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GLOBAL, SMALL RADIALLY SYMMETRIC SOLUTIONS TO NONLINEAR SCHRÖDINGER EQUATIONS AND A GAUGE TRANSFORMATION

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GLOBAL, SMALL RADIALLY SYMMETRIC SOLUTIONS TO NONLINEAR SCHRÖDINGER EQUATIONS AND A GAUGE TRANSFORMATION

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ABSTRACT. This paper proves the global existence of small radially symmetric solutions to the nonlinear Schrödinger equations of the form

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = F(u, \nabla u, \bar{u}, \nabla \bar{u}), & (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0,x) = \varepsilon_0 \phi(|x|), & x \in \mathbb{R}^n, \end{cases}$$

where $n \geq 3$, ε_0 is sufficiently small, $|x| = (\sum_{1 \leq j \leq n} x_j^2)^{1/2}$,

$$\begin{aligned} F = &\sum_{\ell_0 \leq |\alpha| \leq \ell_1} \lambda_\alpha u^{\alpha_1} \bar{u}^{\alpha_2} \\ &+ \sum_{\substack{0 \leq |\alpha| \leq \ell_2 \\ 1 \leq |\beta| \leq \ell_3}} \lambda_{\alpha\beta} u^{\alpha_1} \bar{u}^{\alpha_2} (\sum_{1 \leq j \leq n} |\partial_j u|^2)^{\beta_1} (\sum_{1 \leq j \leq n} (\partial_j u)^2)^{\beta_2} \\ &\qquad \times (\sum_{1 \leq j \leq n} (\partial_j \bar{u})^2)^{\beta_3} \end{aligned}$$

with $\lambda_\alpha, \lambda_{\alpha\beta} \in \mathbb{C}$, $\ell_1, \ell_2, \ell_3 \in \mathbb{N}$, $\ell_0 = 3$ for $n = 3, 4$, and $\ell_0 = 2$ for $n \geq 5$. The method depends on the combination of a gauge transformation and generalized energy estimtes and does not require the condition such that

$$\partial_{\nabla u} F \text{ is pure imaginary}$$

which is needed for the classical energy method.

§1. Introduction. We study the global existence of small radially symmetric solutions to the nonlinear Schrödinger equations of the form

$$
\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = F(u, \nabla u, \bar{u}, \nabla \bar{u}), & (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0,x) = \varepsilon_0 \phi(|x|), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}
$$

where $n \geq 3$, ε_0 is sufficiently small, and F is a polynomial on $\mathbb{C} \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^n$ with complex coefficients, having neither constant nor linear terms. A major difficulty in the problem of global existence of small solutions to (1.1) consists in the so-called derivative loss comining from the first order derivatives in the nonlinear term so long as one makes an attempt to obtain a-priori estimates by energy methods in any generalized sense. Previous works [2,4,5,11,13] concerning global solvability of (1.1) in the usual Sobolev spaces require the condition that

$$
\partial_{\nabla u} F = (\partial_{\partial_1 u} F, \cdots, \partial_{\partial_n u} F) \text{ is pure imaginary} \tag{1.2}
$$

to avoid the derivative loss so that the extra derivatives could be dropped through integration by parts in the energy estimates. The purpose in this paper is to prove the global existence of small solutions to (1.1) without (1.2), provided that everything is radially symmetric. The proof is done by combining a gauge transformation which was used to prove a local existence theorem to nonlinear Schrödinger equations in one space dimension [1,8] and a generalized energy method which was used to prove a global existence theorem to nonlinear Schrödinger equations with the condition (1.2) [2,4,5].

We shall therefore concentrate on the case where F is invariant under space rotations in order that radial symmetry is preserved in unique solutions under the time evolution. This indicates that the contribution of the derivatives $(\nabla u, \nabla \bar{u})$ to the polynomial F is described as a polynomial of $|\nabla u|^2$, $(\nabla u)^2$, and $(\nabla \bar{u})^2$. To be specific, throughout this paper we assume that F is given by

$$
F = F_1(u) + F_2(u)
$$

with

$$
F_1(u) = \sum_{\ell_0 \leq |\alpha| \leq \ell_1} \lambda_\alpha u^{\alpha_1} \bar{u}^{\alpha_2},
$$

$$
F_2(u) = \sum_{\substack{0 \leq |\alpha| \leq \ell_2 \\ 1 \leq |\beta| \leq \ell_3}} \lambda_{\alpha\beta} u^{\alpha_1} \bar{u}^{\alpha_2} \Big(\sum_{1 \leq j \leq n} |\partial_j u|^2 \Big)^{\beta_1} \Big(\sum_{1 \leq j \leq n} (\partial_j u)^2 \Big)^{\beta_2} \Big(\sum_{1 \leq j \leq n} (\partial_j \bar{u})^2 \Big)^{\beta_3},
$$

where $\lambda_\alpha, \lambda_{\alpha\beta} \in \mathbb{C}$, $\ell_0 = 3$ for $n = 3, 4$, and $\ell_0 = 2$ for $n \geq 5$. It should be emphasized that the usual energy estimates are no longer available for the nonlinearity F with this form. The only apparent restriction on F is imposed on ℓ_0 for $n = 3, 4$. This is natural for $n = 3$ since the quadratic nonlinearity in three space dimensions provides only the result of almost global existence [2, 5]. The restriction $\ell_0 = 3$ for $n = 4$ is actually dropped by combining the methods in this paper and in [5], though details are involved and omitted here for simplicity. See the remark below. Now (1.1) is written as

$$
\left\{ \begin{array}{l} i\partial_t u + \frac{1}{2}(\partial_r^2 + \frac{n-1}{r}\partial_r)u = F(u, \partial_r u, \bar{u}, \partial_r \bar{u}), \\ u(0, r) = \varepsilon_0 \phi(r), \end{array} \right. \tag{1.3}
$$

since $\partial_j u = \frac{x_j}{|x|}\partial_r u$ for a radially symmetric function u, where $r = |x| = (\sum_{1 \leq j \leq n} x_j^2)^{1/2}$ and $\partial_r = \frac{x}{|x|} \cdot \nabla$. Note that we have made a little abuse of notation though it might also be used in the sequel when this could cause no confusion.

