



Title	Existence of periodically evolving convex curves moved by anisotropic curvature
Author(s)	Giga, Y.; Mizoguchi, N.
Citation	Hokkaido University Preprint Series in Mathematics, 232, 1-12
Issue Date	1994-3-1
DOI	10.14943/83379
Doc URL	<a href="http://hdl.handle.net/2115/68983">http://hdl.handle.net/2115/68983</a>
Type	bulletin (article)
File Information	pre232.pdf



[Instructions for use](#)

Existence of periodically evolving convex  
curves moved by anisotropic curvature

Y. Giga and N. Mizoguchi

Series #232. March 1994

**HOKKAIDO UNIVERSITY**  
**PREPRINT SERIES IN MATHEMATICS**

- # 203: A. Arai, Characterization of anticommutativity of self-adjoint operators in connection with Clifford algebra and applications, 13 pages. 1993.
- # 204: J. Wierzbicki, An estimation of the depth from an intermediate subfactor, 7 pages. 1993.
- # 205: N. Honda, Vanishing theorem for the tempered distributions, 11 pages. 1993.
- # 206: T. Hibi, Betti number sequences of simplicial complexes, Cohen-Macaulay types and Möbius functions of partially ordered sets, and related topics, 25 pages. 1993.
- # 207: A. Inoue, Regularly varying correlations, 23 pages. 1993.
- # 208: S. Izumiya, B. Li, Overdetermined systems of first order partial differential equations with singular solution, 9 pages. 1993.
- # 209: T. Hibi, Hochster's formula on Betti numbers and Buchsbaum complexes, 7 pages. 1993.
- # 210: T. Hibi, Star-shaped complexes and Ehrhart polynomials, 5 pages. 1993.
- # 211: S. Izumiya, G. T. Kossioris, Geometric singularities for solutions of single conservation laws, 28 pages. 1993.
- # 212: A. Arai, On self-adjointness of Dirac operators in Boson-Fermion Fock spaces, 43 pages. 1993.
- # 213: K. Sugano, Note on non-commutative local field, 3 pages. 1993.
- # 214: A. Hoshiga, Blow-up of the radial solutions to the equations of vibrating membrane, 28 pages. 1993.
- # 215: A. Arai, Scaling limit of anticommuting self-adjoint operators and nonrelativistic limit of Dirac operators, 35 pages. 1993.
- # 216: Y. Giga, N. Mizoguchi, Existence of periodic solutions for equations of evolving curves, 45 pages. 1993.
- # 217: T. Suwa, Indices holomorphic vector fields relative to invariant curves, 10 pages. 1993.
- # 218: S. Izumiya, G. T. Kossioris, Realization theorems of geometric singularities for Hamilton-Jacobi equations, 14 pages. 1993.
- # 219: Y. Giga, K. Yama-uchi, On instability of evolving hypersurfaces, 14 pages. 1993.
- # 220: W. Bruns, T. Hibi, Cohen-Macaulay partially ordered sets with pure resolutions, 11 pages. 1993.
- # 221: S. Jimbo, Y. Morita, Ginzburg Landau equation and stable solutions in a rotational domain, 32 pages. 1993.
- # 222: T. Miyake, Y. Maeda, On a property of Fourier coefficients of cusp forms of half-integral weight, 12 pages. 1993.
- # 223: I. Nakai, Notes on versal deformation of first order PDE and web structure, 34 pages. 1993.
- # 224: I. Tsuda, Can stochastic renewal of maps be a model for cerebral cortex?, 30 pages. 1993.
- # 225: H. Kubo, K. Kubota, Asymptotic behaviors of radial solutions to semilinear wave equations in odd space dimensions, 47 pages. 1994.
- # 226: T. Nakazi, K. Takahashi, Two dimensional representations of uniform algebras, 7 pages. 1994.
- # 227: N. Hayashi, T. Ozawa, Global, small radially symmetric solutions to nonlinear Schrödinger equations and a gauge transformation, 16 pages. 1994.
- # 228: S. Izumiya, Characteristic vector fields for first order partial differential equations, 9 pages. 1994.
- # 229: K. Tsutaya, Lower bounds for the life span of solutions of semilinear wave equations with data of non compact support, 14 pages. 1994.
- # 230: H. Okuda, I. Tsuda, A coupled chaotic system with different time scales: Toward the implication of observation with dynamical systems, 31 pages. 1994.
- # 231: A. Hoshiga, The asymptotic behaviour of radial solutions near the blow-up point to quasi-linear wave equations in two space dimensions, 8 pages. 1994.

