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BLOW-UP OF THE RADIAL SOLUTIONS TO THE EQUATIONS OF VIBRATING MEMBRANE

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1. Introduction.

The vertical motion of a nonlinear vibrating membrane is governed by the equation:

$$u_{tt} - \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0, \qquad (x,t) \in \Omega \times [0,\infty), \tag{1.1}$$

subject to the initial condition at $t = 0$

$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \qquad x \in \Omega. \tag{1.2}$$

The total energy $E(t)$ at time t has the form

$$E(t) = \int_\Omega ({u_t}^2 + \sqrt{1+|\nabla u|^2})dx,$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. Let a solution u to (1.1) satisfy Dirichlet or Neumann boundary condition,

$$u = 0 \quad \text{or} \quad n \cdot \nabla u = 0, \qquad (x,t) \in \partial\Omega \times [0,\infty), \tag{1.3}$$

where n stands for the outer unit normal vector to $\partial\Omega$. Then the conservation law of the energy holds:

$$E(t) = E(0).$$

For the equation of nonlinear vibrating string corresponding to one space dimension, S. Klainerman and A. Majda [9] has proved that a smooth solution with small initial data and with Dirichlet or Neumann boundary condition always develops singularities in the second derivatives at finite time. When the space dimensions are greater than two and $\Omega = \mathbb{R}^n$ ($n \geqq 3$), the results in S. Klainerman [7] guarantee the global in time existence of classical solutions to the initial value problem (1.1), (1.2) with small data of compact support.

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The main aim of this paper is to show the following. Let Ω be a ball B_R centered at the origin with radius R, and let the supports of initial data be contained in B_R. Then a smooth radially symmetric solution to initial-boundary value problem (1.1), (1.2) and (1.3) develops singularities in the second derivatives at finite time provided that the initial data is small and the radius R is sufficiently large. The largeness of R depends only on the initial data. More precisely, blow-up occurs before the disturbances do not reach the boundary. For the purpose we rewrite (1.1) in the radially symmetric form. Setting $u(x,t) = u(r,t), r = |x|$, we get

$$u_{tt} - c^2(u_r)(u_{rr} + \frac{1}{r}u_r) = \frac{1}{r}u_r G(u_r), \qquad (r,t) \in (0,\infty) \times (0,\infty), \tag{1.4}$$

$$u(r,0) = \varepsilon f(r), \quad u_t(r,0) = \varepsilon g(r), \qquad r \in (0,\infty), \tag{1.5}$$

where

$$\begin{aligned} c^2(u_r) =& 1 - \frac{3}{2}{u_r}^2 + O({u_r}^4), \\ G(u_r) =& {u_r}^2 + O(|u_r|^3), \end{aligned}$$

and supports of f, g are contained in $\{x \in \mathbb{R}^2 | \ |x| \leqq M\}$. The term $r^{-1}u_r G(u_r)$ in (1.4) corresponds essentially to the one satisfying strong null condition in (1.1).

When the coefficient c^2 of the Laplacian has the form

$$c^2(u_t) = 1 + au_t + O({u_t}^2) \quad a \neq 0$$

and $G(u_r) \equiv 0$, F. John [4], [5] and [6] has obtained in three space dimensions,

$$\lim_{\varepsilon \to 0} \varepsilon \log(1 + T_\varepsilon) = \frac{1}{H_1},$$

where

$$H_1 = \max_{\rho \in \mathbb{R}} (\frac{a}{2}\mathcal{F}''(\rho)),$$

and in two space dimensions, L. Hörmander [2] and S. Alinhac [1] have shown

$$\lim_{\varepsilon \to 0} \varepsilon T_\varepsilon = (\frac{1}{H_2})^2,$$

where

$$H_2 = \max_{\rho \in \mathbb{R}} (a\mathcal{F}''(\rho)).$$

Here T_ε stands for the lifespan of the initial value problems and the Friedlander radiation field $\mathcal{F}(\rho)$ depending on f, g will be defined later. Although Alinhac's result is more delicate with respect to the order of ε, we do not mention further details here. Concerning these results, we will prove for the lifespan of the initial value problem (1.4) and (1.5)

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log(1 + T_\varepsilon) = \frac{1}{H_0},$$

where

$$H_0 = \max_{\rho \in \mathbb{R}} (\frac{3}{2}\mathcal{F}'(\rho)\mathcal{F}''(\rho)).$$

This fact leads that if we take $T > 1/H_0$, then for sufficiently small $\varepsilon_0 > 0$, we have

$$T_{\varepsilon_0} < \exp(\frac{T}{\varepsilon_0^2}).$$

Thus the solution of (1.3), (1.4) and (1.5) blows up when $\phi = \varepsilon_0 f, \psi = \varepsilon_0 g$ and R is greater than $\exp(T/\varepsilon_0^2) + M$.

2. Statement of Results.

We consider more general initial value problem which involves the initial value problem (1.4) and (1.5):

$$u_{tt} - c^2(u_t, u_r)(u_{rr} + \frac{1}{r}u_r) = \frac{1}{r}u_r G(u_t, u_r), \qquad (r,t) \in (0,\infty) \times (0,\infty), \tag{2.1}$$

$$u(r,0) = \varepsilon f(r), \quad u_t(r,0) = \varepsilon g(r), \qquad r \in (0,\infty). \tag{2.2}$$

Here we assume that $c, G \in C^\infty(\mathbb{R}^2)$,

$$\begin{aligned} c^2(u_t, u_r) &= 1 + a_1 u_t^2 + a_2 u_t u_r + a_3 u_r^2 + O(|u_t|^3 + |u_r|^3), \\ G(u_t, u_r) &= O(u_t^2 + u_r^2), \end{aligned} \tag{2.3}$$

and assume $f, g \in C_0^\infty(\mathbb{R}^2), |f| + |g| \not\equiv 0, \mathrm{supp} f, \mathrm{supp} g \subset \{x \in \mathbb{R}^2 | \ |x| \leqq M\}$ and $f = f(|x|), g = g(|x|)$. Moreover we assume $a_1 - a_2 + a_3 \neq 0$. To state our result, we define the Friedlander radiation field $\mathcal{F}(\rho)$ by

$$\mathcal{F}(\rho) = \lim_{r \to \infty} r^{\frac{1}{2}} u^0(r,t) \qquad \text{for} \qquad \rho = r - t,$$

where $u^0(r,t)$ is the solution of linear wave equation:

$$u_{tt}^0 - u_{rr}^0 - \frac{1}{r}u_r^0 = 0, \qquad (r,t) \in (0,\infty) \times (0,\infty), \tag{2.4}$$

$$u^0(r,0) = f(r), \quad u_t^0(r,0) = g(r), \qquad r \in (0,\infty), \tag{2.5}$$

(e.g. L. Hörmander [2]). $\mathcal{F}$ is explicitly expressed as

$$\mathcal{F}(\rho) = \frac{1}{\sqrt{2}\pi} \int_\rho^\infty (s-\rho)^{-\frac{1}{2}} (R_g(s) - R_f'(s)) ds,$$

where $R_h(s)$ is the Radon transform of $h \in C_0^\infty[0,\infty)$, *i.e.*,

$$R_h(s) = \int_s^\infty \frac{\xi h(\xi)}{\sqrt{\xi^2 - s^2}} d\xi.$$

Moreover, $\mathcal{F}$ has the properties:

$$\left|\frac{d^k}{d\rho^k}\mathcal{F}(\rho)\right| \leqq C_k(1+|\rho|)^{-\frac{1}{2}-k} \quad \text{for} \quad \rho \in \mathbb{R}, \tag{2.6}$$

$$\mathcal{F}(\rho) = 0 \qquad \text{for} \qquad \rho \geqq M. \tag{2.7}$$

Thus the quantity

$$H_0 = \max_{\rho\in\mathbb{R}}\{-(a_1 - a_2 + a_3)\mathcal{F}'(\rho)\mathcal{F}''(\rho)\}$$

is well defined and non-negative. Our assumptions $|f|+|g| \not\equiv 0$ and $a_1-a_2+a_3 \neq 0$ guarantee that $H_0 > 0$, which is shown in [3]. The lifespan T_ε of the solution u to (2.1) and (2.2) means the supremumn of τ such that the solution u exists in $C^\infty((0,\tau)\times\mathbb{R}^2)$.

In this paper we will prove the following

Theorem.

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \log(1+T_\varepsilon) \leqq \frac{1}{H_0}.$$

As a consequence of results in [3], we have

$$\liminf_{\varepsilon\to 0} \varepsilon^2 \log(1+T_\varepsilon) \geqq \frac{1}{H_0}.$$

Thus we obtain

Corollary.

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log(1+T_\varepsilon) = \frac{1}{H_0}.$$

To prove Theorem, we have only to show the following lemma.

Main Lemma. *For any $A > H_0$, there exists an $\varepsilon_A > 0$ such that for $0 < \varepsilon < \varepsilon_A$,*

$$\varepsilon^2 \log(1+T_\varepsilon) \leqq \frac{1}{A} \tag{2.8}$$

holds.

We describe the outline of the proof of Main Lemma. First we fix a constant $B > H_0$. Set $\rho = r - t$, $s = \varepsilon^2 \log(1+t)$ and consider the Burgers' equation:

$$U_s + \frac{a_1 - a_2 + a_3}{6}(U_\rho)^3 = 0, \qquad (\rho, s) \in \mathbb{R}\times[0, \frac{1}{B}],$$

$$U(\rho, 0) = \mathcal{F}(\rho), \qquad \rho \in \mathbb{R}.$$

For the solutions U of the above Burgers' equation and u of the initial value problem (2.1) and (2.2), we will find that

$$|\partial_r^l \partial_t^m u(r,t_0) - \varepsilon r^{-\frac{1}{2}}(-1)^m \partial_\rho^{l+m} U(r-t_0, \frac{1}{B})| \leqq C\varepsilon^{\frac{5}{4}} r^{-\frac{1}{2}}$$

$$\text{for} \quad r - t_0 \geqq -\frac{1}{3\varepsilon} \quad \text{and} \quad l, m \in \mathbb{N} \cup \{0\} \ (l + m \neq 0),$$

where $t_0 = \exp(1/\varepsilon^2 B) - 1$, *i.e.*, $\varepsilon^2 \log(1 + t_0) = 1/B$. Moreover, on characteristic curves Λ in (ρ, s)-plane, we can approximate U by the Friedlander radiation field $\mathcal{F}$ for $0 \leqq s \leqq 1/B$. This gives us approximation for u at $t = t_0$. Next we investigate the behaviour of u after $t = t_0$. If we set $v(r,t) = r^{1/2} u(r,t)$ and

$$w_1(r,t) = \frac{cv_{rr} - v_{rt}}{2c},$$

$$w_2(r,t) = \frac{cv_{rr} + v_{rt}}{2c},$$

the following *a priori* estimates hold:

$$|v(r,t)| < C\varepsilon^{\frac{1}{2}}, \qquad |v_t(r,t)|, |v_r(r,t)| < C\varepsilon,$$

$$|w_2(r,t)| < C\varepsilon^3,$$

as long as u exists. Using these estimates, we will construct an ordinary differenial equation with respect to w_1 along a pseudo-characteristic curve Z connected with Λ at $t = t_0$. Solving the ordinary differential equation, we will find that for $0 < \varepsilon < \varepsilon_A$ w_1 blows up in $t = \exp(1/\varepsilon^2 A) - 1$.

