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Series ♯219. November 1993

HOKKAIDO UNIVERSITY

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On instability of evolving hypersurfaces

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In the memory of Professor Peter Hess

ABSTRACT. A general parabolic evolution equation is considered for a closed hypersurface in Euclidean space. All stationary solutions are shown to be Lyapunov unstable if the normal velocity of a hypersurface depends only on its normal and second fundamental form and is independent of its position. Instability of time periodic solution is also discussed.

1. Introduction. We consider the initial value problem of an evolution of a closed hypersurface Γ_t in $\mathbb{R}^n$

$$V = f(\mathbf{n}, -\mathrm{A}). \tag{1}$$

Here $\mathbf{n}$ is an inward unit normal vector field on Γ_t and V is normal velocity in the direction of $\mathbf{n}$; $\mathrm{A} = -d\,\mathbf{n}$ denotes the second fundamental form. We shall prove that all stationary solutions S of (1) is Lyapunov unstable provided that (1) is (nondegenerate) parabolic. This generalized a recent work of Ei and Yanagida [EY] where they assumed that f depends on A only through its mean curvature. Their method is completely different from ours. They linearized equations around stationary solutions and appealed to spectral analysis. Their method also applies to an equation depending on space variables but invariant under translation while ours does not apply to such an equation. We simply use a distance function of S and appeals to the strong maximum principle. We believe our proof is simpler than theirs for (1) with A replaced by mean curvature.

Our method also applies to instability of periodic solutions of

$$V = f(t, \mathbf{n}, -\mathrm{A}), \tag{2}$$

*Partially supported by the Inamori Foundation

AMS Subject Classifications: 35B10, 35B35, 35K65 where f is time periodic. We show that periodic solutions S_t are unstable unless second fundamental form vanishes somewhere on S_t for all t. As an application, we show that the curvature of periodically evolving curves in the plane are unstable.

2. Parabolic evolution equations. We formulate our equations as in [GG1]. Let E be a bundle over the unit sphere S^{n-1} of the form

$$E = \{ \ (\bar{p}, Q_{\bar{p}}(X)) \in S^{n-1} \times \mathbb{S}_n; X \in \mathbb{S}_n \ \}$$

with $Q_{\bar{p}}(X) = R_{\bar{p}} X R_{\bar{p}}$ and $R_{\bar{p}} = I - \bar{p} \otimes \bar{p}$; $R_{\bar{p}}$ is the projection orthogonal to $\bar{p}$. Here $\mathbb{S}_n$ denotes the space of the $n \times n$ real symmetric matrices. By the standard Euclidean metric the bundle E is identified with the tensor bundle $\mathrm{T}\, S^{n-1} \otimes \mathrm{T}^* \, S^{n-1}$ over S^{n-1}. Let f be a function from $[0, \infty) \times E$ to $\mathbb{R}$. We shall always assume that f is at least continuous. Suppose that a hypersurface Γ is given as a zero level set of u in $\mathbb{R}^n$ such that the gradient $\nabla u \neq 0$ on Γ and $\mathbf{n} = \nabla u / |\nabla u|$. Then as in [GG1], the second fundamental form (in the direction of $\mathbf{n}$) is of the form

$$\mathrm{A} = -Q_{\bar{p}}(\nabla^2 u)/|\nabla u| \qquad \text{with} \qquad \bar{p} = \nabla u / |\nabla u|, \tag{3}$$

where $\nabla^2 u$ denotes the Hessian of u in space variables.

We recall the notion of parabolicity of the equation

$$V = f(t, \mathbf{n}, -\mathrm{A}) \tag{4}$$

for evolving hypersurface Γ_t. It is convenient to introduce the level set equation

$$u_t + F_f(t, \nabla u, \nabla^2 u) = 0 \tag{5}$$

$$\text{with} \qquad F_f(t, p, X) = |p| f(t, p/|p|, Q_{\bar{p}}(X)/|p|). \tag{6}$$

This equation is uniquely determined if each level set of u moves by (4) and a super level set $u > c$ is "inside" the level set $u = c$. Signatures in (6) are different from those in [CGG], [GG1] because our convention of $\mathbf{n}$ is opposite.