# Existence of periodically evolving convex curves moved by anisotropic curvature

YOSHIKAZU GIGA

Department of Mathematics, Hokkaido University,  
Sapporo 060, Japan

AND

NORIKO MIZOGUCHI

Department of Mathematics, Tokyo Gakugei University,  
Koganei, Tokyo 184, Japan

## 1 Introduction.

This note reports our recent results [2] on the existence of time-periodic solution of curvature flow equations in the plane. The present paper includes a natural extension of results in [2].

Let  $\{\Gamma_t\}$  be a smooth one parameter family of closed embedded curves bounding a domain in the plane. Let  $\theta$  be the argument of the inward unit normal  $\mathbf{n}$  of  $\Gamma_t$ . The normal velocity of  $\Gamma_t$  in the direction of  $\mathbf{n}$  denotes  $V$ . We consider an equation of  $\Gamma_t$  of the form

$$V = a(\theta)k - Q(\theta, t), \quad (1)$$

where  $k$  is the inward curvature of  $\Gamma_t$  and  $a$  and  $Q$  are given functions. Since  $\theta$  is argument,  $a$  and  $Q$  are assumed to be  $2\pi$ -periodic in  $\theta$ , i.e.,  $a(\theta + 2\pi) = a(\theta)$  and  $Q(\theta + 2\pi, t) = Q(\theta, t)$ . We assume that  $a$  is strictly positive so that our problem is

parabolic.

**Existence Theorem.** Assume that  $Q$  is  $T$ -periodic in time, i.e.,  $Q(\theta, T + t) = Q(\theta, t)$  for all  $0 \leq \theta < 2\pi, t \in \mathbb{R}$ . Assume that  $a > 0$  and  $Q > 0$  is continuous with partial derivatives  $Q_{\theta\theta}, Q_t, Q_{\theta\theta t}$ . Assume that

$$Q_{\theta\theta} + Q > 0 \quad \text{for all } \theta \text{ and } t. \quad (2)$$

Then there are a constant vector  $c \in \mathbb{R}^2$  and a closed evolving curve  $\Gamma_t$  solving (1) and

$$\Gamma_{t+T} = \Gamma_t + c \quad \text{for all } t \in \mathbb{R}. \quad (3)$$

The curvature of  $\Gamma_t$  is always positive and the quantities in (1) is continuous. If  $Q$  is smooth, so is  $\Gamma_t$ .

This shows the existence of time-periodic solution of (1) when  $Q$  is time-periodic. Several examples of (1) are provided in [3] where a standard form of (1) for thermodynamics is derived. A general motion by anisotropic curvature is described as

$$V = \frac{1}{\beta(\theta)}((\sigma''(\theta) + \sigma(\theta))k - c(t)) \quad (4)$$

where  $\beta > 0$  is called a kinetic coefficient and  $\sigma > 0$  is called the surface energy density of material;  $c(t)$  is the temperature difference. Since the condition (2) is equivalent to say that the Frank diagram of  $Q(\cdot, t)$  has a positive curvature everywhere for  $Q > 0$ , our Existence Theorem yields:

**Corollary.** Assume that  $\beta > 0, \sigma > 0$  and  $c$  are continuous with the second derivative  $\sigma''$ . Assume that  $c$  is  $T$ -periodic and that the Frank diagram of  $\sigma$  and  $1/\beta$  has a positive curvature everywhere. Then there is a closed evolving curve  $\Gamma_t$  solving (4) which is  $T$ -periodic in the sense of (3). The curvature of  $\Gamma_t$  is always positive.