To state our result precisely we introduce some function spaces.

$$
\begin{aligned}
B^{2m} &= \{f \in L^2(\mathbb{R}^n); ||f||_{B^{2m}} = \sum_{0 \leq \ell \leq m} ||(r\partial_r)^\ell f||_{H^{2(m-\ell)}} < \infty\}, \\
X^{2m}(T) &= \{f \in L^\infty(-T, T; L^2(\mathbb{R}^n)); |||f|||_{X^{2m}(T)} = \sup_{t \in [-T, T]} ||f(t)||_{X_t^{2m}} < \infty\}, \\
Y^{2m}(T) &= \{f \in L^\infty(-T, T; L^2(\mathbb{R}^n)); |||f|||_{Y^{2m}(T)} = \sup_{t \in [-T, T]} ||f(t)||_{Y_t^{2m}} < \infty\},
\end{aligned}
$$

where $m \geq 0$ is an integer, H^k is the usual Sobolev space of order k,

$$
\begin{aligned}
||f(t)||_{Y_t^{2m}} &= \sum_{0 \leq \ell \leq m} ||P^\ell f(t)||_{H^{2(m-\ell)}}, \\
||f(t)||_{X_t^{2m}} &= ||f(t)||_{Y_t^{2m}} + ||Qf(t)||_{Y_t^{2(m-1)}} \\
&\quad + (1 + |t|)^{-((4-n)/2)_+} ||Q^2 f(t)||_{Y_t^{(2(m-2))_+}},
\end{aligned}
$$

$P = r\partial_r + 2t\partial_t$, $Q = r\partial_r + it\Delta$, and $a_+ = \max(a, 0)$ for $a \in \mathbb{R}$.

Our main result is

Theorem 1. *Let $n \geq 3$ and let F be as above. Let $\phi \in B^{2m}$ with $2m \geq n + 3$ and let ϕ be radially symmetric. Then, for ε_0 sufficiently small, there exists a unique global solution u of (1.1) such that*

$$
u \in C(\mathbb{R}; H^{2m-1}) \cap L^\infty(\mathbb{R}; H^{2m}).
$$

Furthermore, u satisfies

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{X_t^{2m}} < \infty.$$

Remark. One could include the case $\ell_0 = 2$ for $n = 4$ in the theorem with some modification caused by the introduction of the operator $J = x + it\nabla$ in the function spaces above. The proof proceeds in a way similar to the argument given below with possible modification and requires special treatment. Following [5], one could fill in the details at the cost of simplicity.

We now explain the difficulty arising from the real part of $\partial_{\nabla u}F$, the method of gauge transformation to be used for its compensation, and the necessity of introducing the assumption of radial symmetry. In the proof of global existence of solutions, the main task is the derivation of a-priori estimates of local solutions in $X^{2m}(T)$. To this end, let $\Gamma = \Delta^{m-\ell}P^\ell$ with $0 \le \ell \le m$ and $m \ge 1$ and apply Γ to the equation $Lu = F(u, \nabla u, \bar{u}, \nabla\bar{u})$, where $L = i\partial_t + (1/2)\Delta$. The commutation relation $[L, P] = 2L$ yields

$$L\Gamma u = \Delta^{m-\ell}(P+2)^\ell F(u, \nabla u, \bar{u}, \nabla\bar{u}). \tag{1.4}$$

The right hand side of (1.4) is expressed as

$$\partial_{\nabla u}F \cdot \nabla\Gamma u + \partial_{\nabla\bar{u}}F \cdot \nabla\overline{\Gamma u} + (\text{terms with derivatives of lower order})$$

by expansion based on Leibniz' rule and the commutation relation $[\nabla, P] = \nabla$. The terms with derivatives of the highest order, namely, $\partial_{\nabla u}F \cdot \nabla\Gamma u$ and $\partial_{\nabla\bar{u}}F \cdot \overline{\nabla\Gamma u}$ cause loss of derivatives and deserve special attention. Following the usual energy method, we multiply (1.4) by $\overline{\Gamma u}$ and take the imaginary part of the resulting equation. Then the left hand side equals $(1/2)\partial_t|\Gamma u|^2 + (1/2)\,\mathrm{Im}\,\mathrm{div}(\overline{\Gamma u}\nabla\Gamma u)$ in the divergence form. For the right hand side, the term proportional to $\nabla\overline{\Gamma u}$ is handled on the basis of the identity

$$(\partial_{\nabla\bar{u}}F \cdot \nabla\overline{\Gamma u})\overline{\Gamma u} = \frac{1}{2}\,\mathrm{div}(\partial_{\nabla\bar{u}}F(\overline{\Gamma u})^2) - \frac{1}{2}(\mathrm{div}\,\partial_{\nabla\bar{u}}F)(\overline{\Gamma u})^2.$$

The term proportional to $\nabla\Gamma u$, on the contrary, admits no analogous calculation except when $\partial_{\nabla u}F$ is pure imaginary since one still has

$$\begin{aligned} &\mathrm{Im}((i\,\mathrm{Im}\,\partial_{\nabla u}F) \cdot \nabla\Gamma u)\overline{\Gamma u}) \\ &= \frac{1}{2}\,\mathrm{div}((\mathrm{Im}\,\partial_{\nabla u}F)|\Gamma u|^2) - \frac{1}{2}(\mathrm{Im}\,\mathrm{div}\,\partial_{\nabla u}F)|\Gamma u|^2. \end{aligned}$$

This is the reason why (1.2) is necessary in the previous literature in order that terms with derivatives of the highest order have no effect after integration in space, thereby yielding a differential inequality for $||\Gamma u||_{L^2}$, which is easily integrated to provide the required a-priori estimate of $||\Gamma u||_{L^2}$. In the case where (1.2) is no longer available, the main issue is therefore focused on the treatment of the term proportional to $\nabla\Gamma u$. The method of gauge transformation depends on the existence of the potential function f, depending not necessarily locally on $(u, \nabla u, \bar{u}, \nabla\bar{u})$, such that $\operatorname{Re}\partial_{\nabla u}F = -\nabla f$. Under this condition, the crucial part $L\Gamma u - \partial_{\nabla u}F \cdot \nabla\Gamma u$ of the equation (1.4) is partly linearized by the transformed variable $v = e^f\Gamma u$ as $e^{-f}Lv$ with remainder composed of the terms involving $\nabla\overline{\Gamma u}$ or $\nabla\Gamma u$, both of which are handled by integration by parts technique as before (since the coefficient of $\nabla\Gamma u$ is pure imaginary), and harmless terms involving derivatives of lower order. This enables us to obtain the required a-priori estimate without using (1.2). The problem is thus reduced to finding the potential function for $\operatorname{Re}\partial_{\nabla u}F$. In the one dimensional case this is simply realized by integration from x to infinity [1,3,6,7,15], while higher dimensional case requires a special restriction on $\partial_{\nabla u}F$, that is, the curl free condition of $\partial_{\nabla u}F$, and hence there is little literature on this subject [12,14]. We have therefore restricted ourselves to the radial case, where the potential function is constructed in a way similar to the one dimensional case, to recover a certain amount of generality. Moreover, radial symmetry simplifies the exposition since the angular momentum operators are excluded in the Lie algebra of derivations associated with L, indispensable to the generalized energy estimate.