We say (4) is *strictly parabolic* (uniformly in t) if for each $M > 0$ there is $\mu > 0$ such that

$$F_f(t, \bar{p}, X + Y) - F_f(t, \bar{p}, X) \leq -\mu \, \mathrm{trace}(Q_{\bar{p}}(Y)) \tag{7}$$

for all $Y \geq 0$, $|X| \leq M$, $|\bar{p}| = 1$, $t \in [0, \infty)$, where $|X|$ is the operator norm of X as a self adjoint operator. If (7) holds for $\mu = 0$ we say (4) is (degenerate) *parabolic*. A level set method [CGG], [ES] provides a unique global generalized solution. The following version is taken from [GG1].

2.1. Unique global existence. *Suppose that (4) is parabolic. Let* Γ_0 *be the boundary of a bounded open set in* $\mathbb{R}^n$. *Then there is a unique generalized solution* $\{\Gamma_t\}_{t \geq 0}$ *of (4) starting from* Γ_0.

If f and Γ_0 are smooth enough and (4) is parabolic, there is a local-in - time classical solution Σ_t (see e.g. [GG2]). Moreover Σ_t agrees with Γ_t as far as the former exists [GG2] (see also [ES] for the mean curvature flow). So our generalized solution is a natural extention of classical solution.

3. Instability of stationary solutions. We say that a C^2 hypersurface S is *stationary* for

$$V = f(\mathbf{n}, -\mathrm{A}) \tag{8}$$

if $f(\mathbf{n}, -\mathrm{A}) = 0$ on S. Let $U(\alpha)$ denote a tubular neighborhood of S of the form

$$U(\alpha) = \{\, x \in \mathbb{R}^n;\ \mathrm{dist}(x, S) < \alpha \,\},$$

where dist denotes the distance. We say that S is *Lyapunov stable* for (8) if for each $\epsilon > 0$ there is $\delta > 0$ such that a (generalized) solution Γ_t with initial data Γ_0 stays in $U(\epsilon)$ for all $t > 0$ provided that Γ_0 is contained in $U(\delta)$. If not, S is called *unstable.* If α is a supremum of α' such that every point of $U(\alpha')$ has a unique nearest point on S, α is called the *reach* of S and dented by reach S.

3.1. Instability Theorem. *Suppose that (8) is strictly parabolic. Let S be a stationary C^2 hypersurface of (8) such that $S = \partial D$ for some open set D in $\mathbb{R}^n$. Suppose that* reach $S = \alpha_0 > 0$, $\inf_S |\mathrm{A}| = \sigma > 0$ *and that* $|\mathrm{A}|$ *is bounded on S (if S is not compact). For* $0 < \alpha < \alpha_0$ *let*

$$\Gamma^\alpha = \{\, x \in D; \mathrm{dist}(x, S) = \alpha \,\}. \tag{9}$$

Let Γ_t^α be a (generalized) solution of (8) starting from Γ^α. Then there is $\alpha_1 = \alpha_1(S)$ ($0 < \alpha_1 < \alpha_0$) such that

$$\operatorname{dist}(\Gamma_t^\alpha, S) \geq \alpha(1 + c_0 t) \wedge \alpha_1 \quad \textit{for all } t > 0 \tag{10}$$

with $c_0 > 0$ depending only on f and σ, where $a \wedge b = \min(a, b)$. The same inequality holds if D in (9) is replaced by its complement.

3.2. Corollary. *Suppose that S is a stationary C^2 closed hypersurface for (8) with nonvanishing second fundamental form. Then S is (Lyapunov) unstable provided that (8) is strictly parabolic.*

Of course, Lyapunov instability follows from (10).

3.3. Remark on noncompact surface. Even for an unbounded open set generalized solution can be constructed by the level sets method; see Ilmanen [I] and Ishii and Souganidis [IS].

Following formula for distance function is a key for the proof of Theorem 3.1. Let v be the signed distance function of S, *i.e.*,

$$v(x) = \begin{cases} \operatorname{dist}(x, S) & \text{for } x \in D \\ -\operatorname{dist}(x, S) & \text{otherwise.} \end{cases}$$

If S is C^2, then so is v which is proved in [GT; §14].