In our previous paper [2] these existence results are proved only for (1) with  $a \equiv 1$ . It turns out the method applies to general (1) with a small modification.

**Curvature evolution equations.** Since a solution  $\Gamma_t$  we seek is convex, we may use  $\theta$  as a coordinate to represent  $\Gamma_t$ . In  $\theta$ -coordinate evolution of curvature is described by

$$k_t = k^2(V_{\theta\theta} + V)$$

as in [3]. If  $\Gamma_t$  evolves by (1),  $k$  fulfills

$$k_t = k^2((ak)_{\theta\theta} + ak - (Q_{\theta\theta} + Q)) \quad (5)$$

Since  $\Gamma_t$  is closed,  $k$  fulfills the constraint

$$\int_0^{2\pi} \frac{e^{i\theta}}{k(\theta, t)} d\theta = 0. \quad (6)$$

Of course, since  $\theta$  is the argument of a normal,  $k$  is  $2\pi$ -periodic in  $\theta$ . As in [2] to show Existence Theorem it suffices to find a positive  $T$ -periodic solution  $k$  of (5), (6) with  $2\pi$ -periodicity in  $\theta$ . To simplify the notation we set

$$\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \quad \text{and} \quad K = \mathbb{T} \times (\mathbb{R}/T\mathbb{Z}).$$

By  $h \in C(K)$  we mean that  $h$  is continuous in  $\mathbb{R}^2$  and that  $h(x, t)$  is  $2\pi$ -periodic in  $x$  and  $T$ -periodic in time  $t$ . As in [2], the following existence result implies the existence of  $k$  satisfying (5), (6) by setting  $Q_{\theta\theta} + Q = f$ ,  $\theta = x$ ,  $k = w$  and it yields our Existence Theorem.

**Theorem 1.** Assume that  $a \in C(\mathbb{T})$  is positive. Assume that  $f \in C(K)$  with  $f > 0$  and  $f_t \in C(K)$  satisfies

$$\int_0^{2\pi} f(x, t) e^{ix} dx = 0 \quad \text{for all } t \in \mathbb{R}. \quad (7)$$

Then there is a positive solution  $w \in C(K)$  ( with  $aw \in \bigcap_{p>1} W_p^{2,1}(K)$  )

$$w_t = w^2((aw)_{xx} + aw - f) \quad \text{in } K \quad (8)$$

satisfying

$$\int_0^{2\pi} \frac{e^{ix}}{w(x,t)} dx = 0 \quad \text{for all } t \in \mathbb{R}. \quad (9)$$

Here  $W_p^{2,1}$  denotes  $L^p$  - Sobolev space of order 2 in  $x$ , 1 in  $t$ .

If we set  $u = aw$ ,  $u$  solves

$$au_t = u^2(u_{xx} + u - f).$$

The outline of the proof of Theorem 1 is the same as that of the Main Existence Theorem in [2] where  $a$  is assumed to equal one. Instead of presenting a whole proof, we point out necessary alternations when  $a$  depends on space variable  $x$ .

The main idea is to get a priori lower and upper bounds for approximate penalized equations admitting a solution. The penalty method applies to recover constraint (9).

The biography of [2] includes many references to recent work on equations of form

$$u_t = u^\gamma(u_{xx} + g(u, x, t)), \quad \gamma \geq 1$$

where  $g$  is a given function. We do not repeat it again here.

## 2 Harnack type inequalities.

In this section, we consider the equation

$$au_t = u^\gamma\{u_{xx} + g(u, x, t)\} \quad \text{in } K, \quad (10)$$

where  $\gamma \in \mathbb{R}$  and  $a$  is a continuous positive function on  $K$  with  $a_t \in C(K)$ .

