On a property of Fourier coefficients of cusp forms of half-integral weight

T. Miyake and Y. Maeda

Series ♯222. December 1993

HOKKAIDO UNIVERSITY

PREPRINT SERIES IN MATHEMATICS

♯ 192: J. Seade, T. Suwa, A residue formula for the index of a holomorphic flow, 22 pages. 1993.

♯ 193: H. Kubo, Blow-up of solutions to semilinear wave equations with initial data of slow decay in low space dimensions, 8 pages. 1993.

♯ 194: F. Hiroshima, Scaling limit of a model of quantum electrodynamics, 52 pages. 1993.

♯ 195: T. Ozawa, Y. Tsutsumi, Global existence and asymptotic behavior of solutions for the Zakharov equations in three space dimensions, 34 pages. 1993.

♯ 196: H. Kubo, Asymptotic behaviors of solutions to semilinear wave equations with initial data of slow decay, 25 pages. 1993.

♯ 197: Y. Giga, Motion of a graph by convexified energy, 32 pages. 1993.

♯ 198: T. Ozawa, Local decay estimates for Schrödinger operators with long range potentials, 17 pages. 1993.

♯ 199: A. Arai, N. Tominaga, Quantization of angle-variables, 31 pages. 1993.

♯ 200: S. Izumiya, Y. Kurokawa, Holonomic systems of Clairaut type, 17 pages. 1993.

♯ 201: K.-S. Saito, Y. Watatani, Subdiagonal algebras for subfactors, 7 pages. 1993.

♯ 202: K. Iwata, On Markov properties of Gaussian generalized random fields, 7 pages. 1993.

♯ 203: A. Arai, Characterization of anticommutativity of self-adjoint operators in connection with Clifford algebra and applications, 13 pages. 1993.

♯ 204: J. Wierzbicki, An estimation of the depth from an intermediate subfactor, 7 pages. 1993.

♯ 205: N. Honda, Vanishing theorem for the tempered distributions, 11 pages. 1993.

♯ 206: T. Hibi, Betti number sequences of simplicial complexes, Cohen-Macaulay types and Möbius functions of partially ordered sets, and related topics, 25 pages. 1993.

♯ 207: A. Inoue, Regularly varying correlations, 23 pages. 1993.

♯ 208: S. Izumiya, B. Li, Overdetermined systems of first order partial differential equations with singular solution, 9 pages. 1993.

♯ 209: T. Hibi, Hochster's formula on Betti numbers and Buchsbaum complexes, 7 pages. 1993.

♯ 210: T. Hibi, Star-shaped complexes and Ehrhart polynomials, 5 pages. 1993.

♯ 211: S. Izumiya, G. T. Kossioris, Geometric singularities for solutions of single conservation laws, 28 pages. 1993.

♯ 212: A. Arai, On self-adjointness of Dirac operators in Boson-Fermion Fock spaces, 43 pages. 1993.

♯ 213: K. Sugano, Note on non-commutative local field, 3 pages. 1993.

♯ 214: A. Hoshiga, Blow-up of the radial solitions to the equations of vibrating membrane, 28 pages. 1993.

♯ 215: A. Arai, Scaling limit of anticommuting self-adjoint operators and nonrelativistic limit of Dirac operators, 35 pages. 1993.

♯ 216: Y. Giga, N. Mizoguchi, Existence of periodic solutions for equations of evolving curves, 45 pages. 1993.

♯ 217: T. Suwa, Indices holomorphic vector fields relative to invariant curves, 10 pages. 1993.

♯ 218: S. Izumiya, G. T. Kossioris, Realization theorems of geometric singularities for Hamilton-Jacobi equations, 14 pages. 1993.

♯ 219: Y. Giga, K. Yama-uchi, On instability of evolving hypersurfaces, 14 pages. 1993.

♯ 220: W. Bruns, T. Hibi, Cohen-Macaulay partially ordered sets with pure resolutions, 11 pages. 1993.

♯ 221: S. Jimbo, Y. Morita, Ginzburg Landau equation and stable solutions in a rotational domain, 32 pages. 1993.