3.4. Lemma. *For a general C^2 hypersurface $S = \partial D$ with an open set D*

$$\nabla^2 v(y)(I - v(y)\nabla^2 v(y))^{-1} = \nabla^2 v(x),$$

$$\nabla v(y) = \nabla v(x), \quad y = x + v\mathbf{n}, \ x \in S,$$

for $|v| < \operatorname{reach} S$.

This is also a key in [GG2], where the local existence of classical solution is proved for (2) by a level set method; see [GT; §14 Appendix].

3.5. Proof of Theorem 3.1. We set

$$w(t,x) = v(x) - \rho(t) \quad \text{with} \ \ \rho(t) = \alpha(1 + c_0 t).$$

Our goal is to take $\alpha_1 > 0$ and $c_0 > 0$ so that w is a supersolution of the level set equation of (8):

$$u_t + F(\nabla u, \nabla^2 u) = 0, \qquad F = F_f$$

in a set $U_+(\alpha_1) \setminus \overline{U_+(\alpha/2)}$ for $t > 0$ provided $\alpha \leq \alpha_1$, where

$$U_+(\alpha) = \{ x \in D; \mathrm{dist}(x, S) < \alpha \}.$$

If such a c_0 exists, comparison principle ([CGG], [GG1]) implies that Γ_t^α is contained in $\{w \geq 0\}$. This yields (10) for $\alpha \leq \alpha_1$.

Since S is stationary, we see

$$F(\nabla v, \nabla^2 v) = 0 \quad \text{ on } S.$$

By Lemma 3.4 this yields

$$F(\nabla v, \nabla^2 v(I - v\nabla^2 v)^{-1}) = 0 \quad \text{in } U(\alpha_0).$$

This implies

$$w_t + F(\nabla w, \nabla^2 w) = -c_0\alpha + F(\bar{p}, X) - F(\bar{p}, X(I - vX)^{-1}) \quad \text{in } U_+(\alpha_0). \tag{11}$$

with $\bar{p} = \nabla v$, $X = \nabla^2 v$.

We take $\alpha_1 > 0$ small so that

$$|\nabla^2 v| \geq \sigma/2 \qquad \text{in } \ U_+(\alpha_1), \tag{12}$$

$$(I - v\nabla^2 v)^{-1} \geq I/2 \quad \text{in } \ U_+(\alpha_1). \tag{13}$$

This is possible even if S is noncompact by our assumptions of A and the formula

$$\nabla^2 v(x) = \nabla^2 v(y)(I + v\nabla^2 v(y))^{-1}, \qquad y = x + v\mathrm{n},\, x \in S,$$

which follows from Lemma 3.4.

Since $X\bar{p} \otimes \bar{p} = 0$ by $|\nabla v| = 1$, we see

$$Q_{\bar{p}}[X^2(I - vX)^{-1}] = X^2(I - vX)^{-1} = -\frac{1}{v}(X - X(I - vX)^{-1}).$$

By the strict parabolicity we obtain

$$\begin{aligned} F(\bar p, X) - F(\bar p, X(I - vX)^{-1}) &\ge \mu v \operatorname{trace} Q_{\bar p}[X^2(I - vX)^{-1}] \\ &= \mu v \operatorname{trace} X^2(I - vX)^{-1} \quad \text{in } U_+(\alpha_1). \end{aligned}$$

Using (12), (13) we see

$$\operatorname{trace} X^2(I - vX)^{-1} \ge \frac{1}{2}\left(\frac{\sigma}{2}\right)^2 = c_1$$

which yields

$$F(\bar p, X) - F(\bar p, X(I - vX)^{-1}) \ge \mu c_1 v \quad \text{in } U_+(\alpha_1).$$

If we set $c_0 = \mu c_1/2$, from (11) it follows that w is a classical supersolution of the level set equation of (8) in $U_+(\alpha_1) \setminus \overline{U_+(\alpha/2)}$.

