THE ASYMPTOTIC BEHAVIOUR OF RADIAL SOLUTIONS NEAR THE BLOW-UP POINT TO QUASI-LINEAR WAVE EQUATIONS IN TWO SPACE DIMENSIONS

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THE ASYMPTOTIC BEHAVIOUR OF RADIAL SOLUTIONS NEAR THE BLOW-UP POINT TO QUASI-LINEAR WAVE EQUATIONS IN TWO SPACE DIMENSIONS

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1. Introduction.

Consider the Cauchy problem:

$$u_{tt} - c^2(u_t, u_r)(u_{rr} + \frac{1}{r}u_r) = \frac{1}{r}u_r G(u_t, u_r), \quad (r,t) \in (0,\infty) \times (0, T_\varepsilon), \tag{1.1}$$

$$u(r,0) = \varepsilon f(r), \quad u_t(r,0) = \varepsilon g(r), \quad r \in (0,\infty), \tag{1.2}$$

where

$$\begin{aligned} c(u_t, u_r) =& 1 + \frac{a_1}{2}u_t^2 + \frac{a_2}{2}u_t u_r + \frac{a_3}{2}u_r^2 + O(|u_t|^3 + |u_r|^3), \\ G(u_t, u_r) =& O(u_r^2 + u_t^2), \end{aligned}$$

near $u_t = u_r = 0$. Equation (1.1) is a radially symmetric form of quasi-linear wave equation in two space dimensions which involves the equation of vibrating membrane. In [4], we obtained the following blow up result:

$$\limsup_{\varepsilon \to 0} \varepsilon^2 \log(1 + T_\varepsilon) \leqq \frac{1}{H},$$

where T_ε is the lifespan of the radial solution of the Cauchy problem (1.1), (1.2) and H is a constant depending only on f, g and $\partial^2 c(0,0)$. More precisely, the blow up occurs as follows. If we set

$$w(r,t) = \frac{c(u_t, u_r)v_{rr} - v_{rt}}{2c(u_t, u_r)} \quad \text{with} \quad v(r,t) = r^{\frac{1}{2}}u(r,t),$$

then we find that

$$|w(r,t)| \longrightarrow \infty \quad \text{as} \quad \varepsilon^2 \log(1+t) \to \frac{1}{H}$$

along a pseudo-characteristic curve for sufficiently small ε.

In this paper, we investigate the asymptotic behaviour of $w(r,t)$ when $\varepsilon^2\log(1+t)$ tends to $\frac{1}{H}$.

2. Statement of Results.

As we did in [3], we assume $f,g \in C_0^\infty(\mathbb{R}^2)$, $|f|+|g| \not\equiv 0$ and $f(r)=g(r)=0$ for $r \geqq M$. Moreover we assume $a_1-a_2+a_3=a\neq 0$ which means (1.1) does not satisfy the *null-condition*. Then we can define a positive constant H by

$$\begin{aligned} H &= \max_{\rho\in\mathbb{R}}(-a\mathcal{F}'(\rho)\mathcal{F}''(\rho)) \\ &= -a\mathcal{F}'(\rho_0)\mathcal{F}''(\rho_0), \end{aligned}$$

where $\mathcal{F}(\rho)$ is the Friedlander radiation field which is constructed by f and g (see [4]). We introduce a variable $s=\varepsilon^2\log(1+t)$ and we write $t=t_X$ when $s=X$, *i.e.*,

$$X=\varepsilon^2\log(1+t_X).$$

To state our results, we have to recall the facts which are obtained in [4].

Firstly, for any $B>H$ we consider the Burgers equation:

$$U_{\rho s}+\frac{a}{2}(U_\rho)^2U_{\rho\rho}=0, \quad (\rho,s)\in\mathbb{R}\times[0,\frac{1}{B}],$$

$$U_\rho(\rho,0)=\mathcal{F}'(\rho), \qquad \rho\in\mathbb{R},$$

then, there exists an $\varepsilon(B)>0$ such that the Cauchy problem (1.1), (1.2) has a smooth solution in $0\leqq t\leqq t_{\frac{1}{B}}$ and the following holds.

