



Title	Characteristic vector fields for first order partial differential equations
Author(s)	Izumiya, S.
Citation	Hokkaido University Preprint Series in Mathematics, 228, 1-9
Issue Date	1994-2-1
DOI	10.14943/83375
Doc URL	<a href="http://hdl.handle.net/2115/68979">http://hdl.handle.net/2115/68979</a>
Type	bulletin (article)
File Information	pre228.pdf



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**CHARACTERISTIC VECTOR  
FIELDS FOR FIRST ORDER  
PARTIAL DIFFERENTIAL  
EQUATIONS**

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**Series #228. February 1994**

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# CHARACTERISTIC VECTOR FIELDS FOR FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

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## 0. INTRODUCTION

We consider the following two kinds of first order partial differential equations:

$$(Q) \quad \sum_{i=1}^n a_i(x, y) \frac{\partial y}{\partial x_i} - b(x, y) = 0$$

$$(H) \quad H(x_1, \dots, x_n, \frac{\partial y}{\partial x_1}, \dots, \frac{\partial y}{\partial x_n}) = 0,$$

where  $a_i(x, y)$ ,  $b(x, y)$  and  $H(x, p)$  are  $C^\infty$ -functions. Here the equation (Q) is called a *quasilinear first order partial differential equation* (briefly, a *quasilinear equation*) and (H) is called a *Hamilton-Jacobi equation*. These equations are well studied in several articles ([2],[4],[5],[6],[8],[9],[10],[13],[14],[15],[18], etc.). For the study of quasilinear equations, the theory of entropy solutions has provided the right weak setting (see, for example [9]). For Hamilton-Jacobi equations, the theory of viscosity solutions is appropriate one ([2],[3]). However, these notions of weak solutions have quite different features. Under the some assumptions, the entropy solutions are discontinuous and the viscosity solutions are continuous.

In this note we study the difference between quasilinear equations and Hamilton-Jacobi equations from the view point of the geometric theory of first order partial differential equations.

The goal of the present work is to give characterizations for each equation in terms of characteristic vector fields of equations. In relation to this study, Tsuji[16] considered what a Hamilton-Jacobi equation is. He has given an answer from the view point of the classical theory of first order partial differential equations (see also [17]). Here, we give an answer (Theorem 3.1 and Corollary 3.2) to Tsuji's question from the view point of the geometric theory. Moreover, we also give a characterization of quasilinear equations (Theorem 2.2).

We refer the following two typical examples of these equations.

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*Key words and phrases.* Characteristic vector fields, first order Partial Differential Equations.

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**Example 0.1.** We consider the following equations.

$$(Q') \quad \frac{\partial y}{\partial x_1} + 2y \frac{\partial y}{\partial x_2} = 0$$

$$(H') \quad \frac{\partial y}{\partial x_1} + \left(\frac{\partial y}{\partial x_2}\right)^2 = 0.$$

We can explicitly solve these equations by the classical method of characteristics, when the initial condition is  $y(0, x_2) = \sin x_2$ . The pictures of the graph of geometric (multi-valued) solutions of these equations are given in Figure 1. These pictures are useful to understand the difference between these two equations. We can observe that the geometric solution for (Q') is a smooth submanifold but for (H') is not smooth in the  $(x_1, x_2, y)$ -space.