Putting  $z = u_t/u$ , we have

$$z_x = \frac{u_{xt}}{u} - \frac{u_x u_t}{u^2},$$

$$z_{xx} = \frac{u_{xxt}}{u} - \frac{2u_x z_x}{u} - \frac{u_{xx} u_t}{u^2}.$$

Differentiating  $az = u^{\gamma-1}(u_{xx} + g)$  in  $t$  yields

$$az_t = u^\gamma z_{xx} + 2u^{\gamma-1} u_x z_x + \gamma a z^2 + \{u^{\gamma-1}(g_u u - g) - a_t\}z + u^{\gamma-1} g_t.$$

Let  $(x_0, t_0)$  be a minimizer of  $z$  over  $K$ . Then we have

$$\gamma a z^2 + \{u^{\gamma-1}(g_u u - g) - a_t\}z + u^{\gamma-1} g_t \leq 0 \quad \text{at } (x_0, t_0)$$

and hence

$$z \geq -u^{\gamma-1} \frac{(g_u u - g)_+}{\gamma a} - \frac{(a_t)_+}{\gamma a} - u^{\frac{\gamma-1}{2}} \frac{|g_t|^{1/2}}{(\gamma a)^{1/2}} \quad \text{at } (x_0, t_0).$$

Such differential identity is obtained for (8) with  $a = 1, f = 0$  by Gage [1]. Inequalities of Harnack type in time direction (Lemma1) and in space direction (Lemma2) follow from this estimate of  $\min_K z$  as in [2, §2].

**Lemma 1.** Assume that  $\gamma \geq 1$  and  $\alpha \geq 0$ . Suppose that there are positive constants  $c_0, c_1, c_2$  such that

$$v g_v(v, x, t) - g(v, x, t) \leq c_0, |g_t(v, x, t)|^{1/2} \leq c_1 \quad (11)$$

for all  $(v, x, t) \in (\alpha, \infty) \times K$  and  $\max_K u \geq c_2$  for each positive solution  $u$  of (10). Then there exists  $C = C(c_0, c_1, c_2, \max_K a, \min_K a, \max_K |a_t|, \gamma) > 0$  such that for each solution  $u$  of (10) with  $u > \alpha$

$$u(x, t) \leq u(x, s) \exp(-CM^{\gamma-1}(t-s)) \quad (12)$$

for all  $(x, t), (x, s) \in K$  with  $s - T \leq t \leq s$ , where  $M = \max u$ .

**Lemma 2.** Assume that  $\gamma \geq 1, \alpha \geq 0$  and (11) for  $g$ . If  $u$  is a solution of (10) with  $u > \alpha$ , then

$$u(x, t_0)^\gamma \geq M^\gamma - \frac{\gamma C_M}{2} (x - x_0)^2 \quad \text{in } K,$$



where

$$C_M = \frac{c_0}{\gamma} M^{\gamma-1} + \frac{\max(a_t)_+}{\gamma} + \frac{c_1(\max a)^{1/2}}{\gamma^{1/2}} M^{\frac{\gamma-1}{2}} + M^{\gamma-1} g_M,$$

$$g_M = \max\{(g(v, x, t))_+; \alpha < v < M, (x, t) \in K\}.$$

$$M = \max_K u = u(x_0, t_0),$$

### 3 Upper bounds.

We shall obtain an a priori upper bound for positive smooth solutions of

$$au_t = u^\gamma \{u_{xx} + \varphi(u)(u + \psi(x, u) - f(x, t))\} \quad \text{in } K, \quad (13)$$

where  $\varphi, \psi$  are smooth functions on  $(0, \infty), \mathbb{T} \times (0, \infty)$ , respectively. Here and hereafter,  $a \in C(K)$  is assumed to be time independent. This equation corresponds to the equation (3.1) in [2], in which  $\psi$  is independent of  $x$ . The dependence of  $\psi$  on  $x$  has no effect on proofs in the rest of this paper.