On a property of Fourier coefficients of cusp forms of half-integral weight

Toshitsune Miyake and Yoshitaka Maeda

INTRODUCTION

Since Shimura's epock-making paper [S1] appeared, modular forms of half-integral weight have been recognized to be an important object comparable to those of integral weight. This short note is to show a property of Fourier coefficients of cusp forms of half-integral weight. Before we state our result, we should remind ourself of a property of Fourier coefficients of cusp forms of integral weight. Let $F(z) = \sum_{n=1}^{\infty} A_n e^{2\pi i n z}$ be a primitive form in $S_k(N, \chi)$, then the following fact is well known ([M2],(4.6.17)]):

$$A_n = \chi(n)\overline{A_n} \qquad if \quad (n, N) = 1.$$

Though this is quite simple and easy to prove but sometimes plays an important role. The present note is to show that a parallel (but somewhat weaker) relation for cusp forms of half-integral weight by using this and the main theorem of Waldspurger([W]). We denote by $S_k(N, \chi)$ the space of cusp forms of weight k with level N and character χ, and for an odd integer k we denote by $\mathrm{S}_{k/2}(N, \chi)$ the space of cusp forms of weight $k/2$ (half-integral weight) with level N and character χ.

THEOREM *Let* $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z} \in \mathrm{S}_{k/2}(N, \chi)$ *be a common eigen function of Hecke operators* $\mathrm{T}(p^2)$ *for all prime numbers* p *prime to* N*, and* F *the primitive cusp form in* $S_{k-1}(N, \chi^2)$ *corresponding to* f*. Let* m *and* n *be square-free positive integers relatively prime to* N *satisfying*

$$m/n \in (\mathrm{Q}_p^{\times})^2$$

for all prime numbers p dividing N. If $L((1-k)/2, F, \overline{\chi}\psi_{m,k})$ *and* $L((1-k)/2, F, \overline{\chi}\psi_{n,k})$ *are not 0, then*

$$a_m \overline{a_n} \chi(n) = \overline{a_m} a_n \chi(m).$$

Here $L(s, F, \overline{\chi}\psi_{m,k})$ and $L(s, F, \overline{\chi}\psi_{n,k})$ are Dirichlet series attached to F with character $\overline{\chi}\psi_{m,k}$ and $\overline{\chi}\psi_{n,k}$. For the precise definitions of them and characters $\psi_{m,k}$, $\psi_{n,k}$ see the text. A similar result is also found in [S2]. Since Shimura pointed out in [S1] the importance of Fourier coefficients a_n of cusp forms of half-integral weight, especially for square-free n, several authors obtained interesting formulas expressing a_n (or we rather say a_n^2 or $|a_n|^2$) (see [K-Z], [K], [W], [S2]). Our result is simple but useful to abridge some of those results. In fact, this result is based on a question arizen from a communication with Shimura, to whom authors wish to express their hearty gratitude. In the end of the article, we attached a table of Fourier coefficients of cusp forms of half-integral weight calculated by M.Yamauchi using a table of M.Ueda([U]) by his courtesy.

1. For a positive integer N, we denote by $\Gamma_0(N)$ the congruence modular group

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathrm{Z}) \mid c \equiv 0 \pmod{N} \right\}.$$

For an elemnt $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathrm{Z})$,we put

$$\gamma(z) = \frac{az+b}{cz+d}, \qquad j(\gamma, z) = cz + d.$$

For an odd positive integer k and for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$, we put

$$J_{k/2}(\gamma, z) = \varepsilon_d^{-1} \left(\frac{c}{d}\right) (cz+d)^{1/2} (cz+d)^{(k-1)/2}.$$

Here $\varepsilon_d = 1$ or i according as $d \equiv 1$ or $3 \pmod 4$, and $(cz+d)^{1/2}$ is defined as usual. Let f be a function on the upper half plane $\mathbf{H}$. For an element $\gamma \in SL_2(\mathbf{Z})$ and an integer k, we put

$$\left(f|_k\gamma\right)(z) = j(\gamma, z)^{-k} f(\gamma(z)).$$

We also define, for an odd positive integer k and $\gamma \in \Gamma_0(4)$,

$$\left(f|_{k/2}\gamma\right)(z) = J_{k/2}(\gamma, z)^{-1} f(\gamma(z)).$$

For a positive integer N, we denote by χ a Dirichlet character defined modulo N. We put

$$\chi(\gamma) = \chi(d) \qquad for\ \ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

