| | |
|---|---|
| Title | Note on non-commutative local field |
| Author(s) | Sugano, K. |
| Citation | Hokkaido University Preprint Series in Mathematics, 213, 1-3 |
| Issue Date | 1993-08 |
| DOI | 10.14943/83357 |
| Doc URL | http://hdl.handle.net/2115/68959 |
| Type | bulletin (article) |
| File Information | pre213.pdf |

Note on non-commutative local field

Kozo Sugano

Series ♯213. August 1993

## HOKKAIDO UNIVERSITY

## PREPRINT SERIES IN MATHEMATICS

♯ 186: Y. Kurokawa, On functional moduli for first order ordinary differential equations, 9 pages. 1993.

♯ 187: O. Ogurisu, Unitary equivalence between a spin 1/2 charged particle in a two-dimensional magnetic field and a spin 1/2 neutral particle with an anomalous magnetic moment in a two-dimensional electric field, 4 pages. 1993.

♯ 188: A. Jensen, T. Ozawa, Existence and non-existence results for wave operators for perturbations of the laplacian, 30 pages. 1993.

♯ 189: T. Nakazi, Multipliers of invariant subspaces in the bidisc, 12 pages. 1993.

♯ 190: S. Izumiya, G.T. Kossioris, Semi-local classification of geometric singularities for Hamilton-Jacobi equations, 24 pages. 1993.

♯ 191: N. Hayashi, T. Ozawa, Finite energy solutions of nonlinear Schrödinger equations of derivative type, 21 pages. 1993.

♯ 192: J. Seade, T. Suwa, A residue formula for the index of a holomorphic flow, 22 pages. 1993.

♯ 193: H. Kubo, Blow-up of solutions to semilinear wave equations with initial data of slow decay in low space dimensions, 8 pages. 1993.

♯ 194: F. Hiroshima, Scaling limit of a model of quantum electrodynamics, 52 pages. 1993.

♯ 195: T. Ozawa, Y. Tsutsumi, Global existence and asymptotic behavior of solutions for the Zakharov equations in three space dimensions, 34 pages. 1993.

♯ 196: H. Kubo, Asymptotic behaviors of solutions to semilinear wave equations with initial data of slow decay, 25 pages. 1993.

♯ 197: Y. Giga, Motion of a graph by convexified energy, 32 pages. 1993.

♯ 198: T. Ozawa, Local decay estimates for Schrödinger operators with long range potentials, 17 pages. 1993.

♯ 199: A. Arai, N. Tominaga, Quantization of angle-variables, 31 pages. 1993.

♯ 200: S. Izumiya, Y. Kurokawa, Holonomic systems of Clairaut type, 17 pages. 1993.

♯ 201: K.-S. Saito, Y. Watatani, Subdiagonal algebras for subfactors, 7 pages. 1993.

♯ 202: K. Iwata, On Markov properties of Gaussian generalized random fields, 7 pages. 1993.

♯ 203: A. Arai, Characterization of anticommutativity of self-adjoint operators in connection with Clifford algebra and applications, 13 pages. 1993.

♯ 204: J. Wierzbicki, An estimation of the depth from an intermediate subfactor, 7 pages. 1993.

♯ 205: N. Honda, Vanishing theorem for the tempered distributions, 11 pages. 1993.

♯ 206: T. Hibi, Betti number sequences of simplicial complexes, Cohen-Macaulay types and Möbius functions of partially ordered sets, and related topics, 25 pages. 1993.

♯ 207: A. Inoue, Regularly varying correlations, 23 pages. 1993.

♯ 208: S. Izumiya, B. Li, Overdetermined systems of first order partial differential equations with singular solution, 9 pages. 1993.

♯ 209: T. Hibi, Hochster's formula on Betti numbers and Buchsbaum complexes, 7 pages. 1993.

♯ 210: T. Hibi, Star-shaped complexes and Ehrhart polynomials, 5 pages. 1993.

♯ 211: S. Izumiya, G. T. Kossioris, Geometric singularities for solutions of single conservation laws, 28 pages. 1993.

♯ 212: A. Arai, On self-adjointness of Dirac operators in Boson-Fermion Fock spaces, 43 pages. 1993.

Note on non-commutative local field

Kozo Sugano

Throughout this paper A will be a division ring with non-trivial valuation v, and C will be the center of A. A has the completion with respect to the v-topology, which is also a division ring. We will denote it by $A^*$. For each division subring B of A the closure of B in $A^*$ with respect to the v-topology will be also denoted by $B^*$. $B^*$ is isomorphic to the completion of B as topological ring.

The aim of this paper is to show that $B^*$ coincides with the double centralizer of B in $A^*$ for cach division C-subalgebra B of A, in the case where A is finite over C and $A^*$ is a local field with respect to v.

In this paper we will use the same terminology as [4] and [6]. In particular for each division subring B of A we write

$O(B)=\{x \in B: v(x) \leqq 1\}$, $P(B)=\{x \in B: v(x) < 1\}$ and

Since v is non-trivial, A is locally compact if and only if O(A) is compact (See § 17.5 [4]). In this case A is complete (See the proof of Proposition 17.5 b [4]).

