian solution $i : (L, q_0) \to (J^1(\mathbb{R}^n, \mathbb{R}), z_0)$ of $F = 0$ is called *a singular solution* (in the strict sense) if it satisfies the following condition :

(∗) There exists a representative $\tilde{i} : U \to F^{-1}(0)$ of $i$ such that $\tilde{i}(V)$ is not contained in a leaf of any complete solutions of $F = 0$ for any open subset $V \subset U$.

Then we have the following theorem.

**Theorem 1.1.** *For an equation $F : (J^1(\mathbb{R}^n, \mathbb{R}), z_0) \to (\mathbb{R}^d, 0)$ and a Legendrian solution $i : (L, q_0) \to (J^1(\mathbb{R}^n, \mathbb{R}), z_0)$ of $F = 0$, the following conditions are equivalent.*

(1) *$i$ is a singular solution of $F = 0$ and $F = 0$ is involutory.*

(2) *There exists a complete solution of $F = 0$ such that each leaves are transverse to $i$.*

(3) Image $i \subseteq \Sigma_c(F)$.

In [12] it has been shown that $F = 0$ is completely integrable if and only if it is involutory and $\Sigma_c(F)$ is empty or an $n$-dimensional submanifold. By definition, if $\Sigma_c(F)$ is an $n$-dimensional submanifold, it is automatically a Legendrian solution of $F = 0$. Then we have the following corollary of Theorem 1.1.

**Corollary 1.2.** *An involutory equation $F : (J^1(\mathbb{R}^n, \mathbb{R}), z_0) \to (\mathbb{R}^d, 0)$ has a singular solution if and only if $\Sigma_c(F)$ is an $n$-dimensional submanifold. Moreover, $\Sigma_c(F)$ is a singular solution of $F = 0$.*

The corresponding result for the case $d = 1$ has been obtained in [11]. We call the equation which satisfies the condition in Corollary 1.2 *an equation with singular solution.*

We remark that we can easily check the above condition by the following proposition.

**Proposition 1.3.** *Let $F = (F_1, \dots, F_d) : (J^1(\mathbb{R}^n, \mathbb{R}), z_0) \to (\mathbb{R}^d, 0)$, be a submersion germ. Then*

1) *$(F^{-1}(0), z_0)$ is involutory if and only if $[F_i, F_j]_z = 0$ $(i, j = 1, \dots, d)$ for any $z \in F^{-1}(0)$, where*

$$\begin{aligned}[F,G] =& F \cdot \frac{\partial G}{\partial y} - G \cdot \frac{\partial F}{\partial y} + \sum_{i=1}^{n} \left(\frac{\partial F}{\partial x_i} \cdot \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \cdot \frac{\partial G}{\partial x_i}\right) \\ &+ \sum_{i=1}^{n} p_i \cdot \left(\frac{\partial F}{\partial y} \cdot \frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial y} \cdot \frac{\partial F}{\partial p_i}\right).\end{aligned}$$

2) *$(F^{-1}(0), z_0)$ is contact singular at $z$ if and only if*

$$rank \begin{pmatrix} \frac{\partial F_j}{\partial x_i} + p_i \frac{\partial F_j}{\partial y} \\ \frac{\partial F_j}{\partial p_i} \end{pmatrix} < d$$

at $z$.

*Proof.* For the proof of 1), see Proposition 1.6.3 in [14].

The proof of 2) is given by the fact that $F^{-1}(0)$ is contact singular at $z$ if and only if there exists $(\lambda_1, \ldots, \lambda_d) \neq (0, \ldots, 0)$ in $\mathbb{R}^d$ such that

$$\theta \wedge (\lambda_1 dF_1 + \cdots + \lambda_d dF_d) = 0 \text{ at } z.$$

We can also give the normal form of such equations up to contact diffeomorphism.

**Theorem 1.4.** *Let $F : (J^1(\mathbb{R}^n, \mathbb{R}), z_0) \to (\mathbb{R}^d, 0)$ be an equation with singular solution. Then there is a contact diffeomorphism germ*

$$f : (J^1(\mathbb{R}^n, \mathbb{R}), z_0) \to (J^1(\mathbb{R}^n, \mathbb{R}), 0)$$

*such that $f(F^{-1}(0)) = \{p_1 = \cdots = p_{d-1} = y = 0\}$.*

The following proposition describes a more detailed relation between the singular solution and each member of the corresponding complete solution.