For N, χ and a positive integer k, we denote by $S_k(N,\chi)$ the space of holomorphic functions $f(z)$ on $\mathbf{H}$ satisfying

$$(f|_k\gamma)(z) = \chi(\gamma) f(z)$$

and the usual cuspidal condition. Now assume that N is divisible by 4. For N, χ and an odd integer k, we denote by $\mathrm{S}_{k/2}(N,\chi)$ the space of holomorphic functions $f(z)$ on $\mathbf{H}$ satisfying

$$(f|_{k/2}\gamma)(z) = \chi(\gamma) f(z)$$

and the usual cuspidal condition. Those spaces are called the spaces of cusp forms of integral and half-integral weight, respectively. For prime numbers p not dividing N, we denote by $T(p)$ the Hecke operators acting on $S_k(N,\chi)$ and by $\mathrm{T}(p^2)$ the Hecke operators acting on $\mathrm{S}_{k/2}(N,\chi)$, respectvely (for the definitions of Hecke operators, see [M2] and [S1]).

For a cusp form $f(z)$ of either integral or half-integral weight, we express its Fourier expansion as

$$f(z) = \sum_{n=1}^{\infty} a(n,f) e^{2\pi i n z}.$$

We also define a function f_ρ by

$$f_\rho(z) = \sum_{n=1}^{\infty} \overline{a(n,f)} e^{2\pi i n z}.$$

Then we see easily that

$$f_\rho(z) = \overline{f(-\overline{z})}.$$

Lemma 1. (1) *If $F(z)$ belongs to $S_k(N,\chi)$, then $F_\rho(z)$ belongs to $S_k(N,\overline{\chi})$. Moreover, if $F|_kT(p) = \lambda F$ then $F_\rho|_kT(p) = \overline{\lambda}F_\rho$.*
(2) *Let k be an odd positive integer. If $f(z)$ belongs to $\mathsf{S}_{k/2}(N,\chi)$, then $f_\rho(z)$ belongs to $\mathsf{S}_{k/2}(N,\overline{\chi})$. Moreover, if $f|_{k/2}\mathsf{T}(p^2) = \lambda f$ then $f_\rho|_{k/2}\mathsf{T}(p^2) = \overline{\lambda}f_\rho$.*

Proof The first part of (1) is well known ([M2],Lemma 4.3.2). Let us prove the first part of (2). For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$, we put

$$\gamma' = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$

If $\gamma \in \Gamma_0(N)$, then we see that

$$\begin{aligned}\overline{J_{k/2}(\gamma', -\overline{z})} &= \overline{\varepsilon_d^{-1}\left(\frac{-c}{d}\right)(c\overline{z}+d)^{1/2}(c\overline{z}+d)^{(k-1)/2}} \\ &= \varepsilon_d^{-1}\left(\frac{c}{d}\right)(cz+d)^{1/2}(cz+d)^{(k-1)/2} = J_{k/2}(\gamma, z).\end{aligned}$$

Therefore we obtain that

$$\begin{aligned}f_\rho\left(\frac{az+b}{cz+d}\right) &= \overline{f\left(-\overline{\frac{az+b}{cz+d}}\right)} \\ &= \overline{f\left(\frac{a(-\overline{z})-b}{-c(-\overline{z})+d}\right)} \\ &= \overline{\chi(\gamma)J_{k/2}(\gamma', -\overline{z})f(-\overline{z})} \\ &= \overline{\chi(\gamma)}J_{k/2}(\gamma, z)f_\rho(z).\end{aligned}$$

Since $f_\rho(\gamma(z)) = \overline{f(\gamma'(-\overline{z}))}$, the cuspidal condition is obviously satisfied by f_ρ. The second parts of both (1) and (2) can be directly proved using the Fourier expansion and the definition of Hecke operators. Since we omit the definition of Hecke operators, we also omit the proof. q.e.d.