$$\begin{aligned} &|\partial_r^l\partial_t^m u(r,t_{\frac{1}{B}})-\varepsilon r^{-\frac{1}{2}}(-1)^m\partial_\rho^{l+m}U(r-t_{\frac{1}{B}},\frac{1}{B})| \leqq C_{l,m,B}\varepsilon^{\frac{5}{4}}r^{-\frac{1}{2}} \\ &\qquad \text{for} \quad r-t_{\frac{1}{B}}>-\frac{1}{3\varepsilon} \quad \text{and} \quad l+m\neq 0 \end{aligned} \tag{2.1}$$

for $\varepsilon<\varepsilon(B)$. Moreover, U satisfies

$$\begin{aligned} U(\rho(s),s) &= \mathcal{F}'(\rho_0), \\ U_{\rho\rho}(\rho(s),s) &= \frac{\mathcal{F}''(\rho_0)}{1+a\mathcal{F}'(\rho_0)\mathcal{F}''(\rho_0)s} = \frac{\mathcal{F}''(\rho)}{1-Hs}, \end{aligned} \tag{2.2}$$

for $0\leqq s\leqq\frac{1}{B}$ along the curve Λ_{ρ_0} defined by

$$\frac{d\rho}{ds}=\frac{a}{2}(U_\rho)^2 \quad \text{for} \quad s\geqq 0, \quad \rho=\rho_0 \quad \text{for} \quad s=0.$$

These facts are proved in section 3 of [4] by using the energy inequality and the Klainerman inequality.

Secondly, we define a pseudo-characteristic curve Z by

$$\frac{dr}{dt}=c(u_t,u_r) \quad \text{for} \quad t\geqq t_{\frac{1}{B}}, \quad r=\rho(\frac{1}{B})+t_{\frac{1}{B}} \quad \text{for} \quad t=t_{\frac{1}{B}}$$

and a function w by

$$w(r,t) = \frac{cv_{rr} - v_{rt}}{2c} \qquad \text{with} \qquad v(r,t) = r^{\frac{1}{2}}u(r,t).$$

Then, for any $A < H$ there exists an $\bar{\varepsilon}(A) > 0$ such that if $\varepsilon < \bar{\varepsilon}(A)$, then w should satisfy

$$w'(t) = \alpha_0(t)w(t)^2 + \alpha_1(t)w(t) + \alpha_2(t) \quad \text{for} \quad t_{\frac{1}{B}} \leqq t \leqq t_{\frac{1}{A}}, \tag{2.3}$$

$$w(t_{\frac{1}{B}}) = \varepsilon U_{\rho\rho}(\rho(\frac{1}{B}), \frac{1}{B}) + O(\varepsilon^{\frac{5}{4}}), \tag{2.4}$$

where

$$w(t) = w(r(t),t) \qquad \text{for} \qquad (r(t),t) \in Z$$

and

$$\begin{aligned} \alpha_0(t) &= -a\varepsilon\mathcal{F}'(\rho_0)(1+t)^{-1} + O(\varepsilon^{\frac{5}{4}}(1+t)^{-1}), \\ \alpha_1(t) &= O(\varepsilon^4(1+t)^{-1} + \varepsilon^2(1+t)^{-2}), \\ \alpha_2(t) &= O(\varepsilon(1+t)^{-2}), \end{aligned} \tag{2.5}$$

as long as u exists. Here $X = O(Y)$ means $|X| \leqq CY$ with constant C depending only on B, f, g, ρ_0, a and M. This fact is proved in section 4 and 5 of [4] by using (2.1), (2.2) and *a priori* estimats of u.

Now we state our results.

Theorem. *For any $\delta > 0$ there exists an $\varepsilon_\delta > 0$ such that $w(t)$ is well-defind in $t_{\frac{1}{B}} \leqq t \leqq t_{\frac{1}{H}-\delta}$ for $\varepsilon < \varepsilon_\delta$ and at the point $t = t_{\frac{1}{H}-\delta}$,*

$$\lim_{\varepsilon\to 0}(\frac{1}{H} - \varepsilon^2\log(1+t))\frac{w(t)}{\varepsilon} = \frac{1}{H}\mathcal{F}''(\rho_0)$$

holds.