If we set $c_0 = 0$, by (10) $v - \alpha$ is a supersolution (in $U_+(\alpha_1)$) so that $\operatorname{dist}(\Gamma_t^\alpha, S) \ge \alpha$. Thus (10) holds for $\alpha \ge \alpha_1$.

The proof for the last statement is parallel, so is omitted.

3.6. General instability Theorem. *For (8) there is no stable stationary C^2 closed hypersurface provided that (8) is strictly parabolic and that f is C^1.*

Proof. By Corollary 3.2 we may assume that there is a point on S at which

$$(\mathbf{n}, \mathbf{A}) = (\bar p_0, O),$$

for some $\bar p_0 \in S^{n-1}$. Since S is stationary,

$$f(\bar p_0, O) = 0.$$

The following lemma implies the nonexistence of closed stationary solution, so the proof is complete.

3.7. Nonexistence Lemma. *Suppose that (8) is strictly parabolic and f is C^1. Suppose that $f(\bar{p}_0, O) = 0$ for some $\bar{p}_0 \in S^{n-1}$. Then there is no stationary C^2 closed hypersurface for (8).*

Proof. Let S be a stationary C^2 closed hypersurface. Since S is compact, there is a half space H such that

$$H = \{\, x + c \in \mathbb{R}^n; \quad x \cdot \bar{p}_0 \geq 0 \,\}, \qquad S \subset H \text{ with } c \in S.$$

Note that ∂H is a stationary solution of (8) since $f(\bar{p}_0, O) = 0$. Since (8) is strictly parabolic and f is C^1 we may apply the strong maximum principle [PW] and conclude that S cannot touch ∂H for $t > 0$. This contradicts the existence of stationary closed hypersurface S.

3.8. Remark. If f is C^1 in A , the parabolicity is equivalent to say that $\partial f / \partial \mathrm{A}$ is positive definite.

4. Instability of periodic solutions.

We consider

$$V = f(t, \mathrm{n}, -\mathrm{A}), \tag{14}$$

where $f : [0,T] \times E \to \mathbb{R}$ is continuous and T-periodic, *i.e.* $f(t, \bar{p}, -\mathrm{A}) = f(t + T, \bar{p}, -\mathrm{A})$. We say $S_t\,(-\infty < t < \infty)$ is a T-periodic $C^{2,1}$ solution of (14) such that $S_t = S_{t+T}$ where $C^{2,1}$ implies that C^2 in space and C^1 in time. Note that the signed distance function v of S_t is now a $C^{2,1}$ function.

Let $U(\alpha, t)$ denote the α-tubular neighborhood of S_t. We say S_t is *Lyapunov stable* for (14) if for each $\epsilon > 0$ there is $\delta > 0$ such that a generalized solution Γ_t with $\Gamma_t|_{t=t_0} = \Gamma_0$ always stays in $U(\epsilon, t)$ for all $t > t_0$ provided that $\Gamma_{t_0} \subset U(\delta, t_0)$.

In some cases S_t is called T-periodic even if S_t is T-periodic up to translation $\mathrm{a} \in \mathbb{R}^n$, *i.e.*, $S_{t+T} = S_t + \mathrm{a}$. If we set

$$\Sigma_t = S_t + (t/T)\mathrm{a},$$

then $\Sigma_t = \Sigma_{t+T}$ and Σ_t solves

$$V = f + (\mathrm{n} \cdot \mathrm{a})/T.$$

This equation is included in (14). Thus by T-periodic solution we shall always mean T-periodic without ambiguity of translation.

4.1. Instability Theorem. *Suppose that (14) is strictly parabolic. Let S_t be a T-periodic C^2 solution of (8) of closed hypersurfaces surrounding an bounded open set D in $\mathbb{R}^n$. Suppose that* $\inf_{S_t} |\mathrm{A}| = a(t) \not\equiv 0$. *Let $\alpha_0 > 0$ denote the minimum of* reach S_t *in t. For $0 < \alpha < \alpha_0$ and $t_0 \in \mathbb{R}$ set*