**Proposition 1.5.** *Let $F : (J^1(\mathbb{R}^n, \mathbb{R}), z_0) \to (\mathbb{R}^d, 0)$ be an equation with singular solution. Let $L$ be a leaf of the complete solution which transverses to the singular solution $\Sigma_c(F)$.*
*1) If $\Sigma_c(F)$ is Legendrian non-singular, then $d\pi(T_{z_0}L) \subset d\pi(T_{z_0}\Sigma_c(F))$ for any $z_0 \in L \cap \Sigma_c(F)$.*
*2) If $L$ is Legendrian non-singular, then $d\pi(T_{z_0}\Sigma_c(F)) \subset d\pi(T_{z_0}L)$ for any $z_0 \in L \cap \Sigma_c(F)$.*

The basic ideas of the proofs of theorems are as follows : If $F = 0$ is involutory and $\Sigma_c(F)$ contains an $n$-dimensional submanifold $L$, we can assert that the equation may be considered as a single equation of the form $y = h(x', p')$, where $(x', p') \in T^*\mathbb{R}^{n-d+1}$. (cf., Proposition 2.1). By the theorem of Kostant-Sternberg [8], a neighbourhood of $L_h$ in $T^*\mathbb{R}^{n-d+1}$ has the same structure (i.e. corresponds to the contact structure on $J^1(\mathbb{R}^{n-d+1}, \mathbb{R})$) as a neighbourhood of the zero section of $T^*L_h$, where $L_h$ is a Lagrangian submanifold of $T^*\mathbb{R}^{n-d+1}$ corresponding to $L$. Then we can choose the local foliation around $L_h$ which corresponds to the fibres of $T^*L_h$ as the required complete integral. This is the main part of the proof of Theorem 1.1. Other parts of the proof depend on the (ordinary) classical method of characteristics. For the proof of Theorem 1.3, we also use the Kostant-Sternberg theorem to construct the contact diffeomorphism germ.

## 2. Proof of theorems

In this section we shall give a proof of Theorems 1.1 and 1.3. First we give the proof that (1) implies (3).

*Proof of Theorem 1.1,* (1) $\Rightarrow$ (3). Let $i : (L, q_0) \to (J^1(\mathbb{R}^n, \mathbb{R}), z_0)$ be a singular solution of $F = 0$. Suppose that there exists a point $q_1 \in L$ such that $i(q_1) = z_1 \notin \Sigma_c(F)$. This means that $F = 0$ is contact regular at $z_1$. By the classification theorem of the geometric theory of first order differential equations (see [14], Corollary 2 of Theorem 2.2.7), we may assume that $F(x, y, p) = (p_1, \ldots, p_d)$. Define a submanifold $E_0$ by $x_1 = \cdots = x_d = p_1 = \cdots = p_d = 0$. It is easy to show that $E_0$ has a contact structure $\theta|E_0$. Since the contact Hamiltonian vector field of $p_j = 0$ for $j = 1, \ldots, d$ is given by $X_{p_j} = -\frac{\partial}{\partial x_j}$ (see [6], Theorem 1.4.3) and $i$ is a solution of $p_1 = \cdots = p_d = 0$, then $-\frac{\partial}{\partial x_j} \in Ti(L)$. It follows that $\ell_0 = i(L) \cap \{x_1 = \cdots = x_d = 0\}$ is an $(n-d)$-dimensional submanifold in $E_0$. Since $i$ is a Legendrian immersion, $\ell_0$ is also a Legendrian submanifold of $E_0$. By Darboux's theorem, we can easily show that there exists a foliation on $E_0$ leaves are Legendrian submanifold of $E_0$ such that $\ell_0$ is a leaf of this foliation. Since $-\frac{\partial}{\partial x_j} \notin TE_0$, the leaves are isotropic submanifolds of $J^1(\mathbb{R}^n, \mathbb{R})$ with the non-characteristic property. Thus we can solve the "parametrized" Cauchy problem by the usual characteristic method (see [14] Proposition 1.5.3). Then we have a local complete solution of $p_1 = \cdots = p_d = 0$ such that $i(L)$ is a leaf of this solution. This contradicts the definition of the singular solution in the strict sense.