However, since we are interested in the behaviour of w when $\varepsilon^2\log(1+t)$ tends to $\frac{1}{H}$, we reduce the above result into

Corollary.

$$\lim_{\varepsilon\to 0,\ \varepsilon^2\log(1+t)\to\frac{1}{H}}(\frac{1}{H} - \varepsilon^2\log(1+t))\frac{w(t)}{\varepsilon} = \frac{1}{H}\mathcal{F}''(\rho_0).$$

In three space dimensions, for the radial solution of the Cauchy problem:

$$\begin{aligned} u_{tt} - c^2(u_t)(u_{rr} + \frac{2}{r}u_r) &= 0, \\ u(r,0) = \varepsilon f(r), \quad u_t(r,0) &= \varepsilon g(r), \end{aligned}$$

with $c(u_t) = 1 + au_t + O(u_t^2)$ and $a \neq 0$, F. John [5] and L. Hörmander [2] have shown a blow up result

$$\limsup_{\varepsilon \to 0} \varepsilon \log(1 + T_\varepsilon) \leqq \frac{1}{\max(a\mathcal{F}''(\rho))}.$$

In this case, if we set $H = \max(a\mathcal{F}''(\rho)) = a\mathcal{F}''(\rho_0)$, we also expect

$$\lim_{\varepsilon \to 0,\ \varepsilon \log(1+t) \to \frac{1}{H}} \left(\frac{1}{H} - \varepsilon \log(1+t)\right)\frac{w(t)}{\varepsilon} = \frac{1}{H}\mathcal{F}''(\rho_0),$$

which would be obtained in parallel.

For the non radially symmetric case, S. Alinhac [1] studies the Cauchy problem

$$\partial_t^2 u - \Delta u = \sum_{i,j,k=0}^{2} g_{ij}^k \partial_k u \partial_{ij}^2 u, \quad (x,t) \in \mathbb{R}^2 \times (0, T_\varepsilon),$$

$$u(x,0) = u^0(x;\varepsilon), \quad u_t(x,0) = u^1(x;\varepsilon), \qquad x \in \mathbb{R}^2,$$

where $\partial_0 = \partial_t$ and g_{ij}^k are constants. Note that this problem differs from ours in the power of $\partial_k u$. If u^0, u^1 and g_{ij}^k satisfy the *non degenerate* condition (ND), he finds the *asymptotic lifespan* T_ε^a which satisfies the following: For any $N \in \mathbb{N}$, there exists an $\varepsilon_N > 0$ such that if $\varepsilon < \varepsilon_N$, then

$$T_\varepsilon > T_\varepsilon^a - \varepsilon^N$$

and

$$\frac{1}{C} \leqq (T_\varepsilon^a - t)\|\partial^2 u(t)\|_{L_x^\infty} \leqq C \qquad \text{for} \qquad \frac{C}{\varepsilon^2} \leqq t \leqq T_\varepsilon^a - \varepsilon^N$$

holds for some constant C. Since he estimates $\partial^2 u$ not along a pseud-characteristic curve but in whole space $\mathbb{R}^2$, it seems difficult to determine the constant C.

In the rest of this paper, we concentrate on the proof of Theorem.

3. Proof of Theorem.

In [3], we have proved that there exists an $\varepsilon_1(\delta) > 0$ such that for $\varepsilon < \varepsilon_1$ the Cauchy problem (1.1), (1.2) has a smooth solution u in $0 \leqq t \leqq t_{\frac{1}{H}-\delta}$ and therefore $w(t)$ is well-defined in $t_{\frac{1}{B}} \leqq t \leqq t_{\frac{1}{H}-\delta}$. Thus we have only to prove that for any $\eta > 0$ there exists an $\varepsilon_0(\delta, \eta) > 0$ such that

$$\left|\left(\frac{1}{H} - s\right)\frac{w(t)}{\varepsilon} - \frac{1}{H}\mathcal{F}''(\rho_0)\right| < \eta$$

for $\varepsilon < \varepsilon_0$ and $s = \frac{1}{H} - \delta$. If we take $\frac{1}{A} = \frac{1}{H} + \delta$ in the argument in section 2, there exist an $\varepsilon_2(\delta) > 0$ such that if $\varepsilon < \varepsilon_2$, $w(t)$ should satisfy the ordinary differential equation (2.3), (2.4) in $t_{\frac{1}{B}} \leqq t \leqq t_{\frac{1}{H}+\delta}$ as long as u exists. Thus we find that for $\varepsilon < \min(\varepsilon_1, \varepsilon_2)$ the ordinary differential equation (2.3), (2.4) make sence in $t_{\frac{1}{B}} \leqq t \leqq t_{\frac{1}{H}-\delta}$.