From now on, we denote by k a positive odd integer $k \geq 3$ and by N a positive integer divisible by 4 and χ a Dirichlet character defined modulo N. Let $f \in \mathsf{S}_{k/2}(N, \chi)$ be a common eigen function of Hecke operators $\mathsf{T}(p^2)$ for all primes p ($p \nmid N$) and $f|_{k/2}\mathsf{T}(p^2) = \lambda_p f$. Then there exists a unique primitive form $F \in S_{k-1}(N, \chi^2)$ such that $F|_{k-1}T(p) = \lambda_p F$. We call F the primitive form corresponding to f. For a primitive form $F(z)$ in $S_{k-1}(N, \chi^2)$ of conductor N, we put

$$\mathsf{S}_{k/2}(N, \chi, F) = \left\{ f \in \mathsf{S}_{k/2}(N, \chi) \middle| \begin{array}{l} f|_{k/2}\mathsf{T}(p^2) = a(p, F) f \\ \text{for almost all } p \nmid N \end{array} \right\}.$$

For a modular form f and a Dirichlet character ψ, we define a Dirichlet series $L(s, f, \psi)$ attached to f and ψ by

$$L(s, f, \psi) = \sum_{n=1}^{\infty} a(n, f)\psi(n) n^{-s}.$$

For a square-free positive integer n and an odd integer k, we denote by ψ_n the Dirichlet character corresponding to the quadratic extension $\mathbb{Q}(\sqrt{n})$ and $\psi_{n,k}$ the product of ψ_n with $\left(\frac{-1}{\cdot}\right)^{(k-1)/2}$. Now one of the main results of Waldspurger ([W]) is as follows:

Lemma 2. ([W], Corollaire 2) *The notation and assumptions being as above, let $f \in \mathsf{S}_{k/2}(N, \chi, F)$ and m, n square-free positive integers relatively prime to N satisfying $m/n \in (\mathbb{Q}_p^\times)^2$ for all prime numbers p dividing N. Then*

$$\begin{aligned} &a(m, f)^2 L((1-k)/2, F, \overline{\chi}\psi_{n,k})\chi(n/m) n^{k/2-1} \\ &\qquad = a(n, f)^2 L((1-k)/2, F, \overline{\chi}\psi_{m,k}) m^{k/2-1}. \end{aligned}$$

Corollary 3. *Let the notation and the assumptions be the same* *as in Lemma 2. Assume neither $L((1-k)/2, F, \overline{\chi}\psi_{m,k})$ nor $L((1-k)/2, F, \overline{\chi}\psi_{n,k})$ vanish. Then for any two element $f, g \in \mathrm{S}_{k/2}(N, \chi, F)$,*

$$a(m,f)a(n,g) = a(n,f)a(m,g).$$

Proof By Lemma 2, we easily

$$a(m,f)^2 a(n,g)^2 = a(n,f)^2 a(m,g)^2$$

for any two elements f and g in $\mathrm{S}_{k/2}(N, \chi, F)$. If $a(m,f)a(n,f) = 0$, then $a(m,f)a(n,g) = a(n,f)a(m,g)$ is clear.. Assume $a(m,f)a(n,f) \neq 0$. Since f and g are arbitrary, it also holds if we take $f+g$ in place of g. This implies the equality

$$a(m,f)^2(a(n,f)+a(n,g))^2 = a(n,f)^2(a(m,f)+a(m,g))^2.$$

Therefore we obtain

$$2a(m,f)a(n,f)(a(m,f)a(n,g) - a(n,f)a(m,g)) = 0.$$

By our assumption we see $a(m,f)a(n,g) = a(n,f)a(m,g)$. **q.e.d.**

Proof of the theorem. Let f be an element in $\mathrm{S}_{k/2}(N, \chi)$ which is an eigen function of Hecke operators $\mathrm{T}(p^2)$ for all prime numbers p relatively prime to N. Denote by λ_p the eigen-value of $\mathrm{T}(p^2)$ for f or