We can also prove that (2) implies (1).

*Proof of Theorem 1.1,* (2) $\Rightarrow$ (1). Let $\tilde{i} : U \to F^{-1}(0)$ be a representative of $i$. If there exist an open subset $V \subset U$ and a complete solution such that $\tilde{i}(V)$ is contained in a leaf of such a foliation, then there exists a transversal foliation on $F^{-1}(0)$ around $\tilde{i}(V)$ whose leaves are Legendrian submanifolds. Then $F^{-1}(0)$ is an isotropic submanifold around $\tilde{i}(V)$. This contradicts the fact that dimensions of isotropic submanifolds are at most $n$.

We now prepare for the proof that (3) implies (2).

Let $F : (J^1(\mathbb{R}^n, \mathbb{R}), z_0) \to (\mathbb{R}^d, 0)$ be an equation such that $z_0$ is a contact singular point. The following proposition describes the local normal form of $(F^{-1}(0), z_0)$ up to contact diffeomorphism.

**Proposition 2.1.** ([14],[15]) *Let $F = 0$ be an involutory equation such that $z_0$ is a contact singular point. Then there is a contact diffeomorphism germ*

$$f : (J^1(\mathbb{R}^n, \mathbb{R}), z_0) \to (J^1(\mathbb{R}^n, \mathbb{R}), 0)$$

*such that* $f(F^{-1}(0)) = \{p_1 = \cdots = p_{d-1} = y - h(x', p') = 0\}$, *where* $x' = (x_d, \ldots, x_n)$, $p' = (p_d, \ldots, p_n)$ *and* $h(x', p')$ *is a function germ at* 0.

For our purpose, we may consider the equation of the form in the above proposition. We now define a map germ

$$G_h : (\mathbb{R}^{d-1} \times T^*\mathbb{R}^{n-d+1}, 0) \to (J^1(\mathbb{R}^n, \mathbb{R}), 0)$$

by

$$G_h(x_1, \ldots, x_{d-1}, x', p') = (x_1, \ldots, x_{d-1}, x', h(x', p'), p').$$

We define a 1-form on $T^*\mathbb{R}^{n-d+1}$ by $\theta_h = dh - \sum_{i=d}^{n} p_i dx_i$. Then we have the following one to one correspondence.

$$\{L \mid L \text{ is a solution of } p_1 = \cdots = p_{d-1} = y - h(x', p') = 0\}$$

$$G_h \uparrow\downarrow \Pi_*$$

$$\left\{\mathbb{R}^{d-1} \times L' \mid i : L' \subset T^*\mathbb{R}^{n-d+1} \text{ is a maximal integral submanifold of } \theta_h = 0\right\},$$

where $\Pi(x, y, p) = (x, p')$ and $\Pi_*(L) = \Pi(L)$. It follows that a solution of a equation $p_1 = \cdots = P_{d-1} = y - h(x', p') = 0$ may be regarded as a maximal isotropic submanifold of $(T^*\mathbb{R}^{n-d+1}, \theta_h)$. Since $-d\theta_h = \sum_{i=d}^{n} dp_i \wedge dx_i$ is the canonical symplectic two form, a solution of $p_1 = \cdots = p_{d-1} = y - h(x', p') = 0$ corresponds to a Lagrangian submanifold of $(T^*\mathbb{R}^{n-d+1}, \omega)$, where $\omega = -d\theta_h$. For the definition and properties of Lagrangian submanifolds, see [1]. We now quote the following very important result.