Now the following lemma is useful.

Lemma. *Let $w(t)$ be a solution of the ordinary differential equation*

$$w'(t) = \alpha_0(t)w(t)^2 + \alpha_1(t)w(t) + \alpha_2(t) \quad \text{for} \quad t_0 \leqq t \leqq T$$

and assume

$$\alpha_0(t) \geqq 0 \qquad \text{for} \qquad t_0 \leqq t \leqq T,$$
$$w(t_0) > K$$

where

$$K = \int_{t_0}^{T} |\alpha_2(t)| \exp(-\int_{t_0}^{t} \alpha_1(\tau)d\tau)dt.$$

Then $w(t)$ satisfies

$$w(t)\exp(-\int_{t_0}^{t}\alpha_1(\tau)d\tau) \geqq \frac{w(t_0) - K}{1 - (w(t_0) - K)\int_{t_0}^{t}\alpha_0(\tau)\exp(\int_{t_0}^{\tau}\alpha_1(\xi)d\xi)d\tau}$$

and

$$w(t)\exp(-\int_{t_0}^{t}\alpha_1(\tau)d\tau) \leqq \frac{w(t_0) + K}{1 - (w(t_0) + K)\int_{t_0}^{t}\alpha_0(\tau)\exp(\int_{t_0}^{\tau}\alpha_1(\xi)d\xi)d\tau}.$$

Proof of Lemma. At first we consider the case $\alpha_1(t) \equiv 0$. Let $w_1(t)$ be a solution of

$$w_1'(t) = \alpha_0(t)(w_1(t) - K)^2, \tag{3.1}$$
$$w_1(t_0) = w(t_0) \tag{3.2}$$

and set

$$w_2(t) = \int_{t_0}^{t} |\alpha_2(\tau)|d\tau.$$

Since $\alpha_0(t) \geqq 0$, we find that

$$w_1(t) \geqq w(t_0) > K = w_2(T) \geqq w_2(t)$$

and that

$$\begin{aligned}(w_1(t) - w_2(t))' &= \alpha_0(t)(w_1(t) - K)^2 - |\alpha_2(t)| \\ &\leqq \alpha_0(t)(w_1(t) - w_2(t))^2 + \alpha_2(t), \\ w_1(t_0) - w_2(t_0) &= w(t_0).\end{aligned}$$

Thus the usual comparison theorem leads

$$w_1(t) - w_2(t) \leqq w(t). \tag{3.3}$$

By solving the ordinary differential equation (3.1), (3.2), $w_1(t)$ is represented by

$$w_1(t) = K + \frac{w(t_0) - K}{1 - (w(t_0) - K)\int_{t_0}^{t}\alpha_0(\tau)d\tau}.$$

Substituting this equality into (3.3), we find

$$\begin{aligned} w(t) &\geqq K + \frac{w(t_0) - K}{1 - (w(t_0) - K)\int_{t_0}^{t} \alpha_0(\tau)d\tau} - w_2(t) \\ &\geqq \frac{w(t_0) - K}{1 - (w(t_0) - K)\int_{t_0}^{t} \alpha_0(\tau)d\tau}. \end{aligned}$$

On the other hand, if we let $w_3(t)$ be a solution of

$$w_3'(t) = \alpha_0(t)(w_3(t) + K)^2,$$

$$w_3(t_0) = w(t_0),$$

then we find

$$\begin{aligned} (w_3(t) + w_2(t))' &= \alpha_0(t)(w_3(t) + K)^2 + |\alpha_2(t)| \\ &\geqq \alpha_0(t)(w_3(t) + w_2(t))^2 + \alpha_2(t), \\ w_3(t_0) + w_2(t_0) &= w(t_0). \end{aligned}$$