$$f|_{k/2}\mathrm{T}(p^2) = \lambda_p f$$

and denote by F the corresponding primitive form in $S_{k-1}(N, \chi^2)$. Then we see that

$$F|_{k-1}T(p) = \lambda_p F.$$

Now we put

$$f'(z) = \sum_{n=1}^{\infty} \chi(n)\overline{a(n,f)}e^{2\pi i n z}.$$

and let F' the primitive cusp form in $S_{k-1}(N,\chi^2)$ corresponding to f'.Then by [S1], Lemma 3.6, and the theory of primitive forms ([M2], §4.6.), we see easily that

$$f'|_{k/2}\mathrm{T}(p^2) = \chi(p)^2\overline{\lambda_p}f'$$

and also

$$F'|_{k-1}T(p) = \chi(p)^2\overline{\lambda_p}F'.$$

Since λ_p is the p-th Fourier coefficient of F and $\chi(p)^2\overline{\lambda_p}$ is the p-th Fourier coefficients of F', we see by the property of coefficients of cusp forms of integral weight mentioned in the introduction that

$$\lambda_p = \chi(p)^2\overline{\lambda_p}.$$

This implies that eigen values of $T(p)$ for F and F' are the same for all prime numbers p not dividing N and therefore F and F' must coincide by the theory of primitive forms ([M1],[M2]). Since f and f' correspond to the same primitive form F, we can apply Corollary 3 and obtain our result by taking f' as g there. q.e.d.

Table

The following is a table of Fourier coefficients $a(n)$ of cusp forms f_A, f_B, f_C, and f_D in $\mathrm{S}_{5/2}(28,\chi)$ with character χ of order 3 defined modulo 7. We note that $\dim_{\mathbf{C}}\mathrm{S}_{5/2}(28,\chi) = 4$ and $\{f_A, f_B, f_C, f_D\}$ is a basis of $\mathrm{S}_{5/2}(28,\chi)$ consisting of common eigen functions of Hecke operators $\mathrm{T}(p^2)$ satisfying $(p,28) = 1$. In the table, ω implies $1^{1/3}$. We note f_A belongs to $\mathrm{S}_{5/2}(28,\chi,F_A)$ and f_B belongs to $\mathrm{S}_{5/2}(28,\chi,F_B)$ with primitive forms F_A, $F_B \in S_4(14,\overline{\chi})$ and f_C and f_D belong to $\mathrm{S}_{5/2}(28,\chi,G)$ with a primitive form $G \in S_4(7,\overline{\chi})$.

= 28A =

n	mod 8	mod 7	(*/7)	$\chi(n)$	a(n)	$\chi(n)\overline{a}(n)/a(n)$
(1)	1	1	+	1	1	1
(2)	2	2	+	ω^2	$\omega/2$	1
(3)	3	3	-	ω	$-3+3\omega/2$	-1
(4)	4	4	+	ω	$2+2\omega$	1
(5)	5	5	-	ω^2	0	
(6)	6	6	-	1	$-5-10\omega/2$	-1
(7)	7	0		0	$-3-8\omega/2$	0
(8)	0	1	+	1	$-1-2\omega$	-1
(9)	1	2	+	ω^2	-4ω	1
(10)	2	3	-	ω	$-1+\omega/2$	-1
(11)	3	4	+	ω	$13+13\omega/2$	1
(12)	4	5	-	ω^2	$-6-3\omega$	-1
(13)	5	6	-	1	0	
(14)	6	0		0	$24+15\omega/2$	0
(15)	7	1	+	1	$-17/2$	1
(16)	0	2	+	ω^2	4ω	1
(17)	1	3	-	ω	$-5+5\omega$	-1
(18)	2	4	+	ω	$-1-\omega$	1
(19)	3	5	-	ω^2	$26+13\omega/2$	-1
(20)	4	6	-	1	0	
(21)	5	0		0	0	
(22)	6	1	+	1	$13/2$	1
(23)	7	2	+	ω^2	$-37\omega/2$	1
(24)	0	3	-	ω	$5-5\omega$	-1
(25)	1	4	+	ω	$-2-2\omega$	1
(26)	2	5	-	ω^2	$-28-14\omega$	-1
(27)	3	6	-	1	$3+6\omega/2$	-1
(28)	4	0		0	$5-3\omega$	0
(29)	5	1	+	1	0	
(30)	6	2	+	ω^2	$51\omega/2$	1
(31)	7	3	-	ω	$-13+13\omega/2$	-1
(32)	0	4	+	ω	$-2-2\omega$	1
(33)	1	5	-	ω^2	$14+7\omega$	-1
(34)	2	6	-	1	$-9-18\omega/2$	-1
(35)	3	0		0	$-56-35\omega/2$	0
(36)	4	1	+	1	8	1
(37)	5	2	+	ω^2	0	
(38)	6	3	-	ω	$31-31\omega/2$	-1
(39)	7	4	+	ω	$3+3\omega$	1
(40)	0	5	-	ω^2	$-2-\omega$	-1
(41)	1	6	-	1	$6+12\omega$	-1
(42)	2	0		0	$21+56\omega/2$	0
(43)	3	1	+	1	2	1
(44)	4	2	+	ω^2	13ω	1
(45)	5	3	-	ω	0	
(46)	6	4	+	ω	$-29-29\omega/2$	1
(47)	7	5	-	ω^2	$70+35\omega/2$	-1
(48)	0	6	-	1	$-6-12\omega$	-1
(49)	1	0		0	$-19-18\omega$	0
(50)	2	1	+	1	6	1