**Theorem 2.2.** (Kostant-Sternberg [8]) *Let* $(P, \omega)$ *be a symplectic manifold,* $L$ *a Lagrangian submanifold and* $\alpha$ *a smooth 1-form on a neighbourhood of* $L$ *in* $P$ *with* $\alpha \mid L = 0$ *and* $d\alpha = \omega$. *Then there exists a tubular neighbourhood* $V$ *of* $L$ *in* $P$, *a neighbourhood* $U$ *of zero section* $L$ *in* $T^*L$ *and a unique "local" vector bundle isomorphism* $K : (V, L) \to (U, L)$ *such that* $K$ *is the identity on* $L$ *and* $K^*\theta_L = \alpha$. *Here,* $\theta_L$ *is the canonical 1-form on* $T^*L$.

Now we can prove that (3) implies (2).

*Proof of Theorem 1.1,* (3) $\Rightarrow$ (2). Since Image $i \subset \Sigma_c(F)$, $F = 0$ is contact singular, so we may assume that

$$F^{-1}(0) = \{p_1 = \cdots p_{d-1} = y - h(x', p') = 0\}.$$

In this case $\pi_{T^*\mathbb{R}^{n-d+1}} \circ G_h^{-1}(i(L)) = L'_h$ is a Lagrangian submanifold of $T^*\mathbb{R}^{n-d+1}$, where

$$\pi_{T^*\mathbb{R}^{n-d+1}} : \mathbb{R}^{d-1} \times T^*\mathbb{R}^{n-d+1} \to T^*\mathbb{R}^{n-d+1}$$

is the canonical projection. It follows that $\theta_h$ vanishes on $L'_h$ so that we may apply the Kostant-Sternberg theorem to conclude that there exist a tubular neighbourhood $V$ of $L'_h$ in $T^*\mathbb{R}^{n-d+1}$ and a unique (local) vector bundle isomorphism $K : V \to (T^*L'_h, \theta_{L'_h})$ such that $K$ is the identity on $L'_h$ and $K^*\theta_{L'_h} = -\theta_h$. Since the fibres of the cotangent bundle $T^*L'_h \to L'_h$ are maximal integral submanifolds of $\theta_{L'_h} = 0$, these fibres make a foliation whose leaves are corresponding to solutions of the original equation.

We can also prove Theorem 1.4.

*Proof of Theorem 1.4.* By the same arguments as those of in the proof of Theorem 1.1, we may also assume that $d = 1$. In this case we have $F^{-1}(0) = \{y - h(x,p) = 0\}$ and $G_h^{-1}(\Sigma_c(F)) = L_h$ is a Lagrangian submanifold of $T^*\mathbb{R}^n$ on which $\theta_h$ vanishes, where $\theta_h = dh - \sum_{i=1}^n p_i dx_i$. The Kostant-Sternberg theorem asserts that there exists a tubular neighbourhood $V$ of $L_h$ in $T^*\mathbb{R}^n$ and a unique (local) vector bundle isomorphism $K$ : $V \to T^*L_h$ such that $K$ is identity on $L_h$ and $K^*\theta_{L_h} = -\theta_h$. We denote local coordinates of $L_h$ as $(x'_1, \ldots, x'_n)$ and the corresponding canonical coordinates of $T^*L'_h$ is denoted by $(x'_1, \ldots, x'_n, p'_1, \ldots, p'_n)$. We definer a diffeomorphism germ

$$\Phi : V \times \mathbb{R} \to T^*L_h \times \mathbb{R}$$

by

$$\Phi(x,p,y) = (K(x,p), y - h(x,p)).$$

On the other hand, we have the canonical contact structure on $T^*L_h \times \mathbb{R}$ given by the contact form $dy' - \sum_{i=1}^n p'_i dx'_i$, where $(x'_1, \ldots, x'_n, p'_1, \ldots, p'_n, y')$ is the canonical coordinate on $T^*L_h \times \mathbb{R}$ induced by the previous arguments. It follows that $\Phi^*(dy' - \sum_{i=1}^n p'_i dx'_i) = dy - dh + \theta_h = dy - \sum_{i=1}^n p_i dx_i$. Since $V \times \mathbb{R}$ may be considered as an open set of $J^1(\mathbb{R}^n, \mathbb{R})$, $\Phi$ is a local contact diffeomorphism. By definition, we have $\Phi(\{y = h(x,p)\}) = \{y' = 0\}$ and $\Phi(L_h) = \{p'_1 = \cdots = p'_n = 0\}$. This completes the proof.