Thus we obtain

$$w_3(t) + w_2(t) \geqq w(t).$$

Since $w_3(t)$ is represented by

$$w_3(t) = -K + \frac{w(t_0) + K}{1 - (w(t_0) + K)\int_{t_0}^{t} \alpha_0(\tau)d\tau},$$

we obtain

$$\begin{aligned} w(t) &\leqq -K + \frac{w(t_0) + K}{1 - (w(t_0) + K)\int_{t_0}^{t} \alpha_0(\tau)d\tau} + w_2(t) \\ &\leqq \frac{w(t_0) + K}{1 - (w(t_0) + K)\int_{t_0}^{t} \alpha_0(\tau)d\tau}. \end{aligned}$$

For the general case, setting

$$W(t) = w(t)\exp(-\int_{t_0}^{t} \alpha_1(\tau)d\tau)$$

and applying the result we have just proved to $W(t)$, we obtain the inequalities we wanted.

Now we want to apply Lemma to (2.3), (2.4) as $t_0 = t_{\frac{1}{B}}$ and $T = t_{\frac{1}{H}-\delta}$. By (2.5), we have

$$\begin{aligned} \exp(\pm \int_{t_{\frac{1}{B}}}^{t} \alpha_1(\tau)d\tau) &= \exp(O(\int_{t_{\frac{1}{B}}}^{t} \varepsilon^4(1+\tau)^{-1}d\tau)) \\ &= \exp(O(\varepsilon^4 \log(1+t)) + O(\varepsilon^4 \log(1 + t_{\frac{1}{B}}))) \\ &= \exp(O(\varepsilon^2)) = 1 + O(\varepsilon^2) \quad \text{for} \quad t_{\frac{1}{B}} \leqq t \leqq t_{\frac{1}{H}-\delta}, \end{aligned}$$

$$
\begin{aligned}
K &= \int_{t_{\frac{1}{B}}}^{t_{\frac{1}{H}-\delta}} |\alpha_2(t)| \exp(-\int_{t_{\frac{1}{B}}}^{t} \alpha_1(\tau)d\tau)dt \\
&= O((1+\varepsilon^2)\varepsilon \int_{t_{\frac{1}{B}}}^{t_{\frac{1}{H}-\delta}} (1+t)^{-2}dt) \\
&= O(\varepsilon(1+t_{\frac{1}{B}})^{-1}) + O(\varepsilon(1+t_{\frac{1}{H}-\delta})^{-1}) \\
&= O(\varepsilon^3),
\end{aligned} \tag{3.4}
$$

$$
\begin{aligned}
&\int_{t_{\frac{1}{B}}}^{t} \alpha_0(\tau) \exp(\int_{t_{\frac{1}{B}}}^{\tau} \alpha_1(\xi)d\xi)d\tau \\
&= (1+O(\varepsilon^2))(-a\varepsilon\mathcal{F}'(\rho_0) + O(\varepsilon^{\frac{5}{4}})) \int_{t_{\frac{1}{B}}}^{t} (1+\tau)^{-1}d\tau \\
&= (-a\varepsilon\mathcal{F}'(\rho_0) + O(\varepsilon^{\frac{5}{4}}))(\log(1+t) - \log(1+t_{\frac{1}{B}})) \\
&\qquad \text{for} \quad t_{\frac{1}{B}} \leqq t \leqq t_{\frac{1}{H}-\delta}.
\end{aligned}
$$

Since $H > 0$, $-a\mathcal{F}'(\rho_0)$ and $\mathcal{F}''(\rho_0)$ have the same sign. Without loss of generality, we can assume that both are positive and then it follows from (2.4) and (3.4) that there exists an $\varepsilon_3 > 0$ such that

$$w(t_{\frac{1}{B}}) > K$$

and

$$\alpha_0(t) \geqq 0$$

hold for $\varepsilon < \varepsilon_3$. Thus we can apply Lemma and obtain

$$
\begin{aligned}
&(1+C\varepsilon^2)w(t) \\
&\geqq \frac{w(t_{\frac{1}{B}}) - C\varepsilon^3}{1 - (w(t_{\frac{1}{B}}) - C\varepsilon^3)(-a\varepsilon\mathcal{F}'(\rho_0)U_{\rho\rho}(\frac{1}{B}) - C\varepsilon^{\frac{5}{4}})(\log(1+t) - \log(1+t_{\frac{1}{B}}))} \\
&= \frac{\varepsilon U_{\rho\rho}(\frac{1}{B}) - C\varepsilon^{\frac{5}{4}}}{1 - (-a\mathcal{F}'(\rho_0)U_{\rho\rho}(\frac{1}{B}) - C\varepsilon^{\frac{1}{4}})(s - \frac{1}{B})} \quad \text{for} \quad \frac{1}{B} \leqq s \leqq \frac{1}{H} - \delta,
\end{aligned}
$$