$$\Gamma^\alpha = \{x \in D\,;\, \mathrm{dist}(x, S_{t_0}) = \alpha\}.$$

Let Γ_t^α be a (generalized) solution of (14) for $t > t_0$ with $\Gamma_t^\alpha = \Gamma^\alpha$ at $t = t_0$. Then there is α_1 (depending only on periodic solution) $(0 < \alpha_1 < \alpha_0)$ such that

$$\mathrm{dist}(\Gamma_t^\alpha, S_t) \geq \alpha(1 + \int_{t_0}^t c_0' a^2(\tau)\, \mathrm{d}\,\tau) \wedge \alpha_1 \qquad \textit{for all } t > t_0$$

with some positive constant c_0' depending only on f, S_t. The same inequality holds if D in the definition of Γ^α replaced by $\mathbb{R}^n \setminus D$.

Proof. Since

$$\{\,(t, x)\,;\, x \in S_t,\ t \in K\,\} \qquad \text{with a compact interval } K$$

is compact, it is easy to see that a is continuous. We may assume $t_0 = 0$. As in the proof of Theorem 3.1 it suffices to prove that there are $\alpha_1 > 0$ and $c_0' > 0$ so that

$$w(t, x) = v(t, x) - \rho(t), \qquad \rho = \alpha(1 + c_0' \int_0^t a(\tau)^2\, \mathrm{d}\tau)$$

is a supersolution of the level set equation of (14) in a set $W_+(\alpha_1) \setminus \overline{W_+(\alpha/2)}$ provided $\alpha \leq \alpha_1$. Here v denotes the signed distance function of S_t and

$$W_+(\alpha) = \{\,(t, x)\,;\, \mathrm{dist}(x, S_t) < \alpha,\ x \in D\,\} \subset [0, \infty) \times D.$$

Since S_t solves (14), v solves the level set equation of (14):

$$v_t + F_f(t, \nabla v, \nabla^2 v) = 0 \qquad \text{on} \quad S_t.$$

By Lemma 3.4 and the relation

$$v_t(t, x + v\mathrm{n}) = v_t(t, x) \qquad \text{for} \quad x \in S_t$$

this implies

$$v_t + F(t, \nabla v, \nabla^2 v(I - v\nabla^2 v)^{-1}) = 0 \qquad \text{in } W_+(\alpha_0) \text{ with } F = F_f.$$

As in the proof of Theorem 3.1 this yields

$$\begin{aligned} w_t + F(t, \nabla w, \nabla^2 w) &\geq -c_0'\alpha a^2 + \mu v \,\mathrm{trace}(X^2(I - vX)^{-1}) \qquad \text{in } W_+(\alpha_1) \\ &\geq -c_0'\alpha a^2 + \mu v a^2/8 \end{aligned}$$

by taking α_1 small so that (12) with $\sigma = a$ and (13) hold on $W_+(\alpha_1)$. We now obtain w is a supersolution of (14) in a set $W_+(\alpha_1)\setminus\overline{W_+(\alpha/2)}$ with $c_0' = \mu/16$. The proof is now complete.

4.2. Corollary.*Suppose that (14) is strictly parabolic. If S_t is a T-periodic $C^{2,1}$ solution of (14) consisting of closed hypersurfaces, then S_t is Lyapunov unstable if* $\inf_{S_t} |\mathrm{A}| \not\equiv 0$ *as a function of time.*

5. Instability of curvature of periodically evolving convex curve. Suppose that a closed embedded curve Γ in the plane has a positive curvature k everywhere. Then Γ is parametrized by

$$\mathrm{X}(\theta) = \mathrm{X}_0 + \int_0^\theta \tau(\rho) k^{-1}(\rho)\, \mathrm{d}\,\rho \qquad 0 \leq \theta \leq 2\pi$$

with $\tau(\rho) = (\sin\rho, -\cos\rho)$. This expression is called the Gauss parametrization because θ equals the argument of (inward) normal vector at $\mathrm{X}(\theta)$. It is easy to see that the curvature $\mathrm{d}\theta/\mathrm{d}s$ of X equals k , where s denotes the arc length parameter.