3. Approximation for u by the solution of Burgers' equation.

As we stated in the preceeding section, we consider the following Bergers' equation:

$$U_s + \frac{a_1 - a_2 + a_3}{6} (U_\rho)^3 = 0, \qquad (\rho, s) \in \mathbb{R} \times [0, \frac{1}{B}], \tag{3.1a}$$

$$U(\rho, 0) = \mathcal{F}(\rho), \qquad \rho \in \mathbb{R}, \tag{3.2a}$$

or

$$U_{\rho s} + \frac{a_1 - a_2 + a_3}{2} (U_\rho)^2 U_{\rho\rho} = 0, \quad (\rho, s) \in \mathbb{R} \times [0, \frac{1}{B}], \tag{3.1b}$$

$$U_\rho(\rho, 0) = \mathcal{F}'(\rho), \qquad \rho \in \mathbb{R}, \tag{3.2b}$$

where $B > H_0$ and $\rho = r - t$, $s = \varepsilon^2 \log(1 + t)$. We find that the Cauchy problem (3.1a) and (3.2a) is equivalent to (3.1b) and (3.2b) because there exists a smooth solution U_ρ to (3.1b) and (3.2b) and integral of U_ρ satisfies (3.1a) and (3.2a). For the solutions U of (3.1a) and (3.2a) and u of (2.1) and (2.2), we will prove

$$\begin{gathered} |\partial_r^l \partial_t^m u(r, t_0) - \varepsilon r^{-\frac{1}{2}} (-1)^m \partial_\rho^{l+m} U(r - t_0, \frac{1}{B})| \leqq C_{l,m,B} \varepsilon^{\frac{5}{4}} r^{-\frac{1}{2}} \\ \text{for} \quad r - t_0 > -\frac{1}{3\varepsilon} \quad \text{and} \quad l + m \neq 0, \end{gathered} \tag{3.3}$$

where we denote $t_0 = \exp(1/B\varepsilon^2) - 1$.

The main task in this section is to prove (3.3). To do this, we introduce the vector fields used in S. Klainerman [7] and state some results used through this paper.

$$L_0 = t\partial_t + x_1\partial_{x_1} + x_2\partial_{x_2}, \quad L_i = x_i\partial_t + t\partial_{x_i}, \quad \text{for} \quad i = 1, 2,$$
$$\partial_{x_1}, \quad \partial_{x_2}, \quad \partial_t,$$

named $\Gamma_1, \Gamma_2, \cdots, \Gamma_6$ respectively. These operators satisfy commutation relations:

$$\begin{aligned} [\Gamma_p, \Box] &= \Gamma_p\Box - \Box\Gamma_p = 2\delta_{1p}\Box \quad \text{for} \quad p = 1, 2, \cdots, 6, \\ [\Gamma, \Gamma] &= \overline{\Sigma}\Gamma, \quad [\Gamma, \partial] = \overline{\Sigma}\partial, \end{aligned} \tag{3.4}$$

where $\Box = \partial_t^2 - \Delta$ and $\overline{\Sigma}$ stands for a finite linear combination with constant coefficients. For $\alpha \in \mathbb{Z}_+^6 (\mathbb{Z}_+ = \mathbb{N} \cup \{0\})$ we write $\Gamma^\alpha = \Gamma_1^{\alpha_1}\Gamma_2^{\alpha_2}\cdots\Gamma_6^{\alpha_6}$ and define the norms

$$||v(t)||_k = \sum_{|\alpha| \leqq k} ||\Gamma^\alpha v(t)||_{L_x^2(\mathbb{R}^2)},$$

$$|v(t)|_k = \sum_{|\alpha| \leqq k} ||\Gamma^\alpha v(t)||_{L_x^\infty(\mathbb{R}^2)}.$$

In [3], we proved that

$$|\Gamma^\alpha \partial_r u|, |\Gamma^\alpha \partial_t u| \leqq C_{\alpha,B}\varepsilon(1+t)^{-\frac{1}{2}} \quad \text{for} \quad 0 \leqq t \leqq t_0, \quad \alpha \in \mathbb{Z}_+^6. \tag{3.5}$$

For the solution u^0 to (2.4), (2,5), we set $F(1/r, \rho) = r^{1/2}u^0(r,t)$. Then L. Hörmander showed in [2] that

$$|\partial_z^l \partial_\rho^m F(z,\rho)| \leqq C_{l,m}(1+|\rho|)^{-\frac{1}{2}+l-m} \quad \text{for} \quad 0 < z \leqq \frac{1}{2M} \tag{3.6}$$

and

$$\begin{aligned} &\left|\Gamma^\alpha(\partial_\rho^k F(z,\rho) - \frac{d^k}{d\rho^k}\mathcal{F}(\rho))\right| \leqq C_{\alpha,k,L}(1+|\rho|)^{\frac{1}{2}-k}(1+t)^{-1} \\ &\qquad \text{for} \quad r \geqq Lt \quad \text{and} \quad t \geqq 1. \end{aligned} \tag{3.7}$$

Here $M > 0$ is the radius of support of initial data and $L > 0$. Furthermore,$U(\rho, s)$ satisfies

$$|\partial_\rho^l \partial_s^m U(\rho,s)| \leqq C_{l,m,B}(1+|\rho|)^{-\frac{1}{2}-l-4m} \quad \text{for} \quad 0 \leqq s \leqq \frac{1}{B}, \tag{3.8}$$

$$U(\rho,s) = 0 \quad \text{for} \quad \rho \geqq M, \; 0 \leqq s \leqq \frac{1}{B}, \tag{3.9}$$

which will be proved in Appendix.

We choose a cut-off function $\chi \in C^\infty(\mathbb{R})$ equal to 1 in $(-\infty, 1)$ and 0 in $(2, \infty)$, and define a function $w(r,t)$ by

$$w(r,t) = \varepsilon\chi(\varepsilon t)u^0(r,t) - \varepsilon(1-\chi(\varepsilon t))\chi(-3\varepsilon\rho)r^{-\frac{1}{2}}U(\rho,s).$$

Using (3.5), (3.6), (3.7) and (3,8), we will prove

$$|\Gamma^\alpha w(t)| \leqq C_{\alpha,B}\varepsilon(1+t)^{-\frac{1}{2}}(1+|\rho|)^{-\frac{1}{2}} \quad \text{for} \quad 0 \leqq t \leqq t_0, \tag{3.10}$$

$$\|\Gamma^\alpha J(t)\|_0 \leqq C_{\alpha,B}(\varepsilon^{\frac{5}{4}}(1+t)^{-\frac{4}{5}} + \varepsilon^4(1+t)^{-1}) \quad \text{for} \quad 0 \leqq t \leqq t_0, \tag{3.11}$$

where

$$J(r,t) = \Box w - (a_1 w_t^2 + a_2 w_t w_r + a_3 w_r^2)\Delta w.$$

First we prove (3.10). Since the following decay estimate for u^0 (showed in L. Hörmander [2]) holds

$$|\Gamma^\alpha u^0(r,t)| \leqq C_\alpha(1+t)^{-\frac{1}{2}}(1+|\rho|)^{-\frac{1}{2}}, \tag{3.12}$$

we find that the first term of w satisfies (3.10). On the other hand, we get

$$t+1 \leqq 6r \leqq 6(t+M),$$

in the support of $(1-\chi(\varepsilon t))\chi(-3\varepsilon\rho)U(\rho,s)$. The second term of w satisfies (3.10) if we prove

$$|\Gamma^\beta(r^{-\frac{1}{2}})| \leqq C_\beta(1+t)^{-\frac{1}{2}}, \tag{3.13a}$$

$$|\Gamma^\beta(1-\chi(\varepsilon t))| \leqq C_\beta, \tag{3.13b}$$

$$|\Gamma^\beta(\chi(-3\varepsilon\rho))| \leqq C_\beta, \tag{3.13c}$$

$$|\Gamma^\beta(U(\rho,s))| \leqq C_\beta(1+|\rho|)^{-\frac{1}{2}}. \tag{3.13d}$$

Indeed, (3.13b) follows in principle from the inequalities

$$\begin{aligned} |L_i^k(1-\chi(\varepsilon t))| &\leqq C_k \sum_{j=0}^{k}\sum_{l=0}^{j} \varepsilon^j |x_i|^l t^{j-l} |\chi^{(j)}(\varepsilon t)| \\ &\leqq C_k \sum_{j=0}^{k} \varepsilon^j t^j |\chi^{(j)}(\varepsilon t)| \\ &\leqq C_k \qquad \text{for} \quad i=1,2, \end{aligned}$$

where the last inequality holds since $\varepsilon t \leqq 2$, in the support of $\chi^{(j)}(\varepsilon t)$. (3.13c) follows from the inequalities

$$\begin{aligned} |L_i^k(\chi(-3\varepsilon\rho))| &\leqq C_k \sum_{j=0}^{k}\sum_{l=0}^{j} \varepsilon^j \frac{|x_i|^l t^{j-l}}{r^j} |\rho|^j |\chi^{(j)}(-3\varepsilon\rho)| \\ &\leqq C_k \sum_{j=0}^{k} \varepsilon^j |\rho|^j |\chi^{(j)}(-3\varepsilon\rho)| \\ &\leqq C_k \qquad \text{for} \quad i=1,2, \end{aligned}$$

where the last inequality holds since $\varepsilon|\rho| \leqq 2/3$ in the support of $\chi^{(j)}(-3\varepsilon\rho)$. (3.13d) follows from (3.8) and a similar calculation as above.

Next we show (3.11) by dividing the proof into three cases.

Case 1. $0 \leqq \varepsilon t \leqq 1$. Since

$$w(r,t) = \varepsilon u^0(r,t),$$

we find

$$J(r,t) = -\varepsilon^3(a_1 {u_t^0}^2 + a_2 u_t^0 u_r^0 + a_3 {u_r^0}^2)\Delta u^0.$$

It follows from (3.12) that

$$|\Gamma^\alpha J(r,t)| \leqq C_\alpha \varepsilon^3 (1+t)^{-\frac{3}{2}}.$$

Since

$$\int_{\mathbb{R}^2} |\Gamma^\alpha J(r,t)|^2 dx = 2\pi \int_0^{t+M} |\Gamma^\alpha J(r,t)|^2 r dr,$$

we get

$$\begin{aligned}||\Gamma^\alpha J(r,t)||_0 &\leqq C_\alpha \varepsilon^3 (1+t)^{-\frac{3}{2}}(t+M)\\ &\leqq C_\alpha \varepsilon^3 (1+t)^{-\frac{1}{2}}\\ &\leqq C_\alpha \varepsilon^{\frac{5}{4}} (1+t)^{-\frac{5}{4}},\end{aligned}$$

where the last inequality follows from the fact

$$\varepsilon(1+t) \leqq \varepsilon + 1 \leqq 2.$$

This is what we wanted.