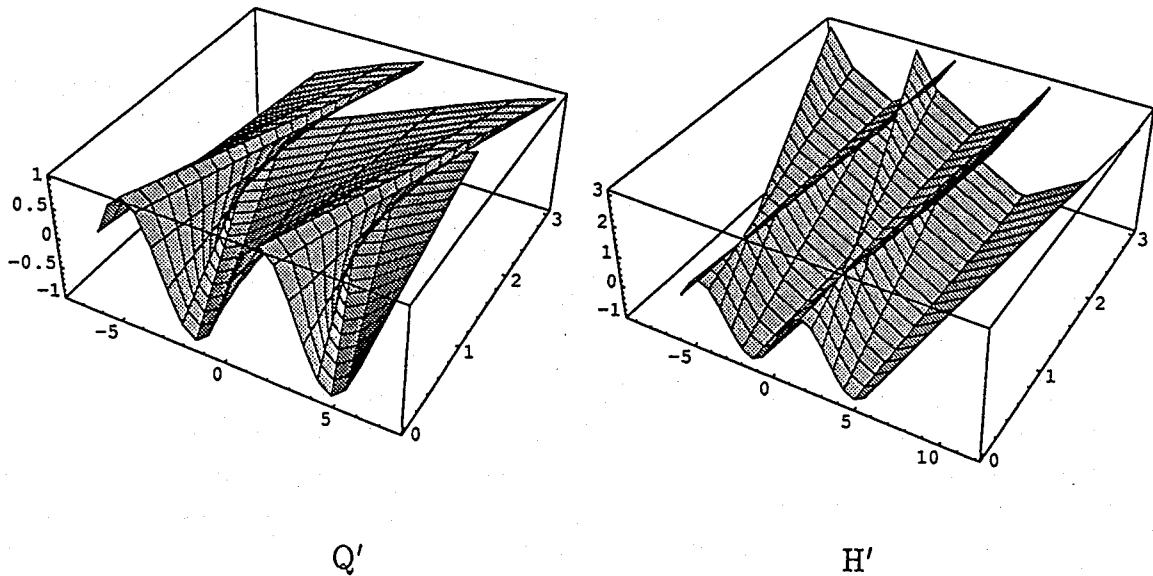


Figure 1

All maps are differentiable of class  $C^\infty$  unless stated otherwise.

## 1. GEOMETRY OF FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

We construct the geometric framework of equations in the projective cotangent bundle  $\pi : PT^*(\mathbb{R}^n \times \mathbb{R}) \rightarrow \mathbb{R}^n \times \mathbb{R}$ . We review geometric properties of this space. Consider the tangent bundle  $\tau : TPT^*(\mathbb{R}^n \times \mathbb{R}) \rightarrow PT^*(\mathbb{R}^n \times \mathbb{R})$  and the differential map  $d\pi : TPT^*(\mathbb{R}^n \times \mathbb{R}) \rightarrow T(\mathbb{R}^n \times \mathbb{R})$  of  $\pi$ .

For any  $X \in TPT^*(\mathbb{R}^n \times \mathbb{R})$ , there exists an element  $\alpha \in T_{(x,y)}^*(\mathbb{R}^n \times \mathbb{R})$  such that  $\tau(X) = [\alpha]$ . For an element  $V \in T_{(x,y)}(\mathbb{R}^n \times \mathbb{R})$ , the property  $\alpha(V) = 0$  does not depend on the choice of representative of the class  $[\alpha]$ . Thus we can define the canonical contact structure on  $PT^*(\mathbb{R}^n \times \mathbb{R})$  by

$$K = \{X \in TPT^*(\mathbb{R}^n \times \mathbb{R}) \mid \tau(X)(d\pi(X)) = 0\}.$$

Because of the trivialization  $PT^*(\mathbb{R}^n \times \mathbb{R}) \cong (\mathbb{R}^n \times \mathbb{R}) \times P(\mathbb{R}^n \times \mathbb{R})^*$ , we call  $((x_1, \dots, x_n, y), [\xi_1; \dots; \xi_n; \eta])$  a homogeneous coordinate, where  $[\xi_1; \dots; \xi_n; \eta]$  is the homogeneous coordinate of the dual projective space  $P(\mathbb{R}^n \times \mathbb{R})^*$ .

It is easy to show that  $X \in K_{((x,y),[\xi;\eta])}$  if and only if  $\sum_{i=1}^n \mu_i \xi_i + \lambda \eta = 0$ , where  $d\tilde{\pi}(X) = \sum_{i=1}^n \mu_i \frac{\partial}{\partial x_i} + \lambda \frac{\partial}{\partial y}$ .

We remark that  $PT^*(\mathbb{R}^n \times \mathbb{R})$  is a fibrewise compactification of the 1-jet space  $J^1(\mathbb{R}^n, \mathbb{R})$  as follows : We consider an open subset  $U_\eta = \{((x,y), [\xi;\eta]) | \eta \neq 0\}$  of  $PT^*(\mathbb{R}^n \times \mathbb{R})$ . For any  $((x,y), [\xi;\eta]) \in U_\eta$ , we have

$$((x_1, \dots, x_n, y), [\xi_1; \dots; \xi_n; \eta]) = ((x_1, \dots, x_n, y), [-\frac{\xi_1}{\eta}; \dots; -\frac{\xi_n}{\eta}; -1])$$