Consider an evolution equation of a plane curve Γ_t of form

$$V = k - q(t, \mathrm{n}), \tag{E}$$

where k is the curvature and q is given. Suppose that k is always positive so that the Gauss parametrization is available. Then evolution equation for the curvature $k(t,\theta)$ becomes

$$k_t = k^2(k_{\theta\theta} + k - f) \tag{K1}$$

with

$$f = \left(\left(\frac{\mathrm{d}}{\mathrm{d}\,\theta}\right)^2 + 1\right) q(t, \cos\theta, \sin\theta).$$

Since Γ_t is closed, k should satisfy the constraint

$$\int_0^{2\pi} \frac{\tau(\rho)}{k(\rho)}\,\mathrm{d}\,\rho = 0 \tag{K2}$$

and the periodic boundary condition

$$k(t,\theta) = k(t, \theta + 2\pi) \tag{K3}$$

see e.g. [GM].

We are interested in instability of time periodic solutions of (K1)-(K3) assuming that q is C^2 and T-periodic, *i.e.*, $q(t,\mathbf{n}) = q(t+T,\mathbf{n})$ for some $T > 0$ so that f is continuous and T-periodic.

5.1. Lemma. *There is no Lyapunov stable T-periodic $C^{2,1}$ solution curve S_t of (E) whose curvature is always positive.*

This follows immediately from Corollary 4.2. In [GM] a periodic positive solution of (K1)-(K3) is constructed provided that f is positive, continuous and that

$$\int_0^{2\pi} f(t,\theta)\tau(\theta)\,\mathrm{d}\,\theta = 0.$$

We would like to discuss whether such a T-periodic solution k is unstable. We say a T- periodic solution k is *(Lyapunov) stable* if for each $\epsilon > 0$ and $0 < t_0 < T$ there is $\delta > 0$ such that

$$\|h - k\|(t_0) < \delta \qquad \text{implies} \quad \|h - k\|(t) < \epsilon.$$

for all $t > t_0$ where h solves (K1)-(K3); here $\|g\|$ denotes the maximum norm of g as a function of θ.

5.2. Instability Theorem. *Suppose that k is a T-periodic, positive solution of (K1)-(K3). Then k is Lyapunov unstable. Here f is assumed to be continuous.*
By a solution we mean that all derivatives appeared in (K1) is in L^p for all $p > 1$ so that the solution itself is continuous. We shall derive Theorem 5.2 from Theorem 4.1. Note that a bound on curvature does not imply that bound of curve itself because of freedom of translation so Theorem 5.2 is not a trivial corollary of Lemma 5.1 (or even of Theorem 4.1).

5.3. Proposition. *Let X_i be a Gauss parametrization of a closed, convex curve in the plane with a curvature k_i, $(i = 1, 2)$. If $\mathrm{X}_1(0) = \mathrm{X}_2(0)$, then*

$$\|\mathrm{X}_1 - \mathrm{X}_2\| \leq \|k_1 - k_2\| (\inf_\theta k_1 k_2)^{-1}.$$

This follows immediately from the parametrization of X_i.

5.4. Lemma. *Suppose that Γ_i is an embedded closed curve parametrized $\mathrm{X}_i(\theta)$ $i = 1, 2$ $(0 \leq \theta < 2\pi)$. Then the Hausdorff distance $\mathrm{d_H}(\Gamma_1, \Gamma_2)$ is dominated by $\|\mathrm{X}_1 - \mathrm{X}_2\|$.*

Proof. Since

$$\mathrm{dist}(\mathrm{X}_i(\theta), \Gamma_j) \leq |\mathrm{X}_1(\theta) - \mathrm{X}_2(\theta)|, \qquad (i, j = 1, 2)$$

we have

$$\begin{aligned} \mathrm{d_H}(\Gamma_1, \Gamma_2) &\equiv \max \left\{ \sup_{x_1 \in \Gamma_1} \mathrm{dist}(x_1, \Gamma_2), \sup_{x_2 \in \Gamma_2} \mathrm{dist}(x_2, \Gamma_1) \right\} \\ &\leq \|\mathrm{X}_1 - \mathrm{X}_2\|. \end{aligned}$$