Case 2. $1 \leqq \varepsilon t \leqq 2$. Since the same estimate holds for nonlinear term $-(a_1 w_t^2 + a_2 w_t w_r + a_3 w_r^2)\Delta w$, we have only to examine

$$\begin{aligned}\Box w =& \varepsilon\Box[(1-\chi(\varepsilon t))\{\chi(-3\varepsilon\rho) r^{-\frac{1}{2}} U(\rho,s) - u^0(r,t)\}]\\ =& \varepsilon\Box\{(1-\chi(\varepsilon t))(\chi(-3\varepsilon\rho)-1)u^0\}\\ &+ \varepsilon\Box\{(1-\chi(\varepsilon t))\chi(-3\varepsilon\rho) r^{-\frac{1}{2}}(U(\rho,s) - F(\frac{1}{r},\rho))\}\\ =& J_1 + J_2,\end{aligned}$$

where $F(1/r,\rho)$ is the one in (3.6) and the last equality is the definition of J_1 and J_2. In the support of $1-\chi(-3\varepsilon\rho)$, we have $6r \leqq 5t$. Hence we find

$$|\Gamma^\alpha \partial_r^l \partial_t^m u^0(r,t)| \leqq C_{\alpha,l,m}(1+t)^{-1-l-m} \quad \text{for} \quad t \geqq 1. \tag{3.14}$$

Since

$$|\partial_r^l \partial_t^m \{(1-\chi(\varepsilon t))(\chi(-3\varepsilon\rho)-1)\}| \leqq C_{l,m}\varepsilon^{l+m}$$

and the support of u^0 is same as that of U, it follows from (3.13b), (3.13c) and (3.14) that

$$\begin{aligned}|\Gamma^\alpha J_1(r,t)| &\leqq C_\alpha(\varepsilon^3(1+t)^{-1}+\varepsilon^2(1+t)^{-2})\\ &\leqq C_\alpha\varepsilon^2(1+t)^{-2},\\ \|\Gamma^\alpha J_1(t)\|_0 &\leqq C_\alpha\varepsilon^2(1+t)^{-1}\\ &\leqq C_\alpha\varepsilon^{\frac{5}{4}}(1+t)^{-\frac{5}{4}}.\end{aligned}$$

On the other hand, in the support of J_2, we have $1+t \leqq 6r \leqq 6(t+M)$ and then obtain (3.13). Moreover we prove that

$$|\Gamma^\alpha(\partial_\rho^l U(\rho,s) - \partial_\rho^l F(\frac{1}{r},\rho))| \leqq C_l(1+|\rho|)^{\frac{1}{2}-l}(1+t)^{-1}, \tag{3.15}$$

for $0 \leqq s \leqq 1/B$, $r \geqq 1/(2M)$. Indeed,

$$\begin{aligned}\partial_\rho^l U(\rho,s) &= \partial_\rho^l U(\rho,0) + \int_0^1 \frac{d}{d\lambda}\partial_\rho^l U(\rho,\lambda s)d\lambda\\ &= \frac{d^l}{d\rho^l}\mathcal{F}(\rho) + \varepsilon^2\log(1+t)\int_0^1 \partial_\rho^l\partial_s U(\rho,\lambda s)d\lambda.\end{aligned}$$

By (3.8), (3.13d), $\varepsilon t \leqq 2$ and the fact

$$|\Gamma^\beta(\varepsilon\log(1+t))| \leqq C_\beta,$$

we find that

$$\begin{aligned}|\Gamma^\alpha(\varepsilon^2\log(1+t)\int_0^1 \partial_\rho^l\partial_s U(\rho,s\lambda)d\lambda)| &\leqq C_\alpha(\varepsilon^2\log(1+t)(1+|\rho|)^{-\frac{1}{2}-l-4})\\ &\leqq C_\alpha(1+|\rho|)^{\frac{1}{2}-l}(1+t)^{-1}.\end{aligned}$$

Thus it follows from (3.7) that

$$|\Gamma^\alpha(\partial_\rho^l U(\rho,s) - \partial_\rho^l F(\frac{1}{r},\rho))| \leqq C_\alpha(1+|\rho|)^{\frac{1}{2}-l}(1+t)^{-1},$$

which implies (3.15). Now, since

$$\Box v = r^{-\frac{1}{2}}(\partial_t^2 - \partial_r^2 - \frac{1}{4r^2})(r^{\frac{1}{2}}v),$$

we get

$$\begin{aligned}J_2 =& \varepsilon r^{-\frac{1}{2}}(\partial_t-\partial_r)(\partial_t+\partial_r)\{(1-\chi(\varepsilon t))\chi(-3\varepsilon\rho)(U-F)\}\\ &+\varepsilon r^{-\frac{5}{2}}(1-\chi(\varepsilon t))\chi(-3\varepsilon\rho)(U-F)\\ =& J_2' + J_2'',\end{aligned}$$

where the last equality is the definition of J_2' and J_2''. By (3.6), we have

$$|\Gamma^\beta F(\frac{1}{r},\rho)| \leqq C_\beta(1+|\rho|)^{-\frac{1}{2}}. \tag{3.16}$$

Then it follows from (3.13), (3.16) and $1+t \leqq 6r$ that

$$|\Gamma^\alpha J_2''| \leqq C_\alpha \varepsilon r^{-\frac{1}{2}}(1+t)^{-2}(1+|\rho|)^{-\frac{1}{2}}.$$

Since $(\partial_t + \partial_r)\rho = 0$, we obtain

$$\begin{aligned}|\Gamma^\alpha J_2'| \leqq & C_\alpha(\varepsilon^2 r^{-\frac{1}{2}}|\Gamma^\alpha\{(\partial_t - \partial_r)(\chi'(\varepsilon t)\chi(-3\varepsilon\rho)(U-F))\}| \\ & + \varepsilon r^{-\frac{1}{2}}(1+t)^{-2}(1+|\rho|)^{-\frac{1}{2}}),\end{aligned}$$

where we have used (3.13), (3.16), $1+t \leqq 6r \leqq 6(t+M)$ and $\varepsilon t \leqq 2$. Moreover using (3.13) and (3.15), we find that

$$\begin{aligned}&|\Gamma^\alpha\{(\partial_t - \partial_r)(\chi'(\varepsilon t)\chi(-3\varepsilon\rho)(U-F))\} \\ \leqq & C_\alpha(\varepsilon(1+t)^{-1}(1+|\rho|)^{\frac{1}{2}} + (1+t)^{-1}(1+|\rho|)^{-\frac{1}{2}}) \\ \leqq & C_\alpha(1+t)^{-1}(1+|\rho|)^{-\frac{1}{2}}.\end{aligned}$$

Thus we get

$$|\Gamma^\alpha J_2'| \leqq C_\alpha \varepsilon r^{-\frac{1}{2}}(1+t)^{-2}(1+|\rho|)^{-\frac{1}{2}}$$

and then we have

$$\begin{aligned}|\Gamma^\alpha J_2| &\leqq C_\alpha \varepsilon r^{-\frac{1}{2}}(1+t)^{-2}(1+|\rho|)^{-\frac{1}{2}}, \\ ||\Gamma^\alpha J_2||_0 &\leqq C_\alpha \varepsilon (1+t)^{-2}(\log(t+M))^{\frac{1}{2}} \\ &\leqq C_\alpha \varepsilon^{\frac{5}{4}}(1+t)^{-\frac{5}{4}},\end{aligned}$$

which implies (3.11) for $1 \leqq \varepsilon t \leqq 2$.

Case 3. $2 \leqq \varepsilon t \leqq \varepsilon t_0$. In this case, we have

$$w(r,t) = \varepsilon r^{-\frac{1}{2}}\chi(-3\varepsilon\rho)U(\rho,s) = \varepsilon r^{-\frac{1}{2}}\hat{U}(\rho,s).$$

We devide J into three parts:

$$\Gamma^\alpha J = Q_1 + Q_2 + Q_3,$$

where

$$\begin{aligned}Q_1 =& \Gamma^\alpha(\Box w + 2\varepsilon^3 r^{-\frac{3}{2}}\hat{U}_{\rho s}), \\ Q_2 =& \Gamma^\alpha(-2\varepsilon^3 r^{-\frac{3}{2}}\hat{U}_{\rho s} - (a_1 - a_2 + a_3)(\hat{U}_\rho)^2\hat{U}_{\rho\rho}), \\ Q_3 =& \Gamma^\alpha((a_1 - a_2 + a_3)\varepsilon^3 r^{-\frac{3}{2}}(\hat{U}_\rho)^2\hat{U}_{\rho\rho} - (a_1 w_t^2 + a_2 w_t w_r + a_3 w_r^2)\Delta w).\end{aligned}$$

Thus our purpose is converted to

$$||Q_i||_0 = O(\varepsilon^{\frac{5}{4}}(1+t)^{-\frac{5}{4}} + \varepsilon^4(1+t)^{-1}) \quad \text{for} \quad i = 1,2,3.$$

In the support of Q_i, we have $1+t \leqq 3r \leqq 3(t+M)$ and then we have (3.13). First we consider Q_1. We get

$$\begin{aligned}\Gamma^\alpha \Box w(r,t) =&\Gamma^\alpha(\varepsilon r^{-\frac{1}{2}}(\partial_t-\partial_r)(\partial_t+\partial_r)\hat{U}(\rho,s)+\frac{1}{4}\varepsilon r^{-\frac{5}{2}}\hat{U}(\rho,s))\\ =&R_1+R_2,\end{aligned}$$

where the last equality is the definition of R_1 and R_2. Using (3.13) and $1+t\leqq 3r$, we get

$$|R_2|\leqq C_\alpha \varepsilon r^{-\frac{1}{2}}(1+t)^{-2}(1+|\rho|)^{-\frac{1}{2}}$$

and then

$$\begin{aligned}||R_2||_0 &\leqq C_\alpha\varepsilon(1+t)^{-2}(\log(t+M))^{\frac{1}{2}}\\ &\leqq C_\alpha \varepsilon^{\frac{5}{4}}(1+t)^{-\frac{5}{4}}.\end{aligned}$$

Since $(\partial_t+\partial_r)\rho=0$, then we have

$$R_1=\Gamma^\alpha(\varepsilon^3 r^{-\frac{1}{2}}(\partial_t-\partial_r)\{(1+t)^{-1}\hat{U}_s(\rho,s)\}).$$

Moreover, by (3.8) and (3.13), we get

$$|R_1+2\varepsilon^3\Gamma^\alpha(r^{-\frac{1}{2}}(1+t)^{-2}\hat{U}_{s\rho}(\rho,s))|\leqq C_\alpha\varepsilon^3 r^{-\frac{1}{2}}(1+t)^{-2}(1+|\rho|)^{-\frac{9}{2}},$$

$$|R_1+2\varepsilon^3\Gamma^\alpha(r^{-\frac{3}{2}}\hat{U}_{s\rho}(\rho,s))|\leqq C_\alpha\varepsilon^3 r^{-\frac{1}{2}}(1+t)^{-2}(1+|\rho|)^{-\frac{7}{2}},$$

where we have used the fact

$$|\Gamma^\alpha(\rho\hat{U}_{s\rho}(\rho,s))|\leqq C_\alpha(1+|\rho|)^{-\frac{7}{2}}.$$