so that we may adopt the corresponding *affine coordinate* as

$$((x_1, \dots, x_n, y), (p_1, \dots, p_n)),$$

where  $p_i = -\frac{\xi_i}{\eta}$ . On  $U_\eta$  we can easily show that  $\theta^{-1}(0) = K|U_\eta$ , where  $\theta = dy - \sum_{i=1}^n p_i dx_i$ . This means that  $U_\eta$  may be identified to the 1-jet space  $J^1(\mathbb{R}^n, \mathbb{R})$ . We call the above coordinate a *canonical coordinate*. Throughout the remainder of this paper, we use this identification so that we have

$$J^1(\mathbb{R}^n, \mathbb{R}) \subset PT^*(\mathbb{R}^n \times \mathbb{R}).$$

A *first order partial differential equation* (briefly, an *equation*) is defined to be a (not necessary closed) hypersurface  $E \subset PT^*(\mathbb{R}^n \times \mathbb{R})$ .

By the philosophy of Lie, we may define the notion of solutions as follows: An immersion  $i : L \rightarrow PT^*(\mathbb{R}^n \times \mathbb{R})$  is said to be a *Legendrian immersion* if  $\dim L = n$  and  $di_q(T_q L) \subset K_{i(q)}$  for any  $q \in L$ . A *geometric solution* of  $E$  is a Legendrian immersion  $i : L \rightarrow PT^*(\mathbb{R}^n \times \mathbb{R})$  with  $\text{Image } L \subset E$ .

We now review the classical method of characteristics to solve equations (cf., [1],[13]). For a submanifold  $N \subset PT^*(\mathbb{R}^n \times \mathbb{R})$ , a point  $z \in N$  is said to be *contact regular* (or *noncharacteristic*) if  $TN$  and  $K$  transversally intersect at  $z$ . The *characteristic direction* at a contact regular point  $z$  in an equation  $E$  is the skew-orthogonal complement of  $T_z E \cap K_z$  in  $K_z$ , where the skew scalar product is defined as  $d\alpha|_{\alpha=0}$  and  $\alpha$  is a local 1-form such that  $\alpha^{-1}(0) = K$ . The *Cauchy problem* for an equation  $E$  with initial submanifold  $N$  consists of determining an integral manifold  $L$  of the contact plane field  $K$ , lying in  $E$  and containing the initial submanifold  $N$ , where  $N$  is an integral submanifold of  $K$  with  $\dim_{\mathbb{R}} N = n - 1$ . Then we have the following classical existence theorem of geometric solutions (cf., [1],[13]).

**Theorem 1.1.** *Let  $z$  be a contact regular point of the initial manifold  $N$ . Then there exists a neighbourhood  $U$  of  $z$  such that the geometric solution of the Cauchy problem for  $E \cap U$  with initial condition  $N \cap U$  exists and is locally unique.*

We do not give the proof of the above theorem, however we remark that the characteristic direction of the equation at each point  $z \in L$  is tangent to  $L$ . The integral curves of the field of characteristic directions on a contact regular equation  $E$  are called *the characteristics of the equation  $E$* , so that the characteristics through the initial manifold  $N$  is contained in the geometric solution  $L$ .

If the equation  $E$  is written in the form  $E \cap J^1(\mathbb{R}^n, \mathbb{R}) = F^{-1}(0)$ , where  $F : J^1(\mathbb{R}^n, \mathbb{R}) \rightarrow \mathbb{R}$  is a submersion, then we can calculate explicitly the differential equation of the characteristics which is given by the following vector field on  $J^1(\mathbb{R}^n, \mathbb{R})$  under the canonical coordinate (cf., [13]):

$$X_F = \sum_{i=1}^n \frac{\partial F}{\partial p_i} \frac{\partial}{\partial x_i} + \sum_{i=1}^n (p_i \frac{\partial F}{\partial p_i} - F) \frac{\partial}{\partial y} - \sum_{i=1}^n (\frac{\partial F}{\partial x_i} + p_i \frac{\partial F}{\partial y}) \frac{\partial}{\partial p_i}.$$

In the last part of this section we refer the following fundamental property of the Legendrian submanifold in  $J^1(\mathbb{R}^n, \mathbb{R})$ .