In [10], a gauge transformation was also used to study local well-posedness of higher order nonlinear dispersive equations in one space dimension which contain the KdV and Benjamin-Ono hierarchies as examples. The local existence of small solutions to (1.1) was proved in [9] by establishing new estimates related to the local smoothing property of the group $\{e^{it\Delta}\}_{-\infty}^{\infty}$, although it is not clear that their method is applicable to the problem of global existence.

§2. Preliminary estimates. In this section we treat radially symmetric functions only. We first give the basic time decay estimates needed in the proof of the main result.

Lemma 2.1. *(1) Let $f \in X^2(T) \cap L^\infty(-T,T;H^m)$ with $m > n/2$. Then for $|t| \le T$*

$$\|f(t)\|_{L^\infty} \le C(1+|t|)^{-a}(\|f(t)\|_{X_t^2} + \|f(t)\|_{H^m}) \quad for \quad n = 3,4, \tag{2.1}$$

where $a = 3/4$ if $n = 3$, and $a = 1-\varepsilon$ with $\varepsilon > 0$ arbitrarily small if $n = 4$.
(2) Let $f \in X^{2m}(T)$ with $m > n/2$. Then for $|t| \le T$

$$\|f(t)\|_{L^\infty} \le C(1+|t|)^{-b}\|f(t)\|_{X_t^{2m}} \quad for \quad n \ge 5, \tag{2.2}$$

where $b = \min(n/4, 2)$ if $n \ne 8$, and $b = 1-\varepsilon$ with $\varepsilon > 0$ arbitrarily small if $n = 8$.
(3) Let $f \in X^4(T) \cap L^\infty(-T,T;H^{m+1})$ with $m > n/2$. Then for $|t| \le T$

(2.3)
$$\|\partial_r f(t)\|_{L^\infty} \le C(1+|t|)^{-9/8}(\|f(t)\|_{X_t^4}^{1/4}\|f(t)\|_{X_t^2}^{3/4} + \|f(t)\|_{H^{m+1}}) \quad for \quad n = 3,$$
(2.4)
$$\|\partial_r f(t)\|_{L^\infty} \le C(1+|t|)^{-3/2}(\|f(t)\|_{X_t^4} + \|f(t)\|_{H^{m+1}}) \quad for \quad n = 4.$$

Proof. The proof is very similar to that of Lemma 2.3 in [2,4,5] and is roughly described as follows. It suffices to consider the estimate away from $t = 0$. As usual, time decay is given by the Gagliardo-Nirenberg inequalities through the gauge transformed derivative $J \equiv x + it\nabla = M(t)it\nabla M(-t)$ with $M(t) = \exp(i|x|^2/2t)$. In order to express the resulting factors in terms of Q, we accompany J with ∇ on every use of J and exploit the relation $\|J\nabla f\|_{L^2} = \|Qf\|_{L^2}$[2, 4], where the component of angular momentum vanishes by assumption of radial symmetry. One application of Q is therefore equivalent to the second derivatives in the Sobolev estimates and to the time decay of order $O(|t|^{-1})$. For instance, for $n = 3$, $\|f\|_{L^\infty} \le C\|\nabla f\|_{L^6}^{3/4}\|f\|_{L^2}^{1/4} \le C|t|^{-3/4}\|J\nabla f\|_{L^2}^{3/4}\|f\|_{L^2}^{1/4}$, which implies (2.1). For $n = 4$, one derivative in $L^{2n/(n-2)} = L^4$ does not dominate the L^∞ norm, so that the estimtes above are then replaced by $\|f\|_{L^{4/\delta}} \le C\|\nabla f\|_{L^4}^{1-\delta}\|f\|_{L^4}^{\delta} \le C|t|^{-1+\delta}\|J\nabla f\|_{L^2}^{1-\delta}\|\nabla f\|_{L^2}^{\delta}$ with $0 < \delta < 1$, and the final estimate is achieved by $\|f\|_{L^\infty} \le C\|\nabla^3 f\|_{L^2}^{\delta/(3+\delta)}\|f\|_{L^{4/\delta}}^{1-\delta/(3+\delta)}$. A similar calculation implies (2.2), (2.3) and (2.4). Q.E.D.

We let $F_1(v)$ and $F_2(v)$ be as in the introduction.

Lemma 2.2. *Let $v \in X^{2m}(T)$, $2m \geq n+3$ and $|||v|||_{X^{2m}(T)} \leq 1$. Then for $|t| \leq T$*

$$\|F_1(v(t))\|_{Y_t^{2m}} \leq C(1+|t|)^{-d}\|v(t)\|_{X_t^{2m}}\|v(t)\|_{Y_t^{2m}}, \tag{2.5}$$

$$\|F_2(v(t))\|_{H^{2m-1}} + \|F_2(v(t))\|_{Y_t^{2(m-1)}} \leq C(1+|t|)^{-d}\|v(t)\|_{X_t^{2m}}\|v(t)\|_{Y_t^{2m}} \tag{2.6}$$

with $d > 1$.

Proof. From Leibniz' rule, Lemma 2.1 ((2.1), (2.2)) and the commutation relations

$$[\partial_j, P] = [\partial_j, Q] = \partial_j, \quad [Q, P] = 0,$$

we have, in view of the lowest power order of F_1,

$$\|F_1(v(t))\|_{Y_t^{2m}} \leq C(1+|t|)^{-d_1}\|v(t)\|_{X_t^{2m}}^{\ell_0-1}\|v(t)\|_{Y_t^{2m}} \tag{2.7}$$

where

$$1 < d_1 < \begin{cases} n/2 & \text{for } n = 3, 4, \\ \min(n/4, 2) & \text{for } n \geq 5. \end{cases}$$

Hence (2.5) follows.

