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Note on non-commutative local field

Kozo Sugano

Throughout this paper A will be a division ring with non-trivial valuation v, and C will be the center of A. A has the completion with respect to the v-topology, which is also a division ring. We will denote it by A^*. For each division subring B of A the closure of B in A^* with respect to the v-topology will be also denoted by B^*. B^* is isomorphic to the completion of B as topological ring.

The aim of this paper is to show that B^* coincides with the double centralizer of B in A^* for each division C-subalgebra B of A, in the case where A is finite over C and A^* is a local field with respect to v.

In this paper we will use the same terminology as [4] and [6]. In particular for each division subring B of A we write

$$O(B)=\{x \varepsilon B : v(x) \leqq 1\},\quad P(B)=\{x \varepsilon B : v(x) < 1\} \text{ and}$$

Since v is non-trivial, A is locally compact if and only if O(A) is compact (See § 17.5 [4]). In this case A is complete (See the proof of Proposition 17.5 b [4]).

Let v be non-archimedian and B an arbitrary division subring of A. O(B) is a local ring with the maximal ideal P(B). Hence O(B)/P(B) is a division ring, which will be denoted by E(B). Write E(B)=K and E(A)=E. Then K is a division subring of E. We will write $f_r(A/B)=[E:K]_r$, $f_l(A/B)=[E:K]_l$ and $e(A/B)=[v(A^\circ):v(B^\circ)]$, where A° and B° are the unit groups of A and B, respectively. In the case where $[E:K]_l=[E:K]_r$, we will write f(A/B) in stead of $f_r(A/B)$ or $f_l(A/B)$. The next lemma is well known in the case where A is a commutative field.

Lemma 1. Let A, B and v be as above, and assume that both e(A/B) and $f_r(A/B)$ are finite. Then we have $e(A/B)f_r(A/B) \leqq [A:B]_r$

Proof. Since $v(ab)=v(a)v(b)=v(b)v(a)=v(ba)$ for any $a,b \varepsilon A$, we can follow the same lines as the proof of Theorem 4.5 Chap. 2 [2].

Now suppose that A is finite dimensional over C. Then for any division C-subalgebra B of A we have $B=V_A(V_A(B))$, and $V_A(B)$ is finite dimensional division C-subalgebra (See e.g., Theorem 12.7 [4]). Therefore A is an H-separable extension of every division C-subalgebra B of A by Theorem 1 [5]. Moreover $(A^*/A, B^*/B)$ have the centralizer property in the sense of [6], and we have $[A:B]=[A^*:B^*]$ and $B^* \supset B$, where $B^*=V_{A^*}(V_{A^*}(B))$ by Theorem 1.3 (4) [6]. In particular C^* is contained in the center of A^*. C is an infinite field, otherwise A must be commutative by Wedderburn's Theorem. Therefore, if furthermore v is non-archimedian and $O(A^*)$ is compact in the v-topology, for any division C-subalgebra B of A we have $e(B^*/C^*)f(B^*/C^*)=[B^*:C^*]<\infty$ by Proposition 17.7 [4], since $O(B^*)$ is also compact and C^* is infinte and closed in B^*.

Now we are ready to have our main theorem.

Theorem 1. Let A be a divsion ring with non-trivial non-archimedian valuation v. Assume A is finite dimensional over its center C, and $O(A^*)$ is compact in the v-topology. Then we have

(1) The closure C^* of C in A^* coincides with the center of A^*, and we have $[A^*:C^*]=[A:C]$

(2) For any division C-subalgebra B of A, we have $B^*=V_{A^*}(V_{A^*}(B))$, $A^*=B^*A=AB^*$ and $B^* \cap A=B$.

Proof. (1). Write $C^*=V_{A^*}(V_{A^*}(C))$. C^* coincides with the center of A^* by Proposition 1.3 [6]. Since v is non-archimedian, we have

$$e(C^*/C)=e(A^*/A)=1 \text{ and } f(A^*/A)=f(C^*/C)=1$$

(See § 17.4 [4]). Since

$$e(A^*/C)=e(A^*/C^*)e(C^*/C)=e(A^*/A)e(A/C),$$

we have $e(A/C)=e(A^*/C^*)$. Similarly we have $f(A^*/C^*)=f(A/C)$. Then by Proposition 17.7 (1) [4] and Lemma 1 we have

$$[A^*:C^*]=e(A^*/C^*)f(A^*/C^*)=e(A/C)f(A/C) \leqq [A:C].$$

But by Theorem 1.7 (3) [6] we have $[A:C]=[A^*:C^*] \leqq [A^*:C^*]$. Hence we have $[A^*:C^*]=[A^*:C^*]$ and $C^*=C^*$.

(2). Put $B^{"}=V_{A^{*}}(V_{A}(B))$. $B^{"}$ is complete, and $[A^{*}:B^{"}]=[A:B]$ by Theorem 1.7 (3) and (4) [6]. On the other hand since $e(B^{*}/B)=f(B^{*}/B)=1$, we have

$$e(A^{*}/B^{"})f(A^{*}/B^{"})=e(A/B)f(A/B)\leqq[A:B]$$

$$e(B^{*}/C^{"})f(B^{*}/C^{"})=e(B/C)f(B/C)\leqq[B:C]$$

where the inequalities $\leqq$'s are due to Lemma 1. Then we have

$$[A^{*}:C^{"}]=e(A^{*}/C^{"})f(A^{*}/C^{"})=e(A^{*}/B^{"})e(B^{*}/C^{"})f(A^{*}/B^{"})f(B^{*}/C^{"})$$

$$\leqq[A:B][B:C]=[A:C]=[A^{*}:C^{*}]$$

This means that $e(A/B)f(A/B)=[A:B]$ and $e(B/C)f(B/C)=[B:C]$. The latter equality shows $[B:C]=e(B^{*}/C^{"})f(B^{*}/C^{"})=[B^{*}:C^{"}]$. Then $[A^{*}:B^{"}]=[A:B]=[A^{*}:B^{"}]$ and we have $B^{*}=B^{"}$. The proof of the remaining part is due to Theorem 1.7 [6].

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