Let v be non-archimedian and B an arbitrary division subring of A. O(B) is a local ring with the maximal ideal P(B). Hence O(B)/P(B) is a division ring, which will be denoted by E(B). Write E(B)=K and E(A)=E. Then K is a division subring of E. We will write $f_r(A/B)=[E:K]_r$, $f_l(A/B)=[E:K]_l$ and $e(A/B)=[v(A^\circ):v(B^\circ)]$, where $A^\circ$ and $B^\circ$ are the unit groups of A and B, respectively. In the case where $[E:K]_l=[E:K]_r$, we will write f(A/B) in stead of $f_r(A/B)$ or $f_l(A/B)$. The next lemma is well known in the case where A is a commutative field.

Lemma 1. Let A, B and v be as above, and assume that both e(A/B) and $f_r(A/B)$ are finite. Then we have $e(A/B)f_r(A/B) \leqq [A:B]_r$

Proof. Since $v(ab)=v(a)v(b)=v(b)v(a)=v(ba)$ for any $a, b \in A$, we can follow the same lines as the proof of Theorem 4.5 Chap. 2 [2].

Now suppose that A is finite dimensional over C. Then for any division C-subalgebra B of A we have $B=V_A(V_A(B))$, and $V_A(B)$ is finite dimensional division C-subalgebra (See e.g., Theorem 12.7 [4]). Therefore A is an H-separable extension of every division C-subalgebra B of A by Theorem 1 [5]. Moreover $\{A^*/A, B^{**}/B\}$ have the centralizer property in the sense of [6], and we have $[A:B]=[A^*:B^{**}]$ and $B^{**}\supset B^*$, where $B^{**}= V_{A^*}(V_{A^*}(B))$ by Theorem 1.3 (4) [6]. In particular $C^*$ is contained in the center of $A^*$. C is an infinite field, otherwise A must be commutative by Wedderburn's Theorem. Therefore, if furthermore v is non-archimedian and $O(A^*)$ is compact in the v-topology, for any division C-subalgebra B of A we have $e(B^*/C^*)f(B^*/C^*)=[B^*:C^*]<\infty$ by Proposition 17.7 [4], since $O(B^*)$ is also compact and $C^*$ is infinte and closed in $B^*$.

Now we are ready to have our main theorem.

Theorem 1. Let A be a divsion ring with non-trivial non-archimedian valuation v. Assume A is finite dimensional over its center C, and $O(A^*)$ is compact in the v-topology. Then we have

(1) The closure $C^*$ of C in $A^*$ coincides with the center of $A^*$, and we have $[A^*:C^*]=[A:C]$

(2) For any division C-subalgebra B of A, we have $B^*=V_{A^*}(V_{A^*}(B))$, $A^*=B^*A=AB^*$ and $B^*\cap A=B$.

Proof. (1). Write $C^{**} = V_{A^*}(V_{A^*}(C))$. $C^{**}$ coincides with the center of $A^*$ by Proposition 1.3 [6]. Since v is non-archimedian, we have

$$e(C^*/C)= e(A^*/A)=1 \text{ and } f(A^*/A)=f(C^*/C)=1$$

(See § 17.4 [4]). Since

$$e(A^*/C)=e(A^*/C^*)e(C^*/C)=e(A^*/A)e(A/C),$$

we have $e(A/C)=e(A^*/C^*)$. Similarly we have $f(A^*/C^*)=f(A/C)$. Then by Proposition 17.7 (1) [4] and Lemma 1 we have

$$[A^*:C^*]= e(A^*/C^*)f(A^*/C^*)=e(A/C)f(A/C)\leqq [A:C].$$

But by Theorem 1.7 (3) [6] we have $[A:C]=[A^*:C^{**}]\leqq [A^*:C^*]$. Hence we have $[A^*:C^*]=[A^*:C^{**}]$ and $C^*=C^{**}$.

(2). Put $B'' = V_{A'}(V_{A'}(B))$. $B''$ is complete, and $[A':B''] = [A:B]$ by Theorem 1.7 (3) and (4) [6]. On the other hand since $e(B'/B) = f(B'/B) = 1$, we have

$$e(A'/B')f(A'/B') = e(A/B)f(A/B) \leqq [A:B]$$

$$e(B'/C')f(B'/C') = e(B/C)f(B/C) \leqq [B:C]$$

where the inequalities $\leqq$'s are due to Lemma 1. Then we have

$$[A':C'] = e(A'/C')f(A'/C') = e(A'/B')e(B'/C')f(A'/B')f(B'/C')$$
$$\leqq [A:B][B:C] = [A:C] = [A':C']$$

This means that $e(A/B)f(A/B) = [A:B]$ and $e(B/C)f(B/C) = [B:C]$. The latter equality shows $[B:C] = e(B'/C')f(B'/C') = [B':C']$. Then $[A':B''] = [A:B] = [A':B']$ and we have $B' = B''$. The proof of the remaining part is due to Theorem 1.7 [6].

## References


[1] E. Artin and G. Whaples, The theory of simple rings, Amer. J. Math., 65 (1943), 87-107.

[2] S. Iyanaga: Number Theorey (in Japanese), Iwanami (1969).

[3] K. Hirata: Some types of separable extensions of rings, Nagoya Math. J., 33 (1968), 107-115.

[4] R. Pierce: Associative Algebras, Springer GTM 88 (1980).

[5] K. Sugano: On H-separable extensions of two-sided simple rings, Hokkaido Math. J., 11 (1982), 246-252.

[6] K. Sugano: On bicommutators of modules over H-separable extension rings III, to appear in Hokkaido Math. J..



Hokkaido University