**Fact 1.2.** *Let  $L \subset J^1(\mathbb{R}^n, \mathbb{R})$  be a Legendrian submanifold. Then  $\text{rank } d(\pi|L)_q < n$  if and only if  $\text{rank } d(\tilde{\pi}|L)_q < n$  for a point  $q \in L$ .*

Here,  $\pi : J^1(\mathbb{R}^n, \mathbb{R}) \rightarrow \mathbb{R}^n \times \mathbb{R}$  is given by  $\pi(x, y, p) = (x, y)$  and  $\tilde{\pi} : J^1(\mathbb{R}^n, \mathbb{R}) \rightarrow \mathbb{R}^n$  is  $\tilde{\pi}(x, y, p) = x$ .

The proof is an easy exercise in the course of basic linear algebra, so that we omit it. However, this fact suggests that the geometric framework for quasilinear equation cannot be formulated in  $J^1(\mathbb{R}^n, \mathbb{R})$  (cf., Example 0.1).

## 2. FIRST ORDER QUASILINEAR PARTIAL DIFFERENTIAL EQUATIONS

In this section we treat the quasilinear equation from the view point of the geometric theory which is introduced in the previous section. A *quasilinear equation* is defined to be a hypersurface

$$E(a_1, \dots, a_n, b) = \{((x, y), [\xi; \eta]) \in PT^*(\mathbb{R}^n \times \mathbb{R}) \mid \sum_{i=1}^n a_i(x, y) \xi_i + b(x, y) \eta = 0\}$$

where  $a_i(x, y), b(x, y)$  are  $C^\infty$ -function with  $(a_1(x, y), \dots, a_n(x, y)) \neq (0, \dots, 0)$ .

We remark that  $\sum_{i=1}^n a_i(x, y) \xi_i + b(x, y) \eta = 0$  if and only if  $\sum_{i=1}^n a_i(x, y) p_i - b(x, y) = 0$  under the canonical coordinate of  $U_\eta = J^1(\mathbb{R}^n, \mathbb{R})$ .

We consider the meaning of the notion of geometric solutions. Let  $S$  be a smooth hypersurface in  $\mathbb{R}^n \times \mathbb{R}$ , then we have a unique Legendrian submanifold  $\hat{S}$  in  $PT^*(\mathbb{R}^n \times \mathbb{R})$  such that  $\pi(\hat{S}) = S$  which is given as follows :

$$\hat{S} = \{((x, y), [\xi; \eta]) \mid \text{the vector } \sum_{i=1}^n \xi_i \frac{\partial}{\partial x_i} + \eta \frac{\partial}{\partial y} \text{ is a normal vector of } S \text{ in } \mathbb{R}^n \times \mathbb{R}\}.$$

It follows that if  $L$  is a geometric solution of  $E(a_1, \dots, a_n, b)$  and  $\pi|L : L \rightarrow \mathbb{R}^n \times \mathbb{R}$  is immersion, then we have  $L = \widehat{\pi(L)}$ .

We consider the condition of a smooth hypersurface  $S$  that  $\hat{S}$  is contained in  $E(a_1, \dots, a_n, b)$ . For any  $(x_0, y_0) \in S$ , there exists a smooth submersion germ  $f : (\mathbb{R}^n \times \mathbb{R}, (x_0, y_0)) \rightarrow (\mathbb{R}, 0)$  such that  $(f^{-1}(0), (x_0, y_0)) = (S, (x_0, y_0))$ . A vector  $\sum_{i=1}^n \mu_i \frac{\partial}{\partial x_i} + \lambda \frac{\partial}{\partial y}$  is tangent to  $S$  at  $(x, y) \in (S, (x_0, y_0))$  if and only if it satisfies  $\sum_{i=1}^n \mu_i \frac{\partial f}{\partial x_i} + \lambda \frac{\partial f}{\partial y} = 0$  at  $(x, y)$ . Then we have the following representation of  $\hat{S}$  :

$$(\hat{S}, ((x_0, y_0); [\xi_0; \eta_0])) = \{((x, y), [\frac{\partial f}{\partial x}; \frac{\partial f}{\partial y}]) \mid (x, y) \in (S, (x_0, y_0))\}.$$

We now consider the following vector field on  $\mathbb{R}^n \times \mathbb{R}$  associated with the quasilinear equation  $E(a_1, \dots, a_n, b)$  :

$$X(a_1, \dots, a_n, b) = \sum_{i=1}^n a_i(x, y) \frac{\partial}{\partial x_i} + b(x, y) \frac{\partial}{\partial y}.$$

Then we have the following proposition.