**Lemma 3.** Suppose that  $\psi \geq 0, f \geq 0$  and  $0 \leq \varphi \leq c_3, v - \varphi(v)v \leq c_4$  on  $(\alpha, \infty)$  with  $\alpha > 0$  for some positive constants  $c_3$  and  $c_4$ . Then for each solution  $u \in C^\infty(K)$  of (13) with  $u > \alpha$

$$\int \int_K u dx dt \leq 2\pi T(c_3 \|f\|_\infty + c_4) \equiv C_1$$

$$\int \int_K \frac{u_t^2}{u^\gamma} dx dt \leq \frac{c_3 C_1 \|f_t\|_\infty}{\min a} \equiv C_2.$$

**Proof.** The first inequality is obtained in the same way as the proof of Lemma 3.1 in [2]. Multiplying  $u_t/u^\gamma$  with (13) and integrating over  $K$  yields

$$\int \int_K a \frac{u_t^2}{u^\gamma} dx dt = - \int \int_K \varphi(u) u_t f dx dt = \int \int_K \Phi(u) f_t dx dt,$$

where

$$\Phi(s) = \int_0^s \varphi(r) dr \quad \text{for } s \in \mathbb{R}.$$

We thus have

$$\int \int_K a \frac{u_t^2}{u^\gamma} dx dt \leq c_3 C_1 \|f_t\|_\infty.$$

This implies the second inequality.  $\square$

Lemmas 2 - 3 yield the following theorem.

**Theorem 2.** Suppose that  $1 \leq \gamma < 3$  and  $\alpha > 0$ . In addition to the hypotheses in Lemma 3, assume that

$$\varphi'(v)(\psi(x, v) - f) + \varphi(v)\psi'(v) \leq 0,$$

$$0 \leq \varphi'(v)v^2 \leq c_5, \varphi(v)(\psi(x, v) - \min_K f) \leq c_6(v + 1)$$

on  $\mathbf{T} \times (\alpha, \infty)$  for some constants  $c_5, c_6 > 0$ . Then there is a positive constant  $M_0$  depending only on  $c_j (3 \leq j \leq 6), T, \|f\|_\infty, \|f_t\|_\infty, \gamma, \min_{\mathbf{T}} a$  such that  $\max_K u \leq M_0$  for each solution  $u \in C^\infty(K)$  with  $u > \alpha$ .

#### 4 Lower bounds.

We consider the equation

$$au_t = u^2 \{u_{xx} + \varphi_\epsilon(u)(u + \psi_\epsilon(x, u) - f_\epsilon)\} \quad \text{in } K \quad (14)$$

in this section. To get a positive lower bound for positive smooth solutions of (14), we investigate the stationary problem

$$U_{xx} + U = F \quad \text{in } \mathbf{T}. \quad (15)$$

The coefficient  $a$  clearly gives no effect when we treat the stationary problem.

The following lemma is a key as in [2].

**Lemma 4.** Let  $b \in \mathbf{R}$  and  $d > 0$ . Suppose that  $V \geq 0$  on  $(b, b + d)$ ,  $V \not\equiv 0$  and  $V_x$  is Lipschitz continuous on  $[b, b + d]$ . If  $V_{xx} + V \geq 0$  on  $(b, b + d)$  with  $V(b) = V_x(b) = 0$  and  $V(b + d) = 0$ , then  $d > \pi$ .