5.5. Lemma. *Suppose that Γ_1 and Γ_2 are two embedded closed curves in the plane. Suppose that Γ_1 encloses Γ_2 and that*

$$\inf \{ \mathrm{dist}(x_1, \Gamma_2) ; x \in \Gamma_1 \} \leq d_0.$$

Then for arbitrary translation $\Gamma_2 + \mathbf{c}$ *of* Γ_2

$$\mathrm{d_H}(\Gamma_1, \Gamma_2 + \mathbf{c}) \geq d_0 + |\mathbf{c}|.$$

Proof. By a rotation we may assume $\mathbf{c} = (c, 0)$ with $c > 0$. Let $\mathbf{p}$ be a leftest point of Γ_2, *i.e.*, $\mathbf{p} = (p_1, p_2) \in \Gamma_2$ and

$$p_1 = \inf\{\, q_1; (q_1, q_2) \in \Gamma_2 \,\}.$$

There is a point $\mathbf{z} = (z_1, p_2) \in \Gamma_1$ such that $z_1 < p_1$. The distance $\mathrm{dist}(\mathbf{z}, \Gamma_2 + \mathbf{c})$ is attained at $\mathbf{p} + \mathbf{c} \in \Gamma_2$ so that

$$\begin{aligned} \mathrm{dist}(\mathbf{z}, \Gamma_2 + \mathbf{c}) &= \mathrm{dist}(\mathbf{z}, \mathbf{p} + \mathbf{c}) \\ &= \mathrm{dist}(\mathbf{z}, \mathbf{p}) + c = \mathrm{dist}(\mathbf{z}, \Gamma_2) + c. \end{aligned}$$

Since $\mathrm{dist}(x, \Gamma_2) \geq d_0$, this yields

$$\mathrm{d_H}(\Gamma_1, \Gamma_2 + \mathbf{c}) \geq \mathrm{dist}(\mathbf{z}, \Gamma_2 + \mathbf{c}) \geq d_0 + c.$$

5.6. Proof of Theorem 5.2. Let k be a T-periodic solution of (K1)-(K3). Let S_t be a T- periodic solution of (E) with curvature $k = k(t, \theta)$. By Theorem 4.1 S_t is known to be unstable.

Suppose that k were stable. Then for each $\epsilon > 0$, $0 \leq t_0 < T$ there would exist $\delta > 0$ such that

$$\|h - k\|(t_0) < \delta \qquad \text{implies} \qquad \|h - k\|(t) < \epsilon$$

for all $t > t_0$ and solution $h > 0$ of (K1)-(K3).

We now apply the behavior of Γ_t^α defined in Theorem 4.1. If α is taken so small then the curvature $h(t_0, \theta)$of Γ^α satisfies $\|h - k\|(t_0) < \delta$ (cf. Lemma 3.4). This would imply

$$\|h - k\|(t) < \epsilon \qquad \text{for all } t > t_0.$$

By Proposition 5.3 and Lemma 5.4

$$\mathrm{d_H}(\Gamma_t^\alpha, S_t + \mathbf{c}(t)) \leq \frac{\epsilon}{k_*(k_* - \epsilon)} = \epsilon', \qquad k_* = \inf_\theta k,$$

where $c(t)$ is taken so that the Gauss parametrization X_1 of Γ_t^α and X_2 of $S_t + c(t)$ have the same initial point, *i.e.* $X_1(0) = X_2(0)$. However, Theorem 4.1 together with Lemma 5.5 implies

$$d_H(\Gamma_t^\alpha, S_t + c(t)) \geq \alpha\left(1 + \int_{t_0}^t c_0(\tau)\, d\tau\right) \wedge \alpha_1, \qquad c_0(\tau) = c_0' a(\tau)^2, \qquad \text{for } t \geq t_0.$$

These two estimates of the Hausdorff distance with $t = t_0 + nT$ yield

$$\epsilon' \geq \alpha(1 + nC_0) \wedge \alpha_1, \qquad C_0 = \int_0^T c_0\, dt > 0 \quad \text{for all } n = 1, 2, \ldots$$

Since α_1 and c_0 is independent of ϵ (although α depends on ϵ), this implies $\epsilon' > \alpha_1$ which leads a contradiction if ϵ is taken sufficiently small.

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