Thus we obtain

$$\begin{aligned}||Q_1||_0 &\leqq ||R_1+\Gamma^\alpha(2\varepsilon^3 r^{-\frac{3}{2}}\hat{U}_{s\rho}(\rho,s))||_0+||R_2||_0\\ &\leqq C_\alpha(\varepsilon^3(1+t)^{-2}+\varepsilon^{\frac{5}{4}}(1+t)^{-\frac{5}{4}})\\ &\leqq C_\alpha\varepsilon^{\frac{5}{4}}(1+t)^{-\frac{5}{4}}.\end{aligned}\tag{3.17}$$

Next we consider Q_3. Using (3.13), we get

$$|\Gamma^\alpha(\partial_t^l\partial_r^m w(r,t)-\varepsilon r^{-\frac{1}{2}}(-1)^l\partial_\rho^{l+m}\hat{U}(\rho,s))|\leqq C_{\alpha,l,m}\varepsilon r^{-\frac{1}{2}}(1+t)^{-1}.$$

This estimate yields

$$|Q_3|\leqq C_\alpha\varepsilon^3 r^{-\frac{3}{2}}(1+t)^{-1}$$

and then we get

$$\begin{aligned}||Q_3||_0 &\leqq C_\alpha\varepsilon^3(1+t)^{-2}(t+M)\\ &\leqq C_\alpha\varepsilon^{\frac{5}{4}}(1+t)^{-\frac{5}{4}}.\end{aligned}\tag{3.18}$$

Finally we estimate Q_2. When $\chi(-3\varepsilon\rho)$ is equal to 1 or 0, we find $Q_2=0$ by (3.1b). Thus we can assume $1\leqq -3\varepsilon\rho\leqq 2$, *i.e.*, $(1+|\rho|)^{-1}\leqq 3\varepsilon$. Using (3.8), (3.12) and $1+t\leqq 3r$, we have

$$\begin{aligned}|Q_2|&\leqq C_\alpha\varepsilon^4 r^{-\frac{1}{2}}(1+t)^{-1}(1+|\rho|)^{-\frac{3}{2}},\\ ||Q_2||_0&\leqq C_\alpha\varepsilon^4(1+t)^{-1}.\end{aligned}\tag{3.19}$$

Combining (3.17), (3.18) and (3.19), we find that (3.11) is valid for $2\leqq\varepsilon t\leqq\varepsilon t_0$ and then that is valid for $0\leqq t\leqq t_0$.

To finish the poof of (3.3), we need the following propositions.

Proposition 3.1. *Let $v \in C^2$ satisfy a wave equation:*

$$\Box v(x,t) = \sum_{\alpha,\beta=0}^{2} \gamma_{\alpha\beta}(x,t)\partial_\alpha\partial_\beta v(x,t) + h(x,t), \qquad (x,t) \in \mathbb{R}^2 \times [0,\infty),$$

where $\partial_0 = \partial_t$ and

$$|\gamma(t)|_0 = \sum_{\alpha,\beta=0}^{2} |\gamma_{\alpha\beta}(t)|_0 < \frac{1}{2} \quad \text{for} \quad 0 \leqq t < T.$$

Assume that for any fixed t, v vanishes for large $|x|$. Then we have for $0 \leqq t < T$

$$||Dv(t)||_0 \leqq 3(||Dv(0)||_0 + \int_0^t ||h(\tau)||_0 d\tau)\exp(\int_0^t |D\gamma(\tau)|_0 d\tau),$$

where

$$Dv = (\partial_0 v, \partial_1 v, \partial_2 v) \quad \text{and} \quad |D\gamma(\tau)|_0 = \sum_{\alpha,\beta,\delta=0}^{2} |\partial_\delta \gamma_{\alpha\beta}(\tau)|_0.$$

Proposition 3.2. *For a smooth function $v(x,t)(x \in \mathbb{R})$,*

$$|v(x,t)| \leqq C_n(1 + |x| + t)^{-\frac{n-1}{2}}(1 + |t - |x||)^{-\frac{1}{2}}||v(t)||_{[\frac{n}{2}]+1},$$

where $[s]$ stands for the largest integer not exceeding s.

Proposition 3.1 is obtained integrating by parts and Gronwall's inequality. Proposition 3.2 is so-called Klainerman's inequality which has proved in S. Klainerman [8] and F. John [6].

If we show that

$$||\Gamma^\alpha D(u(r,t) - w(r,t))||_0 \leqq C_{\alpha,B}\varepsilon^{\frac{5}{4}} \quad \text{for any} \quad \alpha \in \mathbb{Z}_+^6, \tag{3.20}$$

we find that (3.3) is valid. Indeed, it follows from (3.20) and Proposition 3.2 that

$$|\partial_r^l \partial_t^m (u(r,t) - w(r,t))| \leqq C_{l,m,B}\varepsilon^{\frac{5}{4}} r^{-\frac{1}{2}} \quad 0 \leqq t \leqq t_0$$

for any l and m. Moreover when $t \geqq 2/\varepsilon$ and $r - t \geqq -1/3\varepsilon$, $w(r,t) = \varepsilon r^{-1/2}U(\rho,s)$. Then

$$\partial_r^l \partial_t^m w(r,t) = \varepsilon r^{-\frac{1}{2}}(-1)^m \partial_\rho^{l+m} U(\rho,s) + O(\varepsilon r^{-\frac{3}{2}})$$

holds. By combining above inequality and equality, the desired estimate is obtained. Thus we have only to prove (3.20). If we set $v(r,t) = u(r,t) - w(r,t)$, by (2.1) v satisfies

$$\begin{aligned}
\Box v =&(a_1 u_t^2 + a_2 u_t u_r + a_3 u_r^2 + O(|Du|^4))\Delta u + \frac{1}{r}u_r G(u_t, u_r) - J(r,t) \\
&- (a_1 w_t^2 + a_2 w_t w_r + a_3 w_r^2)\Delta w \\
=&(a_1 u_t^2 + a_2 u_t u_r + a_3 u_r^2)\Delta v + O(|Du|^4|\Delta u|) + \frac{1}{r}u_r G(u_r, u_t) \\
&+ \{a_1(u_t + w_t)v_t + a_2(u_t v_r + w_r v_t) + a_3(u_r + w_r)v_r\}\Delta w - J(r,t).
\end{aligned} \tag{3.21}$$

By (3.4), we have for sufficiently small $\varepsilon > 0$

$$|a_1 u_t^2(t) + a_2 u_t(t) u_r(t) + a_3 u_r^2(t)|_0 \leqq \frac{1}{4}.$$

Thus we can apply Proposition 3.1 to (3.21). Since $v(r,0) \equiv 0$, we obtain for $0 \leqq t \leqq t_0$

$$\begin{aligned}||Dv(t)||_0 \leqq & C \int_0^t || \,(|Du(\tau)| + |Dw(\tau)|)|Dv(\tau)| \cdot |\Delta w(\tau)| + |J(\tau)| \\ & + |Du|^4 |D^2 u| + |u_{rr} G(\tau)| \,||_0 d\tau \\ & \times \exp(C \int_0^t |Du(\tau)| \cdot |D^2 u(\tau)| d\tau).\end{aligned}$$

It follows from (3.5), (3.10), (3.11) and $\varepsilon^2 \log(1+t_0) = 1/B$ that

$$\begin{aligned}||Dv(t)||_0 \leqq & C e^{\frac{C}{B}} \int_0^t \{\varepsilon^{\frac{5}{4}}(1+t)^{-\frac{5}{4}} + \varepsilon^4 (1+t)^{-1} + \varepsilon^2 (1+t)^{-1} ||Dv(\tau)||_0\} d\tau \\ \leqq & C\varepsilon^{\frac{5}{4}} + C \int_0^t \varepsilon^2 (1+\tau)^{-1} ||Dv(\tau)||_0 d\tau.\end{aligned}$$

Gronwall's inequality yields

$$||Dv(t)||_0 \leqq C\varepsilon^{\frac{5}{4}} \exp(C \int_0^t \varepsilon^2 (1+t)^{-1} d\tau) \leqq C\varepsilon^{\frac{5}{4}}.$$

This implies that (3.20) is valid for $\alpha = 0$. To prove (3.20) by induction, we assume that (3.20) holds for $|\alpha| = s-1$. For any α ($|\alpha| = s$), (3.3) admits

$$\begin{aligned}\Box \Gamma^\alpha v = & \sum_{|\beta|<|\alpha|} \Gamma^\beta (\Box v) + \Gamma^\alpha \{(a_1 u_t^2 + a_2 u_t u_r + a_3 u_r^2) \Delta v\} \\ & + \Gamma^\alpha \{(a_1 (u_t + w_t) v_t + a_2 (u_t v_r + w_r v_t) + a_3 (u_r + w_r) v_r) \Delta w\} \\ & + O(|Du|^4 |\Delta u|) + \frac{1}{r} u_r G(u_r, u_t) - \Gamma^\alpha J \\ = & (a_1 u_t^2 + a_2 u_t u_r + a_3 u_r^2) \Delta \Gamma^\alpha v + O((|Du| + |Dw|)|\Delta w| \cdot |D\Gamma^\alpha v| \\ & + |\Gamma^{s-1} Dv|(|\Gamma^s Du|^2 + |\Gamma^{s+1} w|^2) + |\Gamma^s J| + |\Gamma^s Du|^4 |\Gamma^s \Delta u| \\ & + |\Gamma^s (\frac{1}{r} u_r G(U_r, u_t))|),\end{aligned}$$

where $Df = (\partial_t f, \partial_r f)$ and $\Gamma^s = \sum_{|\lambda|=s} \Gamma^\lambda$. By Proposition 3.1, we get for $0 \leqq t \leqq t_0$

$$\begin{aligned}||D\Gamma^\alpha v(t)||_0 \leqq & C \int_0^t || \,(|Du(\tau)| + |Dw(\tau)|)|\Delta w(\tau)| \cdot |D\Gamma^\alpha v(\tau)| \\ & + |\Gamma^{s-1} Dv(\tau)|(|\Gamma^s Du(\tau)|^2 + |\Gamma^{s+1} Dw(\tau)|^2) + |\Gamma^s J(\tau)| \\ & |\Gamma^s Du(\tau)|^4 |\Gamma^s \Delta u(\tau)| + |\Gamma^s (\frac{1}{r} u_r G(\tau))| \,||_0 d\tau \\ & \times \exp(C \int_0^t |Du(\tau)| \cdot |D^2 u(\tau)| d\tau).\end{aligned}$$

Proceeding as above argument, by (3.5), (3.10), (3.11), $\varepsilon^2 \log(1+t_0) = 1/B$ and the assumption,

$$\begin{aligned}||D\Gamma^\alpha v(t)||_0 \leqq& C\int_0^t \{\varepsilon^{\frac{5}{4}}(1+\tau)^{-\frac{5}{4}} + \varepsilon^4(1+t)^{-1} + \varepsilon^2(1+t)^{-1}||D\Gamma^\alpha v(\tau)||_0\}d\tau \\ \leqq& C\varepsilon^{\frac{5}{4}} + C\int_0^t \varepsilon^2(1+\tau)^{-1}||D\Gamma^\alpha v(\tau)||_0 d\tau.\end{aligned}$$

Gronwall's inequality yields

$$||D\Gamma^\alpha v(t)||_0 \leqq C\varepsilon^{\frac{5}{4}}\exp(C\int_0^t \varepsilon^2(1+\tau)^{-1}d\tau) \leqq C\varepsilon^{\frac{5}{4}}.$$

Again using (3.4) and the assumption, we obtain

$$||\Gamma^\alpha Dv(t)||_0 \leqq C\varepsilon^{\frac{5}{4}},$$

for any α ($|\alpha| = s$). This completes the proof of (3.20).