In the same way as in the proof of (2.7) we have

$$\|F_2(v(t))\|_{H^{2m-1}} + \|F_2(v(t))\|_{Y_t^{2(m-1)}} \leq C(1+|t|)^{-d_2}\|v(t)\|_{X_t^{2m}}\|v(t)\|_{Y_t^{2m}} \tag{2.8}$$

where

$$1 < d_2 < \begin{cases} 9/8 & \text{for } n = 3, \\ 3/2 & \text{for } n = 4, \\ \min(n/4, 2) & \text{for } n \geq 5. \end{cases}$$

We note here that we have used (2.3), (2.4) instead of (2.1), (2.2) to obtain (2.8).

Q.E.D.

The following lemma is needed to handle terms coming from the use of a gauge transformation.

Lemma 2.3. *Let $f \in H^m$ with $m > \frac{n}{2}$ and $g \in L^\infty$. Then*

$$\int_0^\infty |f||g|dr \leq \|g\|_{L^\infty}(\|f\|_{L^\infty} + \|f\|_{L^2}) \leq C\|g\|_{L^\infty}\|f\|_{H^m}.$$

Proof. We have by Schwarz' inequality

$$\int_0^\infty |f||g|dr = \int_0^1 |f||g|dr + \int_1^\infty |f||g|dr$$
$$\leq ||g||_{L^\infty}(||f||_{L^\infty} + (\int_1^\infty |f|^2 r^{n-1} dr)^{1/2}).$$

The second inequality in the lemma now follows from Sobolev's inequality. Q.E.D.

§3. Proof of Theorem 1. For simplicity we consider only the case $t > 0$, for the other case is treated analogously. Since the main task in the proof of global existence of solutions is the derivation of a-priori estimates of local solutions in $X^{2m}(T)$, we focus our attention on that issue under the assumption that (1.1) is well posed in $X^{2m}(T)$ with T satisfying $T \to \infty$ as $\varepsilon_0 \to 0$. An actual proof of the local existence is given by combining a contraction argument on the partly linearized system for (1.1) as in the argument indicated in [2,4] and the use of gauge transformation as in the argument in [8]. Details are omitted and from now on we concentrate on the derivation of a-priori estimates of local solution u in $X^{2m}(T)$, assuming that $T \geq 1$ and $|||u|||_{X^{2m}(T)} \leq 1$ for simplicity. We note that (1.1) is written as (1.3) with

$$F_2(u) = \sum_{\substack{0 \leq |\alpha| \leq \ell_2 \\ 1 \leq |\beta| \leq \ell_3}} \lambda_{\alpha\beta} u^{\alpha_1} \bar{u}^{\alpha_2} |\partial_r u|^{2\beta_1} (\partial_r u)^{2\beta_2} (\partial_r \bar{u})^{2\beta_3}$$
$$= \sum_{\substack{0 \leq |\alpha| \leq \ell_2 \\ 1 \leq |\beta| \leq \ell_3}} \lambda_{\alpha\beta} u^{\alpha_1} \bar{u}^{\alpha_2} (\partial_r u)^{2\beta_2+\beta_1} (\partial_r \bar{u})^{2\beta_3+\beta_1}.$$

Multiplying both sides of (1.2) by P^ℓ with $0 \leq \ell \leq m$, we obtain

$$LP^\ell u = (P+2)^\ell F \tag{3.1}$$

where

$$L = i\partial_t + \frac{1}{2}\Delta.$$

Hence, by the integral equation associated with (3.1) and the unitarity in L^2 of the propagator for the free Schrödinger equation

$$||u(t)||_{Y_t^{2(m-1)}} \leq C(\varepsilon_0 + \int_0^t ||F(u(\tau))||_{Y_\tau^{2(m-1)}} d\tau). \tag{3.2}$$

We apply Lemma 2.2 (2.6) to (3.2) to get

$$\|u(t)\|_{Y_t^{2(m-1)}} \leq C(\varepsilon_0 + \int_0^t (1+\tau)^{-d} \|u(\tau)\|_{X_\tau^{2m}} \|u(\tau)\|_{Y_\tau^{2m}} d\tau). \tag{3.3}$$

In the same way as in the proof of (3.3) we obtain by Lemma 2.2

$$\|u(t)\|_{H^{2m-1}} \leq C(\varepsilon_0 + \int_0^t (1+\tau)^{-d} \|u(\tau)\|_{X_\tau^{2m}} \|u(\tau)\|_{Y_\tau^{2m}} d\tau). \tag{3.4}$$

We next consider the a-priori estimate of $\|P^m u\|_{L^2}$. By (3.1)

$$LP^m u = (P+2)^m F_1 + (P+2)^m F_2. \tag{3.5}$$

By Leibniz' rule and the commutation relation $[\partial_r, P] = \partial_r$ we decompose the second term of the right hand side of (3.5) into three terms as follows:

$$\begin{aligned}
&(P+2)^m F_2 \\
&= F_3 + \sum_{\substack{0 \leq |\alpha| \leq \ell_2 \\ 1 \leq |\beta| \leq \ell_3}} \lambda_{\alpha\beta}(\beta_1 + 2\beta_2) u^{\alpha_1} \bar{u}^{\alpha_2} (\partial_r \bar{u})^{\beta_1 + 2\beta_3} (\partial_r u)^{\beta_1 + 2\beta_2 - 1} (\partial_r P^m u) \\
&\quad + \sum_{\substack{0 \leq |\alpha| \leq \ell_2 \\ 1 \leq |\beta| \leq \ell_3}} \lambda_{\alpha\beta}(\beta_1 + 2\beta_3) u^{\alpha_1} \bar{u}^{\alpha_2} (\partial_r u)^{\beta_1 + 2\beta_2} (\partial_r \bar{u})^{\beta_1 + 2\beta_3 - 1} (\partial_r P^m \bar{u}) \\
&\equiv F_3 + F_4 \cdot \partial_r P^m u + F_5 \cdot \partial_r P^m \bar{u}.
\end{aligned}$$

Hence (3.5) is written as

$$LP^m u = (P+2)^m F_1 + F_3 + F_4 \cdot \partial_r P^m u + F_5 \cdot \partial_r P^m \bar{u}. \tag{3.6}$$

In order to avoid the derivative loss which comes from the term $F_4 \cdot \partial_r P^m u$ we multiply both sides of (3.6) by

$$S = \exp(\int_r^\infty F_4 dr).$$

Then we have

$$\begin{aligned}
(3.7)\, LSP^m u = (LS)P^m u &+ \sum_{1 \leq j \leq n} \partial_j S \cdot \partial_j P^m u \\
&+ S((P+2)^m F_1 + F_3 + F_4 \cdot \partial_r P^m u + F_5 \cdot \partial_r P^m \bar{u}).
\end{aligned}$$