where $U_{\rho\rho}(\frac{1}{B}) = U_{\rho\rho}(\rho(\frac{1}{B}), \frac{1}{B})$ and C is a constant depending only on B, f, g, ρ_0, a and M and it varies from line to line. By (2.4), we get

$$
\begin{aligned}
\frac{w(t)}{\varepsilon} &\geqq (1 - C\varepsilon^2)\frac{U_{\rho\rho}(\frac{1}{B}) - C\varepsilon^{\frac{1}{4}}}{1 - (-a\mathcal{F}'(\rho_0)U_{\rho\rho}(\frac{1}{B}) - C\varepsilon^{\frac{1}{4}})(s - \frac{1}{B})} \\
&= \frac{U_{\rho\rho}(\frac{1}{B}) - C\varepsilon^{\frac{1}{4}}}{1 - \frac{s - \frac{1}{B}}{\frac{1}{H} - \frac{1}{B}} + C\varepsilon^{\frac{1}{4}}} \\
&= \frac{\frac{1}{H}\mathcal{F}''(\rho_0) - C\varepsilon^{\frac{1}{4}}}{\frac{1}{H} - s + C\varepsilon^{\frac{1}{4}}} \quad \text{for} \quad \frac{1}{B} \leqq s \leqq \frac{1}{H} - \delta.
\end{aligned}
$$

If we set $s = \frac{1}{H} - \delta$, we have

$$\begin{aligned}(\frac{1}{H} - s)\frac{w(t)}{\varepsilon} \geqq & (\frac{1}{H}\mathcal{F}''(\rho_0) - C\varepsilon^{\frac{1}{4}})\frac{\frac{1}{H} - s}{\frac{1}{H} - s + C\varepsilon^{\frac{1}{4}}} \\ = & \frac{1}{H}\mathcal{F}''(\rho_0)\frac{\delta}{\delta + C\varepsilon^{\frac{1}{4}}} - \frac{C\delta\varepsilon^{\frac{1}{4}}}{\delta + C\varepsilon^{\frac{1}{4}}} \\ = & \frac{1}{H}\mathcal{F}''(\rho_0) - \frac{C\varepsilon^{\frac{1}{4}}}{\delta + C\varepsilon^{\frac{1}{4}}} - \frac{C\delta\varepsilon^{\frac{1}{4}}}{\delta + C\varepsilon^{\frac{1}{4}}}.\end{aligned}$$

There exists an $\varepsilon_4(\delta, \eta) > 0$ such that if $\varepsilon < \varepsilon_4$, then

$$\frac{C\varepsilon^{\frac{1}{4}}}{\delta + C\varepsilon^{\frac{1}{4}}} + \frac{C\delta\varepsilon^{\frac{1}{4}}}{\delta + C\varepsilon^{\frac{1}{4}}} < \eta,$$

i.e.,

$$(\frac{1}{H} - s)\frac{w(t)}{\varepsilon} - \frac{1}{H}\mathcal{F}''(\rho_0) > -\eta$$

holds. Similarly, using the other inequality in Lemma we find that there exists an $\varepsilon_5(\delta, \eta) > 0$ such that if $\varepsilon < \varepsilon_5$, then

$$(\frac{1}{H} - s)\frac{w(t)}{\varepsilon} - \frac{1}{H}\mathcal{F}''(\rho_0) < \eta$$

holds. Thus if we take $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5)$, we find that

$$|(\frac{1}{H} - s)\frac{w(t)}{\varepsilon} - \frac{1}{H}\mathcal{F}''(\rho_0)| < \eta$$

holds for $\varepsilon < \varepsilon_0$ and this completes the proof of Theorem.

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