At the end of this section, we investigate the value of the solution $U = U(\rho, s)$ at $s = 1/B$ *i.e.*, $t = t_0$. We assume that the maximum in the definition of H_0 is attained at $\rho = \rho_0$, *i.e.*,

$$H_0 = -(a_1 - a_2 + a_3)\mathcal{F}'(\rho_0)\mathcal{F}''(\rho_0). \tag{3.22}$$

In (ρ, s)-plane, we consider a characteristic curve $\Lambda_q (q \in \mathbb{R})$ which is defined by the solution of the following differential equation:

$$\frac{d\rho}{ds} = \frac{a_1 - a_2 + a_3}{2}(U_\rho(\rho, s))^2 \quad \text{for} \quad s \geqq 0, \qquad \rho = q \quad \text{for} \quad s = 0.$$

If we denote a point on Λ_{ρ_0} by $(\rho(s), s)$, then we find

$$U_\rho(\rho(\frac{1}{B}), \frac{1}{B}) = \mathcal{F}'(\rho_0) \tag{3.23}$$

$$\frac{1}{-(a_1 - a_2 + a_3)\mathcal{F}'(\rho_0)U_{\rho\rho}(\rho(\frac{1}{B}), \frac{1}{B})} = \frac{1}{H_0} - \frac{1}{B}. \tag{3.24}$$

Indeed, by (3.1b) and (3.2b), we have along Λ_{ρ_0}

$$\frac{d}{ds}U_\rho(\rho(s), s) = U_{\rho s} + \frac{a_1 - a_2 + a_3}{2}(U_\rho)^2 U_{\rho\rho} = 0 \quad 0 \leqq s \leqq \frac{1}{B}.$$

Hence we have

$$U_\rho(\rho(s), s) = U_\rho(\rho_0, 0) = \mathcal{F}'(\rho_0) \quad 0 \leqq s \leqq \frac{1}{B}, \tag{3.25}$$

which implies (3.23). Similarly, it follows from (3.1b) and (3.25) that

$$\begin{aligned}\frac{d}{ds}U_{\rho\rho}(\rho(s), s) =& U_{\rho\rho s}(\rho(s), s) + \frac{a_1 - a_2 + a_3}{2}(U_\rho(\rho(s), s))^2 U_{\rho\rho\rho}(\rho(s), s) \\ =& -(a_1 - a_2 + a_3)U_\rho(\rho(s), s)(U_{\rho\rho}(\rho(s), s))^2 \\ =& -(a_1 - a_2 + a_3)\mathcal{F}'(\rho_0)(U_{\rho\rho}(\rho(s), s))^2.\end{aligned}$$

Solving this equation , we obtain by (3.2b) and (3.22)

$$U_{\rho\rho}(\rho(s),s)=\frac{U_{\rho\rho}(\rho_0,0)}{1+(a_1-a_2+a_3)\mathcal{F}'(\rho_0)U_{\rho\rho}(\rho_0,0)s},$$

i.e.,

$$\frac{1}{U_{\rho\rho}(\rho(s),s)}=\frac{1}{\mathcal{F}''(\rho_0)}+(a_1-a_2+a_3)\mathcal{F}'(\rho_0)s,$$

i.e.,

$$\frac{1}{-(a_1-a_2+a_3)\mathcal{F}'(\rho_0)U_{\rho\rho}(\rho(s),s)}=\frac{1}{H_0}-s,$$

for $0\leqq s\leqq 1/B$. Thus (3.24) follows from this equality.

4. *A priori* estimates.

From now on, we investigate the behaviours of u after $t=t_0$. If we set $v(r,t)=r^{\frac{1}{2}}u(r,t)$, the equation (2.1) can be written as

$$v_{tt}-c^2(u_t,u_r)(v_{rr}+\frac{1}{4}r^{-2}v_r)=r^{-\frac{1}{2}}u_rG(u_t,u_r). \tag{4.1}$$

Moreover we define functions $w_1(r,t),w_2(r,t)$ by

$$\begin{aligned}w_1(r,t)=&\frac{cv_{rr}-v_{rt}}{2c}=-\frac{\mathcal{L}_2v_r}{2c},\\ w_2(r,t)=&\frac{cv_{rr}+v_{rt}}{2c}=\frac{\mathcal{L}_1v_r}{2c},\end{aligned}$$

where $\mathcal{L}_1=\partial_t+c\partial_r,\mathcal{L}_2=\partial_t-c\partial_r$. We find that w_1 and w_2 satisfy

$$w_1+w_2=v_{rr},\quad c(w_2-w_1)=v_{rt},$$

and these imply

$$\begin{aligned}u_r=&r^{-\frac{1}{2}}v_r-\frac{1}{2}r^{-\frac{3}{2}}v,\\ u_{rr}=&r^{-\frac{1}{2}}(w_1+w_2)-r^{-\frac{3}{2}}v_r+\frac{3}{4}r^{-\frac{5}{2}}v,\\ u_{rt}=&cr^{-\frac{1}{2}}(w_2-w_1)-\frac{1}{2}r^{-\frac{3}{2}}v_t.\end{aligned} \tag{4.2}$$

Then using (4.2), we obtain the equalities:

$$\begin{aligned}\mathcal{L}_1w_1=&\{c(a_1u_t+\frac{a_2}{2}u_r)-\frac{a_2}{2}u_t-a_3u_r+O(|Du|^3)\}r^{-\frac{1}{2}}w_1^2\\ &+O(\{r^{-\frac{1}{2}}|Du|\cdot|w_2|+r^{-\frac{3}{2}}|Dv|\cdot|Du|+r^{-\frac{5}{2}}|Du|\cdot|v|\}|w_1|\\ &+r^{-\frac{5}{2}}|w_2|\cdot|Du|\cdot|v|+r^{-\frac{3}{2}}|w_2|\cdot|Du|\cdot|Dv|+r^{-2}|Dv|+r^{-3}|v|)\end{aligned} \tag{4.3}$$

$$\begin{aligned}\mathcal{L}_2 w_2 =& O(\{r^{-\frac{1}{2}}|w_2|\cdot|Du| + r^{-\frac{3}{2}}|Du|\cdot|Dv| + r^{-\frac{5}{2}}|Du|\cdot|v|\}|w_1| \\ &+ r^{-\frac{1}{2}}|Du|\cdot|w_2|^2 + r^{-\frac{3}{2}}|Du|\cdot|Dv|\cdot|w_2| + r^{-\frac{5}{2}}|Dw_2|\cdot|Du|\cdot|v| \\ &+ r^{-2}|Dv| + r^{-3}|v|).\end{aligned} \tag{4.4}$$

In what follows, we assume that there exists a T ($t_0 < T < \exp(1/A\varepsilon^2) - 1$) such that the Cauchy problem (2.1), (2.2) has a solution $u(r,t)$ for $0 \leqq t \leqq T$. In (r,t)-plane, we consider pseudo-characteristic curves Z^1_λ and Z^2_μ which are given by solutions of differential equations:

$$Z^1_\lambda : \frac{dr}{dt} = c(u_t, u_r) \quad \text{for} \quad t \geqq t_0, \qquad r = \lambda + t_0 \quad \text{for} \quad t = t_0,$$

$$Z^2_\mu : \frac{dr}{dt} = -c(u_t, u_r) \quad \text{for} \quad t \geqq t_0, \qquad r = \mu - t_0 \quad \text{for} \quad t = t_0.$$

We set

$$\begin{aligned} D =& \{(r,t) \mid t_0 \leqq t \leqq T,\ (r,t) \in Z^1_\lambda,\ -N \leqq \lambda \leqq M\}, \\ D_{t_*} =& D \cap \{(r,t) \mid t_0 \leqq t \leqq t_*\}, \end{aligned}$$

where the constant N is sufficiently greater than $|\rho_0|$. Moreover we define functions

$$\begin{aligned} I(t) &= \max_{t_0 \leqq \tau \leqq t} \int_{r_1(\tau)}^{r_2(\tau)} |w_1(r,\tau)|dr, \\ V(t) &= \max_{(r,\tau)\in D_t} |v(r,\tau)|, \\ \dot{V}(t) &= \max_{(r,\tau)\in D_t} (|v_r(r,\tau)| + |v_t(r,\tau)|), \\ W_2(t) &= \max_{(r,\tau)\in D_t} |w_2(r,\tau)|, \end{aligned}$$

where $(r_1(\tau),\tau) \in Z^1_{-N}$ and $(r_2(\tau),\tau) \in Z^1_M$. Then the purpose of this section is following.

There exist a constant $\hat{C} > 0$ (independent of A) and an $\varepsilon_A > 0$ such that

$$\begin{aligned} I(t) < \hat{C}\varepsilon, \quad & V(t) < \hat{C}\varepsilon^{\frac{1}{2}}, \\ \dot{V}(t) < \hat{C}\varepsilon, \quad & W_2(t) < \hat{C}\varepsilon^3, \quad r > \frac{1+t}{2}, \end{aligned} \tag{4.5}$$

for $(r,t) \in D$ and $0 < \varepsilon < \varepsilon_A$.

To obtain (4.5) we just have to show:
(1) (4.5) holds at $t = t_0$,
(2) If (4.5) holds for $t_0 \leqq t < t_1$, (4.5) also holds at $t = t_1$.
At first we prove (1). If $(r, t_0) \in Z^1_\lambda \cap D$, it follows that

$$r = t_0 + \lambda, \qquad -N \leqq \lambda \leqq M, \tag{4.6}$$

then we find that

$$t_0 - N \leqq r \leqq t_0 + M.$$

If we take ε sufficiently small as

$$t_0 = \exp(\frac{1}{B\varepsilon^2}) - 1 > \max(M-2, 2N+1),$$

then we obtain

$$\frac{1+t_0}{2} < r(t_0) < 2(1+t_0). \tag{4.7}$$

For $(r, t_0) \in Z^1_\lambda$, it follows from (3.5), (4.6) and (4.7) that

$$\begin{aligned}
|u(r,t_0)| =& \left| -\int_r^{t_0+M} \frac{\partial}{\partial\lambda}(u(\lambda, t_0))d\lambda \right| \\
\leqq& |t_0 + M - r| \cdot |u_r(t_0)|_0 \\
\leqq& C\varepsilon(1+t_0)^{-\frac{1}{2}}|\lambda + M| \\
\leqq& C(M+N)\varepsilon(1+t_0)^{-\frac{1}{2}} \\
<& \sqrt{2}C(M+N)\varepsilon r^{-\frac{1}{2}} = C_0\varepsilon r^{-\frac{1}{2}},
\end{aligned}$$

which implies $V(t_0) < C_0\varepsilon^{1/2}$. It follows from (3.4), (4.7) and $V(t_0) < C_0\varepsilon^{1/2}$ that for $(r, t_0) \in D$,

$$\begin{aligned}
|v_r(r,t_0)| =& |r^{\frac{1}{2}}u_r(r,t_0) + \frac{1}{2}r^{-1}v(r,t_0)| \\
\leqq& Cr^{\frac{1}{2}}\varepsilon(1+t_0)^{-\frac{1}{2}} + \frac{1}{2}C_0 r^{-1}\varepsilon^{\frac{1}{2}} \\
<& \sqrt{2}C\varepsilon + \frac{1}{\sqrt{2}}C_0\varepsilon^{\frac{1}{2}}(1+t_0)^{-1}.
\end{aligned}$$