Since

$$\sum_{1\le j\le n} \partial_j S\cdot \partial_j P^m u = \partial_r S\cdot \partial_r P^m u = -SF_4\cdot \partial_r P^m u,$$

we obtain by (3.7)

$$LSP^m u = (LS)P^m u + S((P+2)^m F_1 + F_3 + F_5\cdot \partial_r P^m \bar{u}). \tag{3.8}$$

A direct calculation gives

$$\partial_t S = (\sum_{\substack{0\le|\alpha|\le \ell_2\\ 1\le|\beta|\le\ell_3}} \lambda_{\alpha\beta}(\beta_1+2\beta_2)\sum_{1\le j\le 4} I_j)S,$$

where

$$\begin{aligned}
I_1 &= \int_r^\infty \alpha_1 u^{\alpha_1-1}(\partial_t u)\bar{u}^{\alpha_2}(\partial_r\bar{u})^{\beta_1+2\beta_3}(\partial_r u)^{\beta_1+2\beta_2-1}dr,\\
I_2 &= \int_r^\infty \alpha_2 u^{\alpha_1}\bar{u}^{\alpha_2-1}(\partial_t\bar{u})(\partial_r\bar{u})^{\beta_1+2\beta_3}(\partial_r u)^{\beta_1+2\beta_2-1}dr,\\
I_3 &= \int_r^\infty (\beta_1+2\beta_3)u^{\alpha_1}\bar{u}^{\alpha_2}(\partial_r\bar{u})^{\beta_1+2\beta_3-1}(\partial_r\partial_t\bar{u})(\partial_r u)^{\beta_1+2\beta_2-1}dr,\\
I_4 &= \int_r^\infty (\beta_1+2\beta_2-1)u^{\alpha_1}\bar{u}^{\alpha_2}(\partial_r\bar{u})^{\beta_1+2\beta_3}(\partial_r\partial_t u)(\partial_r u)^{\beta_1+2\beta_2-2}dr.
\end{aligned}$$

We use integration by parts in I_3 and I_4 to obtain

$$\begin{aligned}
I_3 &= -(\beta_1+2\beta_3)u^{\alpha_1}\bar{u}^{\alpha_2}(\partial_r\bar{u})^{\beta_1+2\beta_3-1}(\partial_t\bar{u})(\partial_r u)^{\beta_1+2\beta_2-1}\\
&\quad -\int_r^\infty (\beta_1+2\beta_3)(\partial_r(u^{\alpha_1}\bar{u}^{\alpha_2}(\partial_r\bar{u})^{\beta_1+2\beta_3-1}(\partial_r u)^{\beta_1+2\beta_2-1}))(\partial_t\bar{u})dr,\\
I_4 &= -(\beta_1+2\beta_2-1)u^{\alpha_1}\bar{u}^{\alpha_2}(\partial_r\bar{u})^{\beta_1+2\beta_3}(\partial_t u)(\partial_r u)^{\beta_1+2\beta_2-2}\\
&\quad -\int_r^\infty (\beta_1+2\beta_2-1)(\partial_r(u^{\alpha_1}\bar{u}^{\alpha_2}(\partial_r\bar{u})^{\beta_1+2\beta_3}(\partial_r u)^{\beta_1+2\beta_2-2}))(\partial_t u)dr.
\end{aligned}$$

Hence by Hölder's inequality

$$\begin{aligned}
&||\partial_t S||_{L^\infty} \tag{3.9}\\
&\le C\sum_{\substack{0\le|\alpha|\le \ell_2\\ 1\le|\beta|\le\ell_3}} (\beta_1+2\beta_2)(|\alpha|||u||_{L^\infty}^{(|\alpha|-1)_+}||\partial_r u||_{L^\infty}^{2(|\beta|-1)}\int_r^\infty |\partial_t u||\partial_r u|dr\\
&\quad +(2|\beta|-1)||u||_{L^\infty}^{|\alpha|}||\partial_r u||_{L^\infty}^{2(|\beta|-1)}||\partial_t u||_{L^\infty}\\
&\quad +(2|\beta|-1)|\alpha|||u||_{L^\infty}^{(|\alpha|-1)_+}||\partial_r u||_{L^\infty}^{2(|\beta|-1)}\int_r^\infty |\partial_t u||\partial_r u|dr
\end{aligned}$$

$$+(2|\beta|-1)(2|\beta|-2)||u||_{L^\infty}^{|\alpha|}||u||_{H^{2,\infty}}||\partial_r u||_{L^\infty}^{2(|\beta|-1)}\int_r^\infty |\partial_r u||\partial_t u|dr)||S||_{L^\infty}.$$

From (3.9), Lemma 2.3 and Sobolev's inequality it follows that

$$||\partial_t S||_{L^\infty} \leq C(||\partial_r u||_{L^\infty}||\partial_t u||_{H^m} + ||\partial_t u||_{L^\infty})||S||_{L^\infty}.$$

Hence by Lemma 2.1

$$||\partial_t S||_{L^\infty} \leq C((1+t)^{-d}||u(t)||_{X_t^{2m}}||\partial_t u||_{H^m} + ||\partial_t u||_{L^\infty})||S||_{L^\infty},\ d>1. \tag{3.10}$$

We have by (1.1) and Lemma 2.2

$$||\partial_t u||_{H^m} \leq C(||u||_{H^{m+2}} + ||F(u)||_{H^m}) \leq C||u(t)||_{X_t^{2m}},$$
$$||\partial_t u||_{L^\infty} \leq C(||\Delta u||_{L^\infty} + ||F(u)||_{L^\infty}) \leq C(1+t)^{-d}||u(t)||_{X_t^{2m}},$$

from which with (3.10)

$$||\partial_t S||_{L^\infty} \leq C(1+t)^{-d}||u(t)||_{X_t^{2m}}||S||_{L^\infty},\ d>1. \tag{3.11}$$

We have by Hölder's inequality

$$||S||_{L^\infty} \leq \exp(C\sum_{\substack{0\leq|\alpha|\leq\ell_2\\ 1\leq|\beta|\leq\ell_3}}||u||_{L^\infty}^{|\alpha|}||\partial_r u||_{L^\infty}^{2|\beta|-1}\int_0^\infty |\partial_r u|dr). \tag{3.12}$$