Let  $\{\mu_\varepsilon^\pm\}_{\varepsilon \geq 0}$  be a sequence of positive functions on  $\mathbb{T} \times (0, \infty)$  such that  $\mu_\varepsilon^\pm(x, \cdot)$  is nonincreasing for each  $x \in \mathbb{T}$  and  $\mu_\varepsilon^\pm \rightarrow \mu_0^\pm$  in  $\mathbb{T} \times (0, \infty)$  as  $\varepsilon \rightarrow 0$ . Suppose that  $\mu_\varepsilon^- \rightarrow \mu_0^-$  uniformly in every compact subset of  $\mathbb{T} \times (0, \infty)$  as  $\varepsilon \rightarrow 0$ . Let  $\{h_\varepsilon^-\}_{\varepsilon \geq 0}$  be a sequence in  $L^\infty(0, \infty)$  with  $0 \leq h_\varepsilon^- \leq 1$  such that  $h_\varepsilon^- \rightarrow h_0^- \equiv 1$  uniformly in every compact subset in  $\mathbb{T} \times (0, \infty)$  as  $\varepsilon \rightarrow 0$ . Put  $h_\varepsilon^+ \equiv 1$  for all  $\varepsilon \geq 0$ . For a positive function  $U$  on  $\mathbb{T}$  and  $\varepsilon \geq 0$ , we set

$$A_\varepsilon^\pm(\zeta, U) = \int_0^{2\pi} \sin_\pm(x - \zeta) \mu_\varepsilon^\pm(x, U) h_\varepsilon^\pm(U) dx \quad \text{for } \zeta \in \mathbb{R},$$

where  $\sin_+ z = \max(\sin z, 0)$  and  $\sin_- z = -\min(\sin z, 0)$ .

The following lemma is the same as Lemma 4.2 in [2] except for the dependence of  $\mu_\varepsilon^\pm$  on  $x \in \mathbb{T}$ , which does not affect the proof.

**Lemma 5.** Assume that there are positive constants  $k_j$  ( $0 \leq j \leq 4$ ) such that for each positive solution  $U \in C^2(\mathbb{T})$

- i)  $0 \leq F \leq k_0$ , where  $F = U_{xx} + U$
- ii)  $k_1 \leq \max U \leq k_2$ ,
- iii)  $A_\varepsilon^-(\zeta, U) \leq k_3 A_\varepsilon^+(\zeta, U) + k_4$  for all  $\zeta \in \mathbb{R}$ .

Suppose that

$$\int_0^1 \mu_0^-(x^2) dx = \infty.$$

Then there are positive constants  $\delta_0, \varepsilon_0$  depending only on  $k_j$ 's and  $\{\mu_\varepsilon^\pm\}, \{h_\varepsilon^-\}$  such that  $\min_{\mathbb{T}} U \geq \delta_0$  for each positive solution  $U \in C^2(\mathbb{T})$  of (15) and  $0 \leq \varepsilon \leq \varepsilon_0$ .

The following is the same as Lemma 4.4 in [2].

**Lemma 6.** If  $u \in C(K)$  satisfies (12), then there are  $\lambda, \Lambda > 0$  depending only on  $C, \gamma, M, T$  such that

$$\lambda u(x, t) \leq U(x) \leq \Lambda u(x, t) \quad \text{for } (x, t) \in K,$$

where  $U(x) = \int_0^T u(x, t) dt$ .

Lemma 5.3 in [2] remains valid even if  $\psi_\varepsilon(u)$  is replaced by  $\psi_\varepsilon(x, u)$  as stated below.

**Lemma 7.** Assume that  $f_\varepsilon \in C^\infty(K)$  satisfies (7). If  $0 \leq \varphi_\varepsilon \leq 1$  and  $1 - \varphi_\varepsilon(v) \leq c_7 \varepsilon^2 v^{-1}$  for  $v > \varepsilon^2$  with some positive constant  $c_7$ , then

$$\left| \int \int_K \{ \varphi_\varepsilon(u) \psi_\varepsilon(x, u) + (1 - \varphi_\varepsilon(u)) f_\varepsilon \} \sin(x - \zeta) dx dt \right| \leq 4T c_7 \varepsilon^2 \quad (16)$$

for each solution  $u \in C^\infty(K)$  of (14) with  $u > \varepsilon^2$  and  $\zeta \in \mathbb{R}$ .