If we take ε sufficiently small, we obtain

$$(1+t_0)^{-1} = (\exp(\frac{1}{B\varepsilon^2}))^{-1} < \varepsilon^3. \tag{4.8}$$

Thus we find

$$|v_r(r,t_0)| < \frac{C_1}{2}\varepsilon.$$

Similarly we have

$$|v_t(r,t_0)| < \frac{C_1}{2}\varepsilon.$$

Therefore we obtain $\dot{V}(t_0) < C_1\varepsilon$. Using (3.5), (4.8), $V(t_0) < C_0\varepsilon^{1/2}$, $\dot{V}(t_0) < C_1\varepsilon$ and an equality

$$\partial_t + \partial_r = \frac{1}{t+r}(L_0 + \frac{x_1}{r}L_1 + \frac{x_2}{r}L_2),$$

we have for $(r, t_0) \in D$,

$$\begin{aligned}
|w_2(r,t_0)| =& \left| \frac{v_{tr} + cv_{rr}}{2c} \right| \\
=& \frac{|v_{rt} + v_{rr}|}{2} + O((|v_{rt}| + |v_{rr}|)|Du|^2) \\
=& O((t_0+r)^{-1}|v_r(t_0)|_1 + (|v_{rt}(t_0)|_0 + |v_{rr}(t_0)|_0)|Du(t_0)|_0^2) \\
=& O(\varepsilon(1+t_0)^{-1} + \varepsilon^3(1+t_0)^{-1}) \\
=& O(\varepsilon^4).
\end{aligned}$$

This implies $W_2(t_0) < C_2\varepsilon^3$. Finally we consider $I(t_0)$. It follows from (3.5), (4.8) $V(t_0) < C_0\varepsilon^{1/2}$ and $\dot{V}(t_0) < C_1\varepsilon$ that for $(r, t_0) \in D$,

$$\begin{aligned}|w_1(r,t_0)| =& \left|\frac{v_{rt} - cv_{rr}}{2}\right| \\ \leqq& \frac{|v_{rt}| + |v_{rr}|}{2} + O((|v_{rt}| + |v_{rr}|)|Du|^2) \\ \leqq& C''(\varepsilon + \varepsilon^3(1+t_0)^{-1}) \\ \leqq& C'\varepsilon.\end{aligned}$$

On the other hand, it follows from $(r_1(t_0), t_0) \in Z^1_{-N}, (r_2(t_0), t_0) \in Z^1_M$ and (4.6) that

$$|r_2(t_0) - r_1(t_0)| = |t_0 + M - t_0 + N| = M + N.$$

Then we have

$$I(t_0) = \int_{r_1(t_0)}^{r_2(t_0)} |w_1(r, t_0)|dr \leqq C'(M+N)\varepsilon < C_3\varepsilon.$$

If we take $\hat{C} > 0$

$$\hat{C} > \max\{C_0, C_1, C_2, C_3\},$$

(4.5) is valid at $t = t_0$ for sufficiently small ε. Thus we have proved (1).

To prove (2) we assume that for fixed t_1, (4.5) holds for $t_0 \leqq t < t_1$. The smoothness of the solution u guarantees that the inequalities which are altered $<$ by $\leqq$ in (4.5) hold at $t = t_1$. First we show $r > (1+t_1)/2$ if $\varepsilon < \varepsilon_A$. By (4.5) and the assumption $\varepsilon^2 \log(1+T) < 1/A$, we obtain for $(r(t), t) \in Z^1_\lambda$, $t_0 \leqq t \leqq t_1$

$$\frac{d(r-t)}{dt} = c - 1 = O(|Du|^2) = O(\varepsilon^2(1+t)^{-1}),$$

$$\begin{aligned}|r(t) - t - \lambda| \leqq& C\int_{t_0}^{t} \varepsilon^2(1+\tau)^{-1}d\tau \\ \leqq& C\varepsilon^2\log(1+t) \\ \leqq& \frac{C}{A}.\end{aligned} \tag{4.9}$$

This leads

$$r(t_1) \geqq t_1 + \lambda - \frac{C}{A} \geqq t_1 - M - \frac{C}{A} > \frac{1+t_1}{2},$$

provioded $t_0 > 2M + 2C/A + 1$, which is attained for $0 < \varepsilon < \varepsilon_A$ if ε_A is sufficiently small. Next we estimate $v(r, t_1)$. By (4.5) and (4.9), we obtain for $0 < \varepsilon < \varepsilon_A$,

$$\begin{aligned}|v(r,t_1)| =& \left|-\int_r^{t_1+M} v_r(\lambda, t_1)d\lambda\right| \\ \leqq& \hat{C}\varepsilon|t_1 + M - r| \\ \leqq& \hat{C}(\frac{C}{A} + M + N)\varepsilon \\ <& \hat{C}\varepsilon^{\frac{1}{2}},\end{aligned}$$

if $\varepsilon_A < (C/A + M + N)^{-2}$. Thus $V(t_1) < \hat{C}\varepsilon^{\frac{1}{2}}$ holds.

To prove $I(t_1) < \hat{C}\varepsilon$, we consider exteria derivatives of differential forms $w_1 dr - cw_1 dt$ and $w_2 dr + cw_2 dt$:

$$d(w_1(dr - cdt)) = -(\mathcal{L}_1 w_1 + \frac{\partial c}{\partial r} w_1) dr \wedge dt, \tag{4.10}$$

$$d(w_2(dr + cdt)) = -(\mathcal{L}_2 w_2 - \frac{\partial c}{\partial r} w_2) dr \wedge dt. \tag{4.11}$$

We set

$$\mathcal{K} = \{(r, t_1) \in D_{t_1} | w_1(r, t_1) > 0\},$$
$$\mathcal{K}' = \{(r, t_1) \in D_{t_1} | w_1(r, t_1) < 0\}.$$

Since these are open sets in $\mathbb{R}$, $\mathcal{K}$ and $\mathcal{K}'$ are the unions of at most denumerable families $\{K_i\}$ and $\{K_i'\}$ of open intervals, no two of which have common points. Assume that $\mathcal{K} = \{(r, t_1) | r_1(t_1) \leqq r \leqq r_2(t_1)\}$. Then, integrating (4.10) over D_{t_1} and using Green's formula, we obtain

$$\begin{aligned}
&-\iint_{D_{t_1}} (\mathcal{L}_1 w_1 + \frac{\partial c}{\partial t} w_1) drdt \\
&= \int_{r_1(t_0)}^{r_2(t_0)} w_1 dr + \int_{Z_M^1} w_1(dr - cdt) - \int_{\mathcal{K}} w_1 dr - \int_{Z_{-N}^1} w_1(dr - cdt).
\end{aligned}$$

Since

$$\int_{Z_\lambda^1} w_1(dr - cdt) = 0 \quad \text{for any} \quad \lambda,$$

we have

$$\int_{\mathcal{K}} w_1 dr \leqq \int_{r_1(t_0)}^{r_2(t_0)} |w_1| dr + \iint_{D_{t_1}} \left| \mathcal{L}_1 w_1 + \frac{\partial c}{\partial r} w_1 \right| drdt.$$

Furthermore, assume that $\mathcal{K}' = \{(r, t_1) | r_1(t_1) \leqq r \leqq r_2(t_1)\}$. Then, the same argument gives

$$-\int_{\mathcal{K}'} w_1 dr \leqq \int_{r_1(t_0)}^{r_2(t_0)} |w_1| dr + \iint_{D_{t_1}} \left| \mathcal{L}_1 w_1 + \frac{\partial c}{\partial r} w_1 \right| drdt.$$

Summing up such inequalities corresponding to K_i and K_i', we obtain

$$\begin{aligned}
\int_{r_1(t_1)}^{r_2(t_1)} |w_1| dr \leqq& \int_{r_1(t_0)}^{r_2(t_0)} |w_1| dr + \iint_{D_{t_1}} \left| \mathcal{L}_1 w_1 + \frac{\partial c}{\partial r} w_1 \right| drdt \\
=& I(t_0) + \iint_{D_{t_1}} \left| \mathcal{L}_1 w_1 + \frac{\partial c}{\partial r} w_1 \right| drdt.
\end{aligned} \tag{4.12}$$

It follows from (4.2), (4.3) and (4.5) that

$$\begin{aligned}
\mathcal{L}_1 w_1 + \frac{\partial c}{\partial r} w_1 =& O(\{r^{-\frac{1}{2}} |Du| \cdot |w_2| + r^{-\frac{3}{2}} |Dv| \cdot |Du| + r^{-\frac{5}{2}} |Du| \cdot |v|\} |w_1| \\
&+ r^{-\frac{5}{2}} |w_2| \cdot |Du| \cdot |v| + r^{-\frac{3}{2}} |w_2| \cdot |Du| \cdot |Dv| + r^{-2} |Dv| + r^{-3} |v|) \\
=& O((\varepsilon^4 (1+t)^{-1} + \varepsilon^2 (1+t)^{-2}) |w_1| + \varepsilon (1+t)^{-2}).
\end{aligned}$$

Note that, from (4.9), we have

$$\begin{aligned}|r_1(t)-r_2(t)| \leqq & |r_1(t)-t+N|+|t-r_2(t)+M|+M+N\\ \leqq & \frac{2C}{A}+M+N.\end{aligned}$$

Then it follows from (4.5), (4.8) and the assumption $\varepsilon^2\log(1+T)<1/A$ that

$$\begin{aligned}&\iint_{D_{t_1}}\left|\mathcal{L}_1 w_1+\frac{\partial c}{\partial r}w_1\right|drdt\\ =&O(\int_{t_0}^{t_1}(\varepsilon^4(1+t)^{-1}+\varepsilon^2(1+t)^{-2})dt\int_{r_1(t)}^{r_2(t)}|w_1|dr+\int_{t_0}^{t_1}\varepsilon(1+t)^{-2}dt\int_{r_1(t)}^{r_2(t)}dr)\\ =&O(\varepsilon^5\log(1+t_1)+(\frac{2C}{A}+M+N)\varepsilon(1+t_0)^{-1})\\ =&O(\frac{\varepsilon^3}{A}+(\frac{2C}{A}+M+N)\varepsilon^4).\end{aligned}$$

Thus we obtain

$$I(t_1)<C_3\varepsilon+O(\varepsilon^2)<\hat{C}\varepsilon,$$

for $\varepsilon<\varepsilon_A$ if ε_A is sufficiently small.