**Proposition 2.1.** *Let  $S$  be a smooth hypersurface in  $\mathbb{R}^n \times \mathbb{R}$ . Then  $\hat{S}$  is a geometric solution of  $E(a_1, \dots, a_n, b)$  if and only if the vector field  $X(a_1, \dots, a_n, b)$  is tangent to  $S$ .*

*Proof.* We have a local representation

$$(\hat{S}, ((x_0, y_0); [\xi_0; \eta_0])) = \left\{ \left( (x, y), \left[ \frac{\partial f}{\partial x}; \frac{\partial f}{\partial y} \right] \right) \mid (x, y) \in (S, (x_0, y_0)) \right\},$$

where  $f : (\mathbb{R}^n \times \mathbb{R}, (x_0, y_0)) \rightarrow (\mathbb{R}, 0)$  is a submersion germ with  $f^{-1}(0) = S$ . It follows that  $\hat{S} \subset E(a_1, \dots, a_n, b)$  if and only if

$$\sum_{i=1}^n a_i(x, y) \frac{\partial f}{\partial x_i} + b(x, y) \frac{\partial f}{\partial y} = 0$$

for  $(x, y) \in (S, (x_0, y_0))$ . The last condition is equivalent to the condition that  $X(a_1, \dots, a_n, b)$  is tangent to  $S$  at  $(x, y)$ .

In the classical theory, the vector field  $X(a_1, \dots, a_n, b)$  is called a *characteristic vector field of the quasilinear equation*  $E(a_1, \dots, a_n, b)$  (cf., [1]). However, we already define the notion of characteristic vector fields for general first order partial differential equations in §1. We study the relation between these notions of characteristic vector fields. On  $J^1(\mathbb{R}^n, \mathbb{R})$ , we use the equation  $F(x, y, p) = \sum_{i=1}^n a_i(x, y) p_i - b(x, y) = 0$ . By definition, the characteristic vector field is given as follows :

$$\begin{aligned} X_F &= \sum_{i=1}^n a_i(x, y) \frac{\partial}{\partial x_i} + b(x, y) \frac{\partial}{\partial y} \\ &\quad - \sum_{i=1}^n \left( \sum_{j=1}^n \left( \frac{\partial a_i(x, y)}{\partial x_j} p_j - \frac{\partial b(x, y)}{\partial x_j} \right) + p_i \left( \sum_{j=1}^n \frac{\partial a_i(x, y)}{\partial y} p_j - \frac{\partial b(x, y)}{\partial y} \right) \right) \frac{\partial}{\partial p_i}. \end{aligned}$$

It follows that the projection of the characteristic vector field  $d\pi(X_F)$  is equal to the vector field  $X(a_1, \dots, a_n, b)$ . This fact characterizes the quasilinear equation as the following theorem shows.

**Theorem 2.2.** *Let  $E \subset PT^*(\mathbb{R}^n \times \mathbb{R})$  be a closed equation. Then the following are equivalent.*

- (1)  $E$  is a quasilinear equation.



(2) There exists a function  $F : J^1(\mathbb{R}^n, \mathbb{R}) \rightarrow \mathbb{R}$  with  $(\frac{\partial F}{\partial p_1}, \dots, \frac{\partial F}{\partial p_n}) \neq (0, \dots, 0)$  on  $F^{-1}(0)$  and  $E \cap J^1(\mathbb{R}^n, \mathbb{R}) = F^{-1}(0)$  such that  $d\pi(X_F)$  is a vector field on  $\mathbb{R}^n \times \mathbb{R}$ .