= 28B =

n	mod 8	mod 7	(*/7)	$\chi(n)$	$a(n)$	$\chi(n)\overline{a}(n)/a(n)$
(1)	1	1	+	1	0	
(2)	2	2	+	ω^2	1	ω^2
(3)	3	3	−	ω	$-2-\omega/3$	$-\omega^2$
(4)	4	4	+	ω	0	
(5)	5	5	−	ω^2	$2+4\omega/3$	$-\omega^2$
(6)	6	6	−	1	$-1+\omega$	$-\omega^2$
(7)	7	0		0	$-1-5\omega/3$	0
(8)	0	1	+	1	$-2-2\omega$	ω^2
(9)	1	2	+	ω^2	0	
(10)	2	3	−	ω	$-2-\omega/3$	$-\omega^2$
(11)	3	4	+	ω	ω	ω^2
(12)	4	5	−	ω^2	$2+4\omega/3$	$-\omega^2$
(13)	5	6	−	1	$4-4\omega/3$	$-\omega^2$
(14)	6	0		0	$5+4\omega/3$	0
(15)	7	1	+	1	$3+3\omega$	ω^2
(16)	0	2	+	ω^2	0	
(17)	1	3	−	ω	0	
(18)	2	4	+	ω	-2ω	ω^2
(19)	3	5	−	ω^2	$-5-10\omega/3$	$-\omega^2$
(20)	4	6	−	1	$4-4\omega/3$	$-\omega^2$
(21)	5	0		0	$-8+2\omega/3$	0
(22)	6	1	+	1	$3+3\omega$	ω^2
(23)	7	2	+	ω^2	-3	ω^2
(24)	0	3	−	ω	$4+2\omega$	$-\omega^2$
(25)	1	4	+	ω	0	
(26)	2	5	−	ω^2	$-8-16\omega/3$	$-\omega^2$
(27)	3	6	−	1	$-5+5\omega/3$	$-\omega^2$
(28)	4	0		0	$-8+2\omega/3$	0
(29)	5	1	+	1	$-4-4\omega$	ω^2
(30)	6	2	+	ω^2	-5	ω^2
(31)	7	3	−	ω	$10+5\omega/3$	$-\omega^2$
(32)	0	4	+	ω	4ω	ω^2
(33)	1	5	−	ω^2	0	
(34)	2	6	−	1	$1-\omega/3$	$-\omega^2$
(35)	3	0		0	$25+20\omega/3$	0
(36)	4	1	+	1	0	
(37)	5	2	+	ω^2	2	ω^2
(38)	6	3	−	ω	$-10-5\omega/3$	$-\omega^2$
(39)	7	4	+	ω	-6ω	ω^2
(40)	0	5	−	ω^2	$2+4\omega/3$	$-\omega^2$
(41)	1	6	−	1	0	
(42)	2	0		0	$-1-5\omega$	0
(43)	3	1	+	1	$-4-4\omega$	ω^2
(44)	4	2	+	ω^2	2	ω^2
(45)	5	3	−	ω	$8+4\omega/3$	$-\omega^2$
(46)	6	4	+	ω	ω	ω^2
(47)	7	5	−	ω^2	$5+10\omega/3$	$-\omega^2$
(48)	0	6	−	1	$4-4\omega/3$	$-\omega^2$
(49)	1	0		0	0	
(50)	2	1	+	1	$4+4\omega$	ω^2