Next we estimate v_r. We fix a point $(r,t_1)\in D_{t_1}$, then there exist λ_0 and μ_0 such that $(r,t_1)\in Z^1_{\lambda_0}\cap Z^2_{\mu_0}$. Integrating the following equality

$$\mathcal{L}_1 v_r=v_{rt}+cv_{rr}=2cw_2,$$

along $Z^1_{\lambda_0}$ from t_0 to t_1, we find

$$\begin{aligned}v_r(r,t_1)-v_r(\lambda_0+t_0,t_0)=&\int_{t_0}^{t_1}\frac{d}{dt}(v_r(r(t),t))dt\\ =&\int_{t_0}^{t_1}\mathcal{L}_1 v_r(r(t),t)dt\\ =&2\int_{t_0}^{t_1}cw_2(r(t),t)dt\\ =&O(\int_{t_0}^{t_1}|w_2(r(t),t)|dt),\end{aligned}$$

where $(r(t),t)\in Z^1_{\lambda_0}$. To estimate the last integral in the above equality, we set

$$E=\{(r,t)\in D_{t_1}|(r,t)\in Z^1_\lambda\cap Z^2_\mu,\quad \lambda_0\leqq\lambda\quad\text{and}\quad \mu\leqq\mu_0\}.$$

By the same argument to obtain (4.12), we get from (4.11)

$$\int_{Z^1_\lambda}|w_2|(dr+cdt)\leqq\int_{E\cap\{t=t_0\}}|w_2|dr+\iint_E\left|\mathcal{L}_2 w_2-\frac{\partial c}{\partial r}w_2\right|drdt.$$

It follows from (4.5) and (4.6) that

$$\int_{E\cap\{t=t_0\}} |w_2|dr \leqq \int_{D_{t_0}} |w_2|dr \leqq W_2(t_0)|r_1(t_0)-r_2(t_0)| = O(\varepsilon^3).$$

The same argument to estimate the integral of $|\mathcal{L}_1 w_1 + (\partial c/\partial r)w_1|$ over D_{t_1} gives

$$\iint_E \left|\mathcal{L}_2 w_2 - \frac{\partial c}{\partial r} w_2\right| drdt \leqq \iint_{D_{t_1}} \left|\mathcal{L}_2 w_2 - \frac{\partial c}{\partial r} w_2\right| drdt = O(\varepsilon^2).$$

On the other hand, we find

$$\begin{aligned}\int_{Z^1_\lambda} |w_2|(dr+cdt) &= \int_{t_0}^{t_1} |w_2(r(t),t)|(\frac{dr}{dt}+c)dt\\ &=2\int_{t_0}^{t_1} c|w_2(r(t),t)|dt\\ &\geqq \int_{t_0}^{t_1} |w_2(r(t),t)|dt,\end{aligned}$$

for sufficiently small ε. These imply

$$\int_{t_0}^{t_1} |w_2(r(t),t)|dt = O(\varepsilon^2). \tag{4.13}$$

Thus we obtain

$$v_r(r,t) = v_r(\lambda_0+t_0,t_0) + O(\varepsilon^2), \tag{4.14}$$

and

$$|v_r(r,t_1)| \leqq \frac{C_1}{2}\varepsilon + O(\varepsilon^2) < \frac{\hat{C}}{2}\varepsilon.$$

Similarly we have

$$v_t(r,t_1) = v_t(\lambda_0+t_0,t_0) + O(\varepsilon^2), \tag{4.15}$$

and

$$|v_t(r,t_1)| < \frac{\hat{C}}{2}\varepsilon.$$

Thus $\dot{V}(t_1) < \hat{C}\varepsilon$ holds. More precisely, we have for $(r(t),t) \in Z^1_{\rho(1/B)}$,

$$\partial_r^l \partial_t^m u(r(t),t) = (-1)^m \varepsilon r^{-\frac{1}{2}} \mathcal{F}'(\rho_0) + O(\varepsilon^{\frac{5}{4}} r^{-\frac{1}{2}}) \quad \text{for} \quad l+m=1, \tag{4.16}$$

where $\rho(1/B)$ is the one in (3.23) or (3.24). Indeed, if we write $r_0 = \rho(1/B)+t_0$, (3.3) and (3.23) imply

$$\begin{aligned}r_0^{\frac{1}{2}} \partial_r^l \partial_t^m u(r_0,t_0) &= \varepsilon(-1)^m U_\rho(\rho(\frac{1}{B}),\frac{1}{B}) + O(\varepsilon^{\frac{5}{4}})\\ &= (-1)^m \varepsilon \mathcal{F}'(\rho_0) + O(\varepsilon^{\frac{5}{4}}).\end{aligned} \tag{4.17}$$

When $m = 1$ and $l = 0$, using (4.15) with $\lambda_0 = \rho(1/B)$ and (4.17) we obtain for $(r(t), t) \in Z^1_{\rho(1/B)}$

$$\begin{aligned} r^{\frac{1}{2}} u_t(r(t), t) =& r_0^{\frac{1}{2}} u_t(r_0, t_0) + O(\varepsilon^2) \\ =& -\varepsilon \mathcal{F}'(\rho_0) + O(\varepsilon^{\frac{5}{4}}). \end{aligned}$$

The other case shall be obtained by using (4.14) and (4.17).

Finally we estimate $w_2(r, t_1)$. We fix a point $(r, t_1) \in D_{t_1}$ and take a constant μ such that $(r, t_1) \in Z^2_\mu$. Then, it follows from (4.4), (4.8) and the assumption $\varepsilon^2 \log(1+T) < 1/A$ that for $(r(t), t) \in Z^2_\mu$,

$$\begin{aligned} w_2(r, t_1) - w_2(\mu - t_0, t_0) =& \int_{t_0}^{t_1} \frac{d}{dt} w_2(r(t), t) dt \\ =& \int_{t_0}^{t_1} L_2 w_2(r(t), t) dt \\ =& O(\int_{t_0}^{t_1} \{\varepsilon^7 (1+t)^{-1} + \varepsilon(1+t)^{-2} + \varepsilon^7 |w_1(r(t), t)|\} dt) \\ =& O(\varepsilon^4 + \varepsilon^7 \int_{t_0}^{t_1} |w_1(r(t), t)| dt). \end{aligned}$$

By the same argument to obtain (4.13), we have

$$\int_{t_0}^{t_1} |w_1(r(t), t)| dt = O(\varepsilon) \qquad \text{for} (r(t), t) \in Z^2_\mu.$$

This implies

$$\begin{aligned} |w_2(r, t_1)| =& |w_2(\mu - t_0, t_0)| + O(\varepsilon^4) \\ \leqq& C_3 \varepsilon^3 + O(\varepsilon^4) \\ <& C\varepsilon^3, \end{aligned}$$

for $\varepsilon < \varepsilon_A$ if ε_A is sufficiently small. Thus we have finished proving (2) and then (4.5).

5. Proof of Main Lemma.

The following lemma which corresponds to Lemma 1.4.1 in L. Hörmander [2] play an important role in the proof of (2.8).

Lemma. *Let w be a solution in $[0, T]$ of the ordinary differential equation:*

$$\frac{dw}{dt} = \alpha_0(t) w(t)^2 + \alpha_1(t) w(t) + \alpha_2(t),$$

where α_j are continuous and $\alpha_0 \geqq 0$. Let

$$K = \int_{t_0}^T |\alpha_2(t)| dt \exp(\int_{t_0}^T |\alpha_1(t)| dt).$$

If $w(t_0) > K$, it follows that

$$\int_{t_0}^{T} \alpha_0(t)dt < \frac{1}{w(t_0) - K} \exp(\int_{t_0}^{T} |\alpha_1(t)|dt).$$

By (4.3) and (4.5), we find that $w_1(r(t), t) = w_1(t)$ satisfies

$$\frac{d}{dt} w_1(t) = \alpha_0(t) w_1(t)^2 + \alpha_1(t) w_1(t) + \alpha_2(t),$$

along $Z^1_{\rho(1/B)}$, where

$$\begin{aligned}
\alpha_0(t) =& c(a_1 u_t + \frac{a_2}{2} u_r) - \frac{a_2}{2} u_t - a_3 u_r, \\
\alpha_1(t) =& O(\varepsilon^4 (1+t)^{-1} + \varepsilon^2 (1+t)^{-2}), \\
\alpha_2(t) =& O(\varepsilon (1+t)^{-2}).
\end{aligned}$$

Thus it can be easily seen that $K = O(\varepsilon^4)$. It follows from (4.5), (4.9) and (4.16) that

$$\begin{aligned}
\alpha_0(t) =& -(a_1 - a_2 + a_3)\varepsilon \mathcal{F}'(\rho_0) r^{-1} + O(\varepsilon^{\frac{5}{4}} r^{-1}) \\
=& -(a_1 - a_2 + a_3)\varepsilon \mathcal{F}'(\rho_0)(1+t)^{-1} + O(\varepsilon^{\frac{5}{4}}(1+t)^{-1} + (\frac{1}{r} - \frac{1}{1+t})\varepsilon) \\
=& -(a_1 - a_2 + a_3)\varepsilon \mathcal{F}'(\rho_0)(1+t)^{-1} + O(\varepsilon^{\frac{5}{4}}(1+t)^{-1}).
\end{aligned}$$

Since $H_0 = -(a_1 - a_2 + a_3)\mathcal{F}'(\rho_0)\mathcal{F}''(\rho_0) > 0$, we can assume without loss of generality that $-(a_1 - a_2 + a_3)\mathcal{F}'(\rho_0) > 0$ and $\mathcal{F}''(\rho_0) > 0$. This assumption guarantees $\alpha_0(t) > 0$ for sufficiently small ε. On the other hand, by (3.5) and (4.5)

$$\begin{aligned}
w_1(t_0) =& \frac{c v_{rr}(t_0) - v_{rt}(t_0)}{2c} \\
=& \frac{1}{2} v_{rr}(t_0) - v_{rt}(t_0) + O(|D^2 v||Du|^2) \\
=& \frac{1}{2} r_0^{\frac{1}{2}} u_{rr}(t_0) - \frac{1}{2} r_0^{\frac{1}{2}} u_{rt}(t_0) + \frac{1}{2} r_0^{-\frac{1}{2}}(t_0) - \frac{1}{4} r_0^{-\frac{1}{2}} u_t(t_0) \\
&+ \frac{1}{8} r_0^{-\frac{3}{2}} u + O(\varepsilon^6) \\
=& \frac{1}{2} r_0^{\frac{1}{2}} u_{rr}(t_0) - \frac{1}{2} r_0^{\frac{1}{2}} u_{rt}(t_0) + O(\varepsilon^4),
\end{aligned}$$

where $r_0 = t_0 + \rho(1/B)$. Using (3.2b), we obtain

$$w_1(t_0) = \varepsilon U_{\rho\rho}(\rho(\frac{1}{B}), \frac{1}{B}) + O(\varepsilon^{\frac{5}{4}}).$$

By (3.24), we have $U_{\rho\rho}(\rho(1/B), 1/B) > 0$ and therefore $w_1(t_0) > K$. Thus, applying Lemma to $w = w_1$, we find that T must satisfy

$$\int_{t_0}^{T} \alpha_0(t)dt < \frac{1}{w_1(t_0) - K} \exp(\int_{t_0}^{T} |\alpha_1(t)|dt).$$

Using the estimates for α_0, α_1 and w_1, we have

$$-(a_1-a_2+a_3)\mathcal{F}'(\rho_0)\varepsilon(\log(1+T)-\log(1+t_0))+O(\varepsilon^{\frac{5}{4}}\log(1+T))$$
$$<\frac{1}{\varepsilon U_{\rho\rho}(\rho(\frac{1}{B}),\frac{1}{B})+O(\varepsilon^{\frac{5}{4}})}(1+O(\varepsilon^2)),$$

which implies

$$\varepsilon^2\log(1+T)-\frac{1}{B}<\frac{1}{-(a_1-a_2+a_3)\mathcal{F}'(\rho_0)U_{\rho\rho}(\rho(\frac{1}{B}),\frac{1}{B})}+O(\varepsilon^{\frac{1}{4}}\frac{1}{A}).$$

By (3.24) we have

$$\begin{aligned}\varepsilon^2\log(1+T)-\frac{1}{B}&<\frac{1}{H}-\frac{1}{B}+O(\varepsilon^{\frac{1}{8}})\\&<\frac{1}{A}-\frac{1}{B}\qquad\text{for}\qquad 0<\varepsilon<\varepsilon_A,\end{aligned}$$

i.e.,

$$\varepsilon^2\log(1+T)<\frac{1}{A}\qquad\text{for}\qquad 0<\varepsilon<\varepsilon_A.$$

This completes the proof of Main Lemma.