*Proof.* Assume that  $d\pi(X_F)$  is a vector field on  $\mathbb{R}^n \times \mathbb{R}$ . Since

$$d\pi(X_F) = \sum_{i=1}^n \frac{\partial F}{\partial p_i} \frac{\partial}{\partial x_i} + \sum_{i=1}^n (p_i \frac{\partial F}{\partial p_i} - F) \frac{\partial}{\partial y},$$

the assumption means that  $\frac{\partial F}{\partial p_i}(x, y, p)$  and  $\sum_{i=1}^n (p_i \frac{\partial F}{\partial p_i} - F)(x, y, p)$  are independent on  $(p_1, \dots, p_n)$ -variables, so that we may put  $a_i(x, y) = \frac{\partial F}{\partial p_i}(x, y, p)$  for  $i = 1, \dots, n$ . It follows that

$$\begin{aligned} F(x, y, p) &= F(x, y, 0) + \int_0^1 \frac{dF(x, y, tp)}{dt} dt \\ &= F(x, y, 0) + \int_0^1 \sum_{i=1}^n \frac{\partial F}{\partial p_i}(x, y, tp) p_i dt \\ &= F(x, y, 0) + \sum_{i=1}^n \int_0^1 a_i(x, y) dt p_i \\ &= F(x, y, 0) + \sum_{i=1}^n a_i(x, y) p_i. \end{aligned}$$

If we put  $b(x, y) = -F(x, y, 0)$ , then the equation has the form  $F(x, y, p) = \sum_{i=1}^n a_i(x, y) p_i - b(x, y)$ . Since  $(\frac{\partial F}{\partial p_1}, \dots, \frac{\partial F}{\partial p_n}) \neq (0, \dots, 0)$ ,  $(a_1(x, y), \dots, a_n(x, y)) \neq (0, \dots, 0)$ .

Thus we have  $E(a_1, \dots, a_n, b) = \overline{F^{-1}(0)} = E$ . The proof of the converse direction is already proved by the previous arguments.

### 3. HAMILTON-JACOBI EQUATIONS

In this section we study the Hamilton-Jacobi equations by the similar view point as the case for quasilinear equations. Let  $H(x_1, \dots, x_n, p_1, \dots, p_n) = 0$  be a Hamilton-Jacobi equation, then the characteristic vector field is given as follows :

$$X_H = \sum_{i=1}^n \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x_i} - \sum_{i=1}^n \frac{\partial H}{\partial x_i} \frac{\partial}{\partial p_i},$$

so that it exists on the  $(x, p)$ -space (i.e., the cotangent bundle  $T^*\mathbb{R}^n$ ). Since we have the canonical decomposition  $J^1(\mathbb{R}^n, \mathbb{R}) \cong T^*\mathbb{R}^n \times \mathbb{R}$ , we give the geometric definition of the Hamilton-Jacobi equation as follows : Let  $E \subset J^1(\mathbb{R}^n, \mathbb{R}) \subset PT^*(\mathbb{R}^n \times \mathbb{R})$  be an equation. We say that  $E$  is a *Hamilton-Jacobi equation* if there exists a submersion  $H : T^*\mathbb{R}^n \rightarrow \mathbb{R}$  such that  $E = H^{-1}(0)$ . We introduce another important class of equations. We say that  $E$  is a *graphlike equation* if there exists a function  $H : T^*\mathbb{R}^n \rightarrow \mathbb{R}$  such that  $E = \{(x, y, p) \in J^1(\mathbb{R}^n, \mathbb{R}) | y = H(x, p)\}$ . The notion of graphlike equations is introduced by Kossowski [11] in order to study singular

solutions of equations. The most famous example is the Clairaut equation which is given by

$$y = \sum_{i=1}^n x_i p_i + f(p_1, \dots, p_n).$$

It is well known that the Clairaut equation is completely integrable with singular solution and the singular solution is the envelope of the family of graphs of the complete solution. In [7] we have shown that the set of contact singular points of an equation  $E$  is an  $n$ -dimensional submanifold if and only if the equation has a singular solution (in the strict sense) and the equation is (locally) graphlike near the singular solution.

We have the following characterization theorem.

**Theorem 3.1.** *Let  $E \subset J^1(\mathbb{R}^n, \mathbb{R}) \subset PT^*(\mathbb{R}^n \times \mathbb{R})$  be an equation. Then the following are equivalent.*

(1)  $E$  is a Hamilton-Jacobi equation or a graphlike equation.

(2) There exists a submersion  $F : J^1(\mathbb{R}^n, \mathbb{R}) \rightarrow \mathbb{R}$  such that  $F^{-1}(0) = E$  and  $d\Pi(X_F)$  is a vector field on  $T^*\mathbb{R}^n$ ,

where  $d\Pi : J^1(\mathbb{R}^n, \mathbb{R}) \rightarrow T^*\mathbb{R}^n$  is the canonical projection.