= 28C =

n	mod 8	mod 7	(*/7)	$\chi(n)$	$a(n)$	$\chi(n)\bar{a}(n)/a(n)$
(1)	1	1	+	1	1	1
(2)	2	2	+	ω^2	$-\omega/2$	1
(3)	3	3	−	ω	$-1+\omega/2$	−1
(4)	4	4	+	ω	$2+2\omega$	1
(5)	5	5	−	ω^2	$-4-2\omega$	−1
(6)	6	6	−	1	$-3-6\omega/2$	−1
(7)	7	0		0	$7/2$	0
(8)	0	1	+	1	-3	1
(9)	1	2	+	ω^2	4ω	1
(10)	2	3	−	ω	$-7+7\omega/2$	−1
(11)	3	4	+	ω	$7+7\omega/2$	1
(12)	4	5	−	ω^2	$6+3\omega$	−1
(13)	5	6	−	1	$-4-8\omega$	−1
(14)	6	0		0	$-7\omega/2$	0
(15)	7	1	+	1	$-3/2$	1
(16)	0	2	+	ω^2	-12ω	1
(17)	1	3	−	ω	$3-3\omega$	−1
(18)	2	4	+	ω	$5+5\omega$	1
(19)	3	5	−	ω^2	$-18-9\omega/2$	−1
(20)	4	6	−	1	$4+8\omega$	−1
(21)	5	0		0	$14+14\omega$	0
(22)	6	1	+	1	$3/2$	1
(23)	7	2	+	ω^2	$9\omega/2$	1
(24)	0	3	−	ω	$-9+9\omega$	−1
(25)	1	4	+	ω	$-2-2\omega$	1
(26)	2	5	−	ω^2	$4+2\omega$	−1
(27)	3	6	−	1	$-7-14\omega/2$	−1
(28)	4	0		0	$-21-21\omega$	0
(29)	5	1	+	1	4	1
(30)	6	2	+	ω^2	$-19\omega/2$	1
(31)	7	3	−	ω	$9-9\omega/2$	−1
(32)	0	4	+	ω	$2+2\omega$	1
(33)	1	5	−	ω^2	$-18-9\omega$	−1
(34)	2	6	−	1	$17+34\omega/2$	−1
(35)	3	0		0	$7\omega/2$	0
(36)	4	1	+	1	-8	1
(37)	5	2	+	ω^2	30ω	1
(38)	6	3	−	ω	$1-\omega/2$	−1
(39)	7	4	+	ω	$-11-11\omega$	1
(40)	0	5	−	ω^2	$42+21\omega$	−1
(41)	1	6	−	1	$-2-4\omega$	−1
(42)	2	0		0	$-21/2$	0
(43)	3	1	+	1	14	1
(44)	4	2	+	ω^2	-21ω	1
(45)	5	3	−	ω	$20-20\omega$	−1
(46)	6	4	+	ω	$-27-27\omega/2$	1
(47)	7	5	−	ω^2	$-14-7\omega/2$	−1
(48)	0	6	−	1	$-2-4\omega$	−1
(49)	1	0		0	$21+14\omega$	0
(50)	2	1	+	1	-6	1