From Lemma 2.3 it follows that

$$\int_0^\infty |\partial_r u|dr \leq ||\partial_r u||_{L^\infty} + ||\partial_r u||_{L^2} \leq C||u||_{H^m}.$$

Thus we obtain by (3.12)

$$||S||_{L^\infty} \leq \exp(C||u(t)||_{H^m}). \tag{3.13}$$

We next consider ΔS. By a simple calculation

$$\partial_r S = (-\sum_{\substack{0\leq|\alpha|\leq\ell_2\\ 1\leq|\beta|\leq\ell_3}}\lambda_{\alpha\beta}(\beta_1+2\beta_2)u^{\alpha_1}\bar{u}^{\alpha_2}(\partial_r\bar{u})^{\beta_1+2\beta_3}(\partial_r u)^{\beta_1+2\beta_2-1})S$$
$$\equiv I_5 S,$$

$$\begin{aligned}\partial_r^2 S = (I_5^2 - \sum_{\substack{0\le|\alpha|\le\ell_2\\1\le|\beta|\le\ell_3}} & \lambda_{\alpha\beta}(\beta_1+2\beta_2)(\alpha_1 u^{\alpha_1-1}\bar{u}^{\alpha_2}(\partial_r\bar{u})^{\beta_1+2\beta_3}(\partial_r u)^{\beta_1+2\beta_2} \\ & + \alpha_2 u^{\alpha_1}\bar{u}^{\alpha_2-1}(\partial_r\bar{u})^{\beta_1+2\beta_3+1}(\partial_r u)^{\beta_1+2\beta_2-1} \\ & + (\beta_1+2\beta_3)u^{\alpha_1}\bar{u}^{\alpha_2}(\partial_r\bar{u})^{\beta_1+2\beta_3-1}(\partial_r u)^{\beta_1+2\beta_2-1}\partial_r^2\bar{u} \\ & + (\beta_1+2\beta_2-1)u^{\alpha_1}\bar{u}^{\alpha_2}(\partial_r\bar{u})^{\beta_1+2\beta_3}(\partial_r u)^{\beta_1+2\beta_2-2}\partial_r^2 u))S.\end{aligned}$$

Hence

$$\begin{aligned}\triangle S &= (\partial_r^2 + \frac{n-1}{r}\partial_r)S \\ &= (I_5^2 - \sum_{\substack{0\le|\alpha|\le\ell_2\\1\le|\beta|\le\ell_3}} \lambda_{\alpha\beta}(\beta_1+2\beta_2)(\alpha_1 u^{\alpha_1-1}\bar{u}^{\alpha_2}(\partial_r\bar{u})^{\beta_1+2\beta_3}(\partial_r u)^{\beta_1+2\beta_2} \\ &+ \alpha_2 u^{\alpha_1}\bar{u}^{\alpha_2-1}(\partial_r\bar{u})^{\beta_1+2\beta_3+1}(\partial_r u)^{\beta_1+2\beta_2-1} \\ &+ u^{\alpha_1}\bar{u}^{\alpha_2}(\partial_r\bar{u})^{\beta_1+2\beta_3-1}(\partial_r u)^{\beta_1+2\beta_2-1}\triangle\bar{u} \\ &+ u^{\alpha_1}\bar{u}^{\alpha_2}(\partial_r\bar{u})^{\beta_1+2\beta_3}(\partial_r u)^{\beta_1+2\beta_2-2}\triangle u \\ &+ (\beta_1+2\beta_3-1)u^{\alpha_1}\bar{u}^{\alpha_2}(\partial_r\bar{u})^{\beta_1+2\beta_3-2}(\partial_r u)^{\beta_1+2\beta_2-1}\frac{1}{2}\partial_r((\partial_r\bar{u})^2) \\ &+ (\beta_1+2\beta_2-2)u^{\alpha_1}\bar{u}^{\alpha_2}(\partial_r\bar{u})^{\beta_1+2\beta_3}(\partial_r u)^{\beta_1+2\beta_2-3}\frac{1}{2}\partial_r((\partial_r u)^2)))S.\end{aligned}$$

By Hölder's inequality

$$\begin{aligned}(3.14)\quad & \|\triangle S\|_{L^\infty} \\ & \le C \sum_{\substack{0\le|\alpha|\le\ell_2\\1\le|\beta|\le\ell_3}} (\|u\|_{L^\infty}^{2|\alpha|}\|\partial_r u\|_{L^\infty}^{4|\beta|-2} + |\alpha|\|u\|_{L^\infty}^{(|\alpha|-1)_+}\|\partial_r u\|_{L^\infty}^{2|\beta|} \\ & + \|u\|_{L^\infty}^{|\alpha|}\|\partial_r u\|_{L^\infty}^{2(|\beta|-1)}\|\triangle u\|_{L^\infty} \\ & + (2|\beta|-2)\|u\|_{L^\infty}^{|\alpha|}\|\partial_r u\|_{L^\infty}^{(2|\beta|-3)_+}\sum_{1\le j,k\le n}\|\partial_j\partial_k u\|_{L^2}\|\partial_j\partial_k u\|_{L^\infty})\|S\|_\infty.\end{aligned}$$

We apply (3.13) and Lemma 2.1 to the right hand side of (3.14) to get

$$(3.15)\qquad \|\triangle S\|_{L^\infty} \le C(1+t)^{-d}\|u(t)\|_{X_t^{2m}},\ d>1.$$

We now turn to the a-priori estimate of $\|P^m u\|_{L^2}$ through that of $\|SP^m u\|_{L^2}$. Multiplying both sides of (3.8) by $\overline{SP^m u}$, integrating in x and taking the imaginary part, we obtain

$$(3.16)\quad \|SP^m u(t)\|_{L^2}^2$$

$$\leq C\varepsilon_0^2 + 2\int_0^t (\|(LS)^{1/2}S^{1/2}P^m u(\tau)\|_{L^2}^2 + \|S(P+2)^m F_1\|_{L^2}\|SP^m u(\tau)\|_{L^2}$$

$$+ \|SF_3\|_{L^2}\|SP^m u(\tau)\|_{L^2} + |\operatorname{Im}(SF_5 \cdot \partial_r P^m \bar{u}, SP^m u)|)d\tau.$$