*Proof.* Suppose that the condition (2) holds, then the characteristic vector field is given as follows:

$$X_F = \sum_{i=1}^n \frac{\partial F}{\partial p_i} \frac{\partial}{\partial x_i} + \sum_{i=1}^n (p_i \frac{\partial F}{\partial p_i} - F) \frac{\partial}{\partial y} - \sum_{i=1}^n (\frac{\partial F}{\partial x_i} + p_i \frac{\partial F}{\partial y}) \frac{\partial}{\partial p_i}.$$

It follows that  $d\Pi(X_F)$  is a vector field on  $T^*\mathbb{R}^n$  if and only if  $\frac{\partial F}{\partial p_i}(x, y, p)$  and  $\frac{\partial F}{\partial x_i}(x, y, p) + p_i \frac{\partial F}{\partial y}(x, y, p)$  depends only on  $(x, p)$ -variables for  $i = 1, \dots, n$ .

We have

$$\begin{aligned} F(x, y, p) &= F(x, y, 0) + \int_0^1 \frac{dF(x, y, tp)}{dt} dt \\ &= F(x, y, 0) + \int_0^1 \sum_{i=1}^n \frac{\partial F}{\partial p_i}(x, y, tp) p_i dt \\ &= F(x, y, 0) + \sum_{i=1}^n \left( \int_0^1 \frac{\partial F}{\partial p_i}(x, y, tp) dt \right) p_i. \end{aligned}$$

Since  $\frac{\partial F}{\partial p_i}(x, y, p)$  are independent on the  $y$ -variable, we may put

$$F(x, y, 0) = a(x, y) \quad \text{and} \quad \sum_{i=1}^n \left( \int_0^1 \frac{\partial F}{\partial p_i}(x, y, tp) dt \right) p_i = h(x, p).$$

Moreover, by the condition that  $\frac{\partial F}{\partial x_i}(x, y, p) + p_i \frac{\partial F}{\partial y}(x, y, p)$  are independent on the  $y$ -variable,

$$\frac{\partial a}{\partial x_i}(x, y) + \frac{\partial h}{\partial x_i}(x, p) + p_i \frac{\partial a}{\partial y}(x, y)$$

are independent on the  $y$ -variable. It is equivalent to the following condition:

$$\frac{\partial^2 a}{\partial x_i \partial y}(x, y) + p_i \frac{\partial^2 a}{\partial y^2}(x, y) \equiv 0$$

for  $i = 1, \dots, n$ . Differentiate the above equalities with respect to  $p_i$ -variables, we have  $\frac{\partial^2 a}{\partial x_i \partial y}(x, y) \equiv 0$  and  $\frac{\partial^2 a}{\partial y^2}(x, y) \equiv 0$ , so that  $\frac{\partial a}{\partial x_i}(x, y)$  and  $\frac{\partial a}{\partial y}(x, y)$  depend only on  $(x_1, \dots, x_n)$ -variables for  $i = 1, \dots, n$ . Then we may put  $\frac{\partial a}{\partial x_i}(x, y) = b_i(x)$  and  $\frac{\partial a}{\partial y}(x, y) = c(x)$ . Thus we have

$$\begin{aligned} a(x, y) &= a(0, 0) + \int_0^1 \frac{da(tx, ty)}{dt} dt \\ &= a(0, 0) + \int_0^1 \left( \sum_{i=1}^n \frac{\partial a}{\partial x_i}(tx, ty) x_i + \frac{\partial a}{\partial y}(tx, ty) \right) dt \\ &= a(0, 0) + \sum_{i=1}^n \left( \int_0^1 \frac{\partial a}{\partial x_i}(tx, ty) dt \right) x_i + \left( \int_0^1 \frac{\partial a}{\partial y}(tx, ty) dt \right) y \\ &= a(0, 0) + \sum_{i=1}^n \left( \int_0^1 b_i(tx) dt \right) x_i + \left( \int_0^1 c(tx) dt \right) y \\ &= B(x) + C(x)y, \end{aligned}$$

where we put  $B(x) = a(0, 0) + \sum_{i=1}^n \left( \int_0^1 b_i(tx) dt \right) x_i$  and  $C(x) = \int_0^1 c(tx) dt$ .