= 28D =

n	mod 8	mod 7	(*/7)	χ(n)	a(n)	χ(n)$\overline{a}$(n)/a(n)
(1)	1	1	+	1	1	1
(2)	2	2	+	ω^2	$3\omega/2$	1
(3)	3	3	−	ω	$3-3\omega/2$	-1
(4)	4	4	+	ω	$-6-6\omega$	1
(5)	5	5	−	ω^2	$4+2\omega$	-1
(6)	6	6	−	1	$9+18\omega/2$	-1
(7)	7	0		0	$-21/2$	0
(8)	0	1	+	1	1	1
(9)	1	2	+	ω^2	4ω	1
(10)	2	3	−	ω	$21-21\omega/2$	-1
(11)	3	4	+	ω	$-21-21\omega/2$	1
(12)	4	5	−	ω^2	$-2-\omega$	-1
(13)	5	6	−	1	$4+8\omega$	-1
(14)	6	0		0	$21\omega/2$	0
(15)	7	1	+	1	$9/2$	1
(16)	0	2	+	ω^2	4ω	1
(17)	1	3	−	ω	$3-3\omega$	-1
(18)	2	4	+	ω	$-15-15\omega$	1
(19)	3	5	−	ω^2	$54+27\omega/2$	-1
(20)	4	6	−	1	$4+8\omega$	-1
(21)	5	0		0	$-14-14\omega$	0
(22)	6	1	+	1	$-9/2$	1
(23)	7	2	+	ω^2	$-27\omega/2$	1
(24)	0	3	−	ω	$3-3\omega$	-1
(25)	1	4	+	ω	$-2-2\omega$	1
(26)	2	5	−	ω^2	$-12-6\omega$	-1
(27)	3	6	−	1	$21+42\omega/2$	-1
(28)	4	0		0	$7+7\omega$	0
(29)	5	1	+	1	-4	1
(30)	6	2	+	ω^2	$57\omega/2$	1
(31)	7	3	−	ω	$-27+27\omega/2$	-1
(32)	0	4	+	ω	$10+10\omega$	1
(33)	1	5	−	ω^2	$-18-9\omega$	-1
(34)	2	6	−	1	$-51-102\omega/2$	-1
(35)	3	0		0	$-21\omega/2$	0
(36)	4	1	+	1	24	1
(37)	5	2	+	ω^2	-30ω	1
(38)	6	3	−	ω	$-3+3\omega/2$	-1
(39)	7	4	+	ω	$33+33\omega$	1
(40)	0	5	−	ω^2	$-14-7\omega$	-1
(41)	1	6	−	1	$-2-4\omega$	-1
(42)	2	0		0	$63/2$	0
(43)	3	1	+	1	-42	1
(44)	4	2	+	ω^2	7ω	1
(45)	5	3	−	ω	$-20+20\omega$	-1
(46)	6	4	+	ω	$81+81\omega/2$	1
(47)	7	5	−	ω^2	$42+21\omega/2$	-1
(48)	0	6	−	1	$-10-20\omega$	-1
(49)	1	0		0	$21+14\omega$	0
(50)	2	1	+	1	18	1

References

[K] W.Kohnen, Fourier coefficients of modular forms of half integral weight, Math. Ann. 271(1985),237-268.

[K-Z] W.Kohnen and D.Zagier, Values of L-series of modular forms at the center of the critical strip, Inv. Math. 64(1981),175-198.

[M1] T.Miyake, On automorphic forms on GL_2 and Hecke operators, Ann. of Math. 94(1971), 174-189.

[M2] T.Miyake, *Modular Forms*, Springer-Verlag, 1989.

[S1] G.Shimura, On modular forms of half integral weight, Ann. of Math. 97(1973), 440-481.

[S2] G.Shimura, On the Fourier coefficients of Hilbert modular forms of half-integral weight, preprint.

[W] J.-L. Waldspurger, Sur les coefficients de Fourier des formes modulares de poids demi-entier, J.Math.pures et appl. 60(1981), 375-484.

[U] M. Ueda, *Table for modular forms of weight* $\frac{3}{2}$, Argo Lecture Notes 1-4, 1987, 1989

Hokkaido University
Sapporo, 060, Japan