Appendix.

It remains to prove

$$|\partial_\rho^l\partial_s^m U(\rho,s)|\leqq C_{l,m,B}(1+|\rho|)^{-\frac{1}{2}-l-4m},\qquad\text{for}\quad 0\leqq s\leqq\frac{1}{B},\tag{3.8}$$

$$U(\rho,s)=0\qquad\text{for}\qquad\rho\geqq M,\tag{3.9}$$

for the solution $U(\rho,s)$ of the initial value problem (3.1a), (3.2a). Along the same argument to obtain (3.23) we get for $(\rho(s),s)\in\Lambda_q$

$$U_\rho(\rho(s),s)=U_\rho(q,0)=\mathcal{F}'(q),\quad\text{for}\quad 0\leqq s\leqq\frac{1}{B}.\tag{A.1}$$

Hence, by the definition of characteristic curves Λ_q, $\rho(s)$ can be written as

$$\rho(s)=q+\frac{a_1-a_2+a_3}{2}(\mathcal{F}'(q))^2s,\quad\text{for}\quad 0\leqq s\leqq\frac{1}{B}.\tag{A.2}$$

On the other hand, it has been known that $\mathcal{F}$ satisfies

$$\left|\frac{d^k}{d\rho^k}\mathcal{F}(\rho)\right|\leqq\widetilde{C}_k(1+|\rho|)^{-\frac{1}{2}-k}\qquad\text{for}\qquad\rho\in\mathbb{R},\tag{A.3}$$

$$\mathcal{F}(\rho)=0 \qquad \text{for} \qquad \rho \geqq M, \tag{A.4}$$

(e.g. L. Hörmander [2]). Then we have

$$\left|\frac{a_1-a_2+a_3}{2}(\mathcal{F}'(q))^2 s\right| \leqq \frac{|a_1-a_2+a_3|\widetilde{C}_1^2}{2B}=C_1',$$

where the last inequality is the definition of C_1'. At first we prove (3.8) for $l=1$ and $m=0$. When $|\rho(s)| \leqq 2C_1'$, we find that for $(\rho(s),s)\in\Lambda_q$

$$\begin{aligned}|U_\rho(\rho(s),s)| =&|\mathcal{F}_\rho(q)| \leqq \widetilde{C}_1 \leqq \widetilde{C}_1(1+2C_1')^{\frac{3}{2}}(1+2C_1')^{-\frac{3}{2}}\\ \leqq&\widetilde{C}_1(1+2C_1')^{\frac{3}{2}}(1+|\rho|)^{-\frac{3}{2}}.\end{aligned}$$

When $|\rho(s)| \geqq 2C_1'$, it follows from (A.2) that

$$|q|=\left|\rho(s)-\frac{a_1-a_2+a_3}{2}(\mathcal{F}'(q))^2 s\right| \geqq |\rho|-C_1' \geqq \frac{1}{2}|\rho|.$$

Thus we obtain

$$|U_\rho(\rho(s),s)|=|\mathcal{F}'(q)| \leqq \widetilde{C}_1(1+|q|)^{-\frac{3}{2}} \leqq 2\sqrt{2}\widetilde{C}_1(1+|\rho|)^{-\frac{3}{2}}.$$

Therefore if we take $C_{1,0,B}=\widetilde{C}_1(1+2C_1')^{3/2}+2\sqrt{2}\widetilde{C}_1$, we find that (3.8) is valid for $l=1$ and $m=0$. When $l=0$ and $m=0$, (3.1a) and (3.2a) imply that for any $(\rho,s)\in\mathbb{R}\times[0,1/B]$

$$\begin{aligned}U(\rho,s) =&U(\rho,0)+\int_0^s \frac{\partial}{\partial s}U(\rho,s)ds\\ =&\mathcal{F}(\rho)-\frac{a_1-a_2+a_3}{6}\int_0^s (U_\rho(\rho,s))^3 ds.\end{aligned}$$

Thus we obtain

$$\begin{aligned}|U(\rho,s)| \leqq&\widetilde{C}_0(1+|\rho|)^{-\frac{1}{2}}+\frac{|a_1-a_2+a_3|}{6B}C_{1,0,B}^3(1+|\rho|)^{-\frac{9}{2}}\\ \leqq&(\widetilde{C}_0+\frac{|a_1-a_2+a_3|}{6B}C_{1,0,B}^3)(1+|\rho|)^{-\frac{1}{2}}.\end{aligned}$$

This implies that (3.8) is valid for $l=0$ and $m=0$ if we take

$$C_{0,0,B}=\widetilde{C}_0+\frac{|a_1-a_2+a_3|}{6B}C_{1,0,B}^3.$$

Next we prove (3.8) for general $l \geqq 2$ and $m=0$. Let s $(0 \leqq s \leqq 1/B)$ be fixed arbitrary. Then for any point (ρ,s), there exist a smooth curve $q=q_s(\rho)$ such that $(\rho,s)\in\Lambda_q$. Differentiating (A.1) with respect to ρ, we find that for $l \geqq 2$

$$\partial_\rho^l U(\rho,s)=\sum_{j=1}^{l-1}\mathcal{F}^{(j+1)}(q)\sum_{m(j)\in X} C_{m(j)}(\frac{\partial q}{\partial\rho})^{m_1(j)}(\frac{\partial^2 q}{\partial\rho^2})^{m_2(j)}\cdots(\frac{\partial^{l-1}q}{\partial\rho^{l-1}})^{m_{l-1}(j)}, \tag{A.5}$$

where

$$X = \{m(j) \in \mathbb{Z}_+^{l-1} |\ m_1(j) + m_2(j) + \cdots + m_{l-1}(j) = j,\\ m_1(j) + 2m_2(j) + \cdots + (l-1)m_{l-1}(j) = l-1\}.$$

On the other hand, differentiating $\partial q/\partial \rho = (\partial \rho/\partial q)^{-1}$ with respect to ρ, we find that for $k \geqq 2$

$$\frac{\partial^k q}{\partial \rho^k} = \sum_{j=2}^{k} \frac{\partial^j \rho}{\partial q^j} \sum_{N(j)\in Y} C_{N(j)} (\frac{\partial q}{\partial \rho})^{N_1(j)} (\frac{\partial^2 q}{\partial \rho^2})^{N_2(j)} \cdots (\frac{\partial^{k-1} q}{\partial \rho^{k-1}})^{N_{k-1}(j)}, \tag{A.6}$$

where

$$Y = \{N(j) \in \mathbb{Z}_+^{k-1} |\ N_1(j) + N_2(j) + \cdots + N_{k-1}(j) = j+1,\\ N_1(j) + 2N_2(j) + \cdots + (k-1)N_{k-1}(j) = k+1\}.$$

Moreover by (A.2), (A.3) and the same argument in the case $l = 1$ and $m = 0$, we obtain

$$\begin{aligned} \left|\frac{\partial \rho}{\partial q}\right| &\leqq \hat{C}_1, \\ \left|\frac{\partial^k \rho}{\partial q^k}\right| &\leqq \hat{C}_k (1+|\rho|)^{-3-k} \quad \text{for} \quad k \geqq 2. \end{aligned} \tag{A.7}$$

Using (A.7), we get

$$\begin{aligned} \left|\frac{\partial q}{\partial \rho}\right| &\leqq \overline{C}_1, \\ \left|\frac{\partial^k q}{\partial \rho^k}\right| &\leqq \overline{C}_k (1+|\rho|)^{-3-k} \quad \text{for} \quad k \geqq 2. \end{aligned} \tag{A.8}$$

Thus it follows from (A.5) and (A.8) that

$$\begin{aligned} |\partial_\rho^l U(\rho,s)| &\leqq C_{l,B} \sum_{j=2}^{l-1} (1+|\rho|)^{-\frac{1}{2}-j-1} \prod_{k=2}^{l-1} (1+|\rho|)^{(-k-3)m_k(j)} \\ &\leqq C_{l,B} \sum_{j=1}^{l-1} (1+|\rho|)^{-\frac{1}{2}-j-1} (1+|\rho|)^{-\sum_{k=2}^{l-1}(k-1)m_k(j) - 4\sum_{k=2}^{l-1} m_k(j)}. \end{aligned}$$

Since

$$m_2(j) + 2m_3(j) + \cdots + (l-2)m_{l-1}(j) = l - j - 1,\\ m_2(j) + m_3(j) + \cdots + m_{l-1}(j) \geqq 0,$$

we have

$$\begin{aligned} |\partial_\rho^l U(\rho,s)| &\leqq C_{l,0,B} (1+|\rho|)^{-\frac{1}{2}-j-1-l+1+j} \\ &\leqq C_{l,0,B} (1+|\rho|)^{-\frac{1}{2}-l}. \end{aligned}$$

Next we assume that (3.8) holds for any l and $0 \leqq m \leqq k-1$. Differentiating the equation (3.1a), we have

$$\partial_\rho^l \partial_s^k U(\rho,s) = \sum C \partial_\rho^{\alpha_1}\partial_s^{1+\beta} U(\rho,s)\partial_\rho^{\alpha_2}\partial_s^{1+\beta_2} U(\rho,s)\partial_\rho^{\alpha_3}\partial_s^{1+\beta_3} U(\rho,s),$$

where

$$\alpha_1+\alpha_2+\alpha_3 = l \quad \text{and} \quad \beta_1+\beta_2+\beta_3 = k-1.$$

Thus we have

$$\begin{aligned} |\partial_\rho^l \partial_s^k U(\rho,s)| &\leqq C_{l,k,B}(1+|\rho|)^{-\frac{3}{2}-4(k-1)-3-l} \\ &\leqq C_{l,k,B}(1+|\rho|)^{-\frac{1}{2}-4k-l}. \end{aligned}$$

This comletes the proof of (3.8).

Finally we prove (3.9). If $\rho \geqq M$ and $(\rho,s) \in \Lambda_q$, we find $q \geqq M$ because of the uniqueness of Λ_q. It follows from (A.2) and (A.4) that

$$U_\rho(\rho,s) = \mathcal{F}'(q) = 0 \quad \text{for} \quad \rho \geqq M,\ 0 \leqq s \leqq \frac{1}{B}.$$

Thus we have

$$U(\rho,s) = 0 \qquad \text{for} \qquad \rho \geqq M,\ 0 \leqq s \leqq \frac{1}{B},$$

which implies (3.9).

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