By (3.11), (3.13), (3.15) and Lemma 2.2, the right hand side of (3.16) is bounded by

$$(3.17)\quad C\varepsilon_0^2 + C\int_0^t (1+\tau)^{-d}\|u(\tau)\|_{X_r^{2m}}\|u(\tau)\|_{Y_r^{2m}}^2 + |\operatorname{Im}(SF_5 \cdot \partial_r P^m \bar{u}, SP^m u)|d\tau.$$

Integration by parts gives

$$(SF_5 \cdot \partial_r P^m \bar{u}, SP^m u)$$

$$= \sum_{\substack{0\leq|\alpha|\leq \ell_2 \\ 1\leq|\beta|\leq\ell_3}} \lambda_{\alpha\beta}(\beta_1+2\beta_3)\int_0^\infty u^{\alpha_1}\bar{u}^{\alpha_2}(\partial_r u)^{\beta_1+2\beta_2}(\partial_r\bar{u})^{\beta_1+2\beta_3-1}|S|^2(\partial_r P^m\bar{u})P^m\bar{u}r^{n-1}dr$$

$$= -\frac{1}{2}\sum_{\substack{0\leq|\alpha|\leq \ell_2 \\ 1\leq|\beta|\leq\ell_3}} \lambda_{\alpha\beta}(\beta_1+2\beta_3)\int_0^\infty (\partial_r(|S|^2u^{\alpha_1}\bar{u}^{\alpha_2}(\partial_r u)^{\beta_1+2\beta_2}(\partial_r\bar{u})^{\beta_1+2\beta_3-1})(P^m\bar{u})^2r^{n-1}$$

$$+ |S|^2u^{\alpha_2}\bar{u}^{\alpha_2}(\partial_r u)^{\beta_1+2\beta_2}(\partial_r\bar{u})^{\beta_1+2\beta_3-1}\frac{(n-1)}{r}(P^m\bar{u})^2r^{n-1})dr$$

$$= -\frac{1}{2}\sum_{\substack{0\leq|\alpha|\leq \ell_2 \\ 1\leq|\beta|\leq\ell_3}} \lambda_{\alpha\beta}(\beta_1+2\beta_3)\int_0^\infty I_6(r)r^{n-1}dr.$$

We have

$$\begin{aligned} I_6(r) = \{&((\partial_r S)\bar{S} + \bar{S}\partial_r S)u^{\alpha_1}\bar{u}^{\alpha_2}(\partial_r u)^{\beta_1+2\beta_2}(\partial_r\bar{u})^{\beta_1+2\beta_3-1} \\ &+ |S|^2(\alpha_1 u^{\alpha_1-1}\bar{u}^{\alpha_2}(\partial_r u)^{\beta_2+2\beta_2+1}(\partial_r\bar{u})^{\beta_1+2\beta_3-1} \\ &+ \alpha_2 u^{\alpha_1}\bar{u}^{\alpha_2-1}(\partial_r u)^{\beta_1+2\beta_2}(\partial_r\bar{u})^{\beta_1+2\beta_3} \\ &+ (\beta_1+2\beta_2)u^{\alpha_1}\bar{u}^{\alpha_2}(\partial_r u)^{\beta_1+2\beta_2-1}(\partial_r\bar{u})^{\beta_1+2\beta_3-1}\partial_r^2 u \\ &+ (\beta_1+2\beta_3-1)u^{\alpha_1}\bar{u}^{\alpha_2}(\partial_r u)^{\beta_1+2\beta_2}(\partial_r\bar{u})^{\beta_1+2\beta_3-2}\partial_r^2\bar{u} \\ &+ u^{\alpha_1}\bar{u}^{\alpha_2}(\partial_r u)^{\beta_1+2\beta_2}(\partial_r\bar{u})^{\beta_1+2\beta_3-1}\frac{(n-1)}{r})\}(P^m\bar{u})^2 \\ \equiv \{&I_7 + |S|^2(I_8+I_9+I_{10}+I_{11}+I_{12})\}(P^m\bar{u})^2. \end{aligned}$$

Since $\triangle = \partial_r^2 + \frac{n-1}{r}\partial_r$, I_{12} and parts of I_{10} and I_{11} are combined to give

$$I_{10}+I_{11}+I_{12} = u^{\alpha_1}\bar{u}^{\alpha_2}((\partial_r u)^{\beta_1+2\beta_2-1}(\partial_r\bar{u})^{\beta_1+2\beta_3-1}\triangle u$$

$$+ (\partial_r u)^{\beta_1+2\beta_2}(\partial_r \bar{u})^{\beta_1+2\beta_3-2}\Delta\bar{u}$$
$$+ (\beta_1+2\beta_2-1)(\partial_r u)^{\beta_1+2\beta_2-2}(\partial_r \bar{u})^{\beta_1+2\beta_3-1}\frac{1}{2}\partial_r(\partial_r u)^2$$
$$+ (\beta_1+2\beta_3-2)(\partial_r u)^{\beta_1+2\beta_2}(\partial_r \bar{u})^{\beta_1+2\beta_3-3}\frac{1}{2}\partial_r(\partial_r \bar{u})^2).$$

Hence by (3.12) and Lemma 2.1

$$\begin{aligned}(3.18)\quad & |\operatorname{Im}(SF_5\cdot\partial_r P^m\bar{u}, SP^m u)| \\ &\le C\sum_{\substack{0\le|\alpha|\le \ell_2\\ 1\le|\beta|\le\ell_3}}(\|S\|_{L^\infty}\|\partial_r S\|_{L^\infty}\|u\|_{L^\infty}^{|\alpha|}\|\partial_r u\|_{L^\infty}^{2|\beta|-1} \\ &\quad+\|S\|_{L^\infty}^2(|\alpha|\|u\|_{L^\infty}^{(|\alpha|-1)_+}\|\partial_r u\|_{L^\infty}^{2|\beta|}+\|u\|_{L^\infty}^{|\alpha|}\|\partial_r u\|_{L^\infty}^{2(|\beta|-1)}\|\Delta u\|_{L^\infty} \\ &\quad+(2|\beta|-1)\|u\|_{L^\infty}^{|\alpha|}\|\partial_r u\|_{L^\infty}^{2(|\beta|-1)}\sum_{1\le j,k\le n}\|\partial_j\partial_k u\|_{L^\infty}))\|P^m u\|_{L^2}^2 \\ &\le C(1+t)^{-d}\|u(t)\|_{X_t^{2m}}\|P^m u\|_{L^2}^2 \quad \text{with } d>1.\end{aligned}$$