Since  $\frac{\partial a}{\partial y}(x, y) = C(x)$ , we have  $\frac{\partial C}{\partial x_i}(x) = \frac{\partial^2 a}{\partial y \partial x_i}(x, y) = \frac{\partial^2 a}{\partial x_i \partial y}(x, y) \equiv 0$ , so that  $C(x) = c$  (constant). We put  $H(x, p) = B(x) + h(x, p)$ , then we have

$$F(x, y, p) = cy + H(x, p).$$

If  $c = 0$ , then the equation is a Hamilton-Jacobi equation. If  $c \neq 0$ , then we have

$$F^{-1}(0) = \left\{ (x, y, p) \mid y + \frac{1}{c} H(x, p) = 0 \right\},$$

it is a graphlike equation. The proof for converse direction is given by direct calculations. This completes the proof.

We also have the following characterization theorem of Hamilton-Jacobi equations as a corollary of the above theorem.

**Corollary 3.2.** *Under the same assumption as in Theorem 3.1, the following are equivalent.*

(1)  $E$  is a Hamilton-Jacobi equation.

(2) There exists a submersion  $F : J^1(\mathbb{R}^n, \mathbb{R}) \rightarrow \mathbb{R}$  such that  $F^{-1}(0) = E$  and the characteristic vector field  $X_F$  is a vector field on  $T^*\mathbb{R}^n$ .

*Proof.* Since  $X_F$  is a vector field on  $T^*\mathbb{R}^n$ , then  $d\Pi(X_F) = X_F$ , so that the equation is written in the form  $F(x, y, p) = cy + H(x, p)$ , by the proof of Theorem 3.1. If  $c \neq 0$ , then  $X_F$  is not a vector field on  $T^*\mathbb{R}^n$ . The proof for converse direction is also given by direct calculations.

## REFERENCES

1. V. I. Arnol'd, *Geometric Methods in the Theory of Ordinary Differential Equations*, Springer-Verlag, 1983.
2. G. Crandall and P. -Lions, *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc. **277** (1983), 1-42.
3. M. G. Crandall, H. Ishii and P.-Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. **27** (1992), 1-67.
4. J. J. Duistermaat, *Oscillatory integrals, Lagrange immersions and Unfolding of singularities*, Commu. pure and applied Math. **27** (1974), 207-281.
5. J. Guckenheimer, *Solving a single conservation law*, Lecture notes in Mathematics 468, Springer Verlag, New York, 1975, pp. 108-134.
6. S. Izumiya, *Perestroikas of optical wave fronts and graphlike Legendrian unfoldings*, J. of Differential Geometry **38** (1993), 485-500.
7. S. Izumiya, *Singular solutions of first-order differential equations*, Bull. London Math. Soc. (to appear).
8. S. Izumiya and G. Kossioris, *Semi-local classification of geometric singularities for Hamilton-Jacobi equations*, J. Differential Equations (to appear).
9. S. Izumiya and G. Kossioris, *Geometric singularities for solutions of single conservation laws*, preprint.
10. G. Jennings, *Piecewise smooth solutions of a single conservation law exist*, Adv. in Math **33** (1979), 192-205.
11. M. Kossowski, *first-order partial differential equations with singular solution*, Indiana Univ. Math. Jour. **35** (1986), 209-223.
12. S. N. Kruzkov, *First order quasilinear equations in several independent variables*, Math. USSR Sb. **10** (1970), 217-243.
13. V. V. Lychagin, *Local classification of non-linear first-order partial differential equations*, Russian Math. Surveys **30** (1975), 105-175.
14. S. Nakane, *Formation of shocks for a single conservation law*, SIAM J. Math. Anal. **19** (1988), 1391-1408.
15. M. Tsuji, *Formation of singularities for Hamilton-Jacobi equation II*, J. Math. Kyoto Univ. **26** (1986), 299-308.
16. M. Tsuji, *Prolongation of classical solutions and singularities of generalized solutions*, Ann. Inst. Henri Poincaré (Analyse non linéaire) **7** (1990), 505-523.
17. M. Tsuji and T. T. Li, *Remarks on Characteristics of Partial Differential Equation of First Order*, Funkcialaj Ekvacioj **32** (1989), 157-162.
18. D. Wagner, *The Riemann problem in two space dimensions for a single conservation law*, SIAM J. Math. Ann. **14** (1983), 534-559.