*Proof of Proposition 1.5.* Let $L$ be a leaf of the complete solution which is transverse to $\Sigma_c(F)$ in $F^{-1}(0)$. Firstly, we prove the assertion 1). If $\pi|\Sigma_c(F)$ does not contain $\pi|L$, then they are transversal at a point $z_0$. It follows that there exists a vector $v \in T_{z_0}L$ such that $d\pi(v) \notin d\pi(T_{z_0}\Sigma_c(F))$. Since $\operatorname{rank} d\pi|\Sigma_c(F) = n$, we have $d\pi(\langle v \rangle_{\mathbb{R}} + T_{z_0}\Sigma_c(F)) = T_{\pi(z_0)}\mathbb{R}^n \times \mathbb{R}$. However, we have $\Sigma_c(F) \subset \Sigma_\pi(F)$ so that $\operatorname{rank} d\pi|F^{-1}(0) \leq n$ at $z_0$. This contradicts to the above fact.

The assertion 2) can be proved by the same arguments as the above.

## 3. EXAMPLES

In this section we shall give typical examples of systems of equations with singular solution.

**Examples 3.1.** (The Clairaut system) Consider the following equation around the origin:

$$p_1 = \cdots = p_{d-1} = y - \sum_{i=d}^{n} x_i p_i + f(p_d, \ldots, p_n) = 0,$$

where $f$ is a smooth function. The complete solution is given by $f(t, x) = \sum_{i=d}^{n} x_i t_{i-d+1} + f(t_1, \ldots, t_{n-d+1})$. Each of them is Legendrian non-singular. The contact singular set is given by

$$\{\frac{\partial f}{\partial p_d} - p_d = \cdots = \frac{\partial f}{\partial p_n} - p_n = p_1 = \cdots = p_{d-1} = y - \sum_{i=d}^{n} x_i p_i + f(p_d, \ldots, p_n) = 0\}.$$

It is easy to show that $\Sigma_c(F) = \Sigma_\pi(F)$ and $d\pi(T_{z_0}\Sigma_c(F)) \subset d\pi(T_{z_0}L)$ for any $z_0 \in L \cap \Sigma_c(F)$. Here, $L$ is a leaf of the complete solution.

**Example 3.2.** Consider the following equation around the origin:

$$p_1 = \cdots = p_{d-1} = y - f(x_1, \ldots, x_n) = 0.$$

The complete solution is given by

$$L_t = \{(u_1, \ldots, u_{d-1}, t, f(t), 0, u_d, \ldots, u_n) | (t, u) \in \mathbb{R}^{n-d+1} \times \mathbb{R}^n\},$$

where $t = (t_1, \ldots, t_{n-d+1})$ is the parameter.

The contact singular set is given by

$$\{p_1 = \cdots p_{d-1} = y - f(x_d, \ldots, x_n) = \frac{\partial f}{\partial x_d} - p_d = \cdots = \frac{\partial f}{\partial x_n} - p_n = 0\}.$$

It is easy to show that $\pi(L_t) \subset \pi(\Sigma_c(F))$.

**Example 3.3.** Consider the following equation around the origin:

$$p_1 = y - 2p_2^3 = 0 \quad (n = 2).$$

We have a complete solution $s : (\mathbb{R} \times \mathbb{R}^2, 0) \to J^1(\mathbb{R}^2, \mathbb{R})$ defined by $s(t, u, v) = (3u^2 + t, v, 2u^3, 0, u)$, where $t$ is the parameter. In this case the contact singular set is $\{p_1 = p_2 = y = 0\}$.

**Example 3.4.** Consider the following equation around the origin:

$$y - xp^2 - p^3 = 0 \qquad (n = 1).$$

We can calculate that $\Sigma_\pi(F) = \{(x, y, p) | y = x \cdot p^2 + p^3 \text{ and } p \cdot (2x + 3p) = 0\}$ and $\Sigma_c(F) = \{(x, y, p) | y = p = 0\}$. Then we have $\Sigma_c(F) \subsetneqq \Sigma_\pi(F)$.

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