From (3.16)-(3.18) it follows that

$$(3.19)\qquad \|SP^m u(t)\|_{L^2}^2 \le C(\varepsilon_0^2+\int_0^t(1+\tau)^{-d}\|u(\tau)\|_{Y_\tau^{2m}}^2 d\tau).$$

Similarly, for ℓ with $1\le\ell\le m$,

$$(3.20)\qquad \|SP^{m-\ell}\Delta^\ell u(t)\|_{L^2}^2 \le C(\varepsilon_0^2+\int_0^t(1+\tau)^{-d}\|u(\tau)\|_{Y_\tau^{2m}}^2 d\tau).$$

Let

$$\|u(t)\|_{\tilde{Y}_t^{2m}} = \|u(t)\|_{Y_t^{2(m-1)}}+\|u(t)\|_{H^{2m-1}}+\sum_{0\le\ell\le m}\|SP^{m-\ell}\Delta^\ell u(t)\|_{L^2}.$$

Then (3.3), (3.4), (3.19) and (3.20) give

$$(3.21)\qquad \|u(t)\|_{\tilde{Y}_t^{2m}}^2 \le C(\varepsilon_0^2+\int_0^t(1+\tau)^{-d}\|u(\tau)\|_{Y_\tau^{2m}}^2 d\tau).$$

We prove

$$(3.22)\qquad \|u(t)\|_{X_t^{2m}} \le C\|u(t)\|_{\tilde{Y}_t^{2m}}.$$

By (1.1)

$$(3.23)\qquad Qu(t) = Pu(t)+2itF.$$

Then by Lemma 2.2 (2.6)

$$\begin{aligned} \|Qu(t)\|_{Y_t^{2(m-1)}} &\leq \|u(t)\|_{Y_t^{2m}} + C|t|\|F\|_{Y_t^{2(m-1)}} \\ &\leq \|u(t)\|_{Y_t^{2m}} + C\|u(t)\|_{X_t^{2m}}\|u(t)\|_{Y_t^{2m}}. \end{aligned} \tag{3.24}$$

Similarly, by (3.23), (3.24), Lemma 2.2

$$\begin{aligned} &(1+t)^{-((4-n)/2)_+}\|Q^2u(t)\|_{Y_t^{2(m-2)}} \\ &\leq \|Qu(t)\|_{Y_t^{2(m-1)}} + C(1+t)^{((4-n)/2)_+}|t|\|QF\|_{Y_t^{2(m-2)}} \\ &\leq \|u(t)\|_{Y_t^{2m}} + C\|u(t)\|_{X_t^{2m}}\|u(t)\|_{Y_t^{2m}} \end{aligned} \tag{3.25}$$

(for the details of the derivation of (3.25) see [4, (3.23)]).

Combining (3.24) and (3.25), we have

$$\|u(t)\|_{X_t^{2m}} \leq C(\|u(t)\|_{Y_t^{2m}} + \|u(t)\|_{X_t^{2m}}\|u(t)\|_{Y_t^{2m}})$$

which implies

$$\|u(t)\|_{X_t^{2m}} \leq C\|u(t)\|_{Y_t^{2m}}$$

by assumption. By (3.13)

$$\|u(t)\|_{Y_t^{2m}} \leq \exp(C\|u(t)\|_{H^m})\|u(t)\|_{\tilde{Y}_t^{2m}}$$

from which we have (3.22). We use (3.22) in (3.21) to have

$$\|u(t)\|^2_{\tilde{Y}_t^{2m}} \leq C(\varepsilon_0^2 + \int_0^t (1+\tau)^{-d}\|u(\tau)\|^2_{\tilde{Y}_\tau^{2m}}d\tau)$$

since $\|u(t)\|_{Y_t^{2m}} \leq \|u(t)\|_{X_t^{2m}}$ by definition. Gronwall's inequality yields

$$\begin{aligned} \|u(t)\|^2_{\tilde{Y}_t^{2m}} &\leq C\varepsilon_0^2\exp(C\int_0^t (1+\tau)^{-d}d\tau) \\ &\leq M\varepsilon_0^2. \end{aligned} \tag{3.26}$$

From (3.22) and (3.26) we have the desired a-priori estimate, that is,

$$\|u(t)\|_{X_t^{2m}} \leq C\varepsilon_0.$$

Q.E.D.

REFERENCES

1. H. Chihara, *Local existence for the semilinear Schrödinger equations in one space dimension*, preprint (1993).
2. J. Ginibre and N. Hayashi, *Almost global solutions to quadratic nonlinear Schrödinger equations*, Math. Z. (in press)..
3. N. Hayashi, *The initial value problem for the derivative nonlinear Schrödinger equation in the energy space*, Nonlinear Analysis, TMA **32** (1993), 823-833.
4. N. Hayashi, *Global existence of small solutions to quadratic nonlinear Schrödinger equations*, Comm. P.D.E. **18** (1993), 1109-1124.
5. N. Hayashi, *Global and almost global solutions to quadratic nonlinear Schrödinger equations with small initial data*, preprint (1992).
6. N. Hayashi and T. Ozawa, *On the derivative nonlinear Schrödinger equation*, Physica D **55** (1992), 14-36.
7. N. Hayashi and T. Ozawa, *Finite energy solutions of nonlinear Schrödinger equations of derivative type*, SIAM J. Math. Anal. (in press).
8. N. Hayashi and T. Ozawa, *Remarks on nonlinear Schrödinger equations in one space dimension*, Diff. and Integral Eq. (in press).
9. C.E. Kenig, G. Ponce and L. Vega, *Small solutions to nonlinear Schrödinger equations*, Ann. Inst. Henri Poincaré, Analyse non linéaire **10** (1993), 255-288.
10. C.E. Kenig, G. Ponce and L. Vega, *Higher order non-linear dispersive equations*, preprint (1992).
11. S. Klainerman and G. Ponce, *Global, small amplitude solutions to nonlinear evolution equations*, Comm. Pure Appl. Math. **36** (1983), 133-141.
12. T. Ozawa, *Remarks on quadratic nonlinear Schrödinger equations*, preprint (1992).
13. J. Shatah, *Global existence of small solutions to nonlinear evolution equations*, J. Diff. Eq. **46** (1982), 409-425.
14. A. Soyeur, *The Cauchy problem for the Ishimori equations*, J. Funct. Anal. **105** (1992), 232-255.
15. Tan Shaobin and Zhang Linghai, *On weak solution of the mixed nonlinear Schrödinger equations*, preprint (1992).