Using Lemmas 5-7, we can prove our lower bound theorem in the same way as the proof of Theorem 5.7 in [2].

**Theorem 3.** Assume that  $f_\varepsilon \in C^\infty(K)$  satisfies (7) with  $f_\varepsilon > 0$  and that  $\varphi_\varepsilon, \psi_\varepsilon$  fulfill

$$0 \leq \varphi_\varepsilon(v) \leq 1, 0 \leq \varphi_{\varepsilon v}(x, v) \leq 2, \varepsilon^2 \leq 2v(1 - \varphi_\varepsilon(v)) \leq 2\varepsilon^2 \leq 2 \quad \text{for } v > \varepsilon^2$$

$$\min_{\varepsilon > 0} \min_K (f_\varepsilon - \psi_\varepsilon(x, v)) > 0, \psi_{\varepsilon v}(x, v) \leq 0, \quad \text{for } v > \varepsilon^2, x \in \mathbb{T}.$$

Then there are positive constants  $\varepsilon_0, \delta_0$  depending only on  $T, \|f\|_\infty, \|f_t\|_\infty, \min_K f_\varepsilon, \min_{\mathbb{T}} a, \max_{\mathbb{T}} a$  such that  $\min_K u \geq \delta_0$  for each solution  $u \in C^\infty(K)$  of (14) with  $u > \varepsilon^2$  and  $0 < \varepsilon \leq \varepsilon_0$ .

## 5 Existence of periodic solutions.

We start with approximate equations

$$aw_t = (w + \varepsilon^2)^2 \left\{ w_{xx} + \frac{w^2}{(w + \varepsilon^2)^2} \left( w + \frac{\varepsilon a}{\xi_\varepsilon(x, aw + \varepsilon^2)} - f \right) \right\} \quad \text{in } K, \quad (17)$$

where  $a \in C^\infty(\mathbb{T}), f \in C^\infty(K), \xi_\varepsilon : \mathbb{T} \times (0, \infty) \rightarrow (0, \infty)$  is a smooth function such that  $\xi_\varepsilon(x, \cdot)$  is nondecreasing for every  $x \in \mathbb{T}$ ,

$$\xi_\varepsilon(x, v) = v \quad \text{for } v \geq m\varepsilon a, x \in \mathbb{T},$$

$$v \vee (m\epsilon a) \leq \xi_\epsilon(x, v) \leq l(v \vee (m\epsilon a)) \quad \text{for } v \geq 0, x \in \mathbf{T}$$

with some  $1 < l < 2$  and

$$\min_K f - \frac{1}{m} \geq \frac{1}{2} \min_K f.$$

To solve (17), we need the following fact, in which the coefficient  $a(v)$  of  $v_{xx}$  in Lemma 6.1 in [2] is replaced by  $a(v, x, t)$  and we can prove in the same way as the proof of Lemma 6.1

**Lemma 8.** Assume that  $b$  is a positive constant and that  $a$  is a continuous function on  $\mathbf{R} \times K$  such that  $a(\sigma, x, t) \geq a_0$  for all  $\sigma \in \mathbf{R}$  on  $K$  with some positive constant  $a_0$ . Then for each  $h \in C(K)$  there exists a unique solution  $v \in \bigcap_{q>1} W_q^{2,1}(K) \subset C(K)$  of

$$v_t = a(v, x, t)(v_{xx} - bv + h) \quad \text{in } K.$$

Moreover the solution operator  $h \mapsto v$  is a continuous, compact operator from  $C(K)$  into itself. There are positive constants  $\theta_0, C_0$  depending only on  $a_0, \|h\|_\infty, b, T, \sup_K a$  such that

$$\|v\|_{W_p^{2,1}} \leq C_0 \|h\|_\infty \quad \text{for } 2 \leq p \leq 2 + \theta_0, h \in C(K).$$

Take  $b > 0$  such that

$$\phi(w, x, t) = bw_+ + \frac{(w_+)^2}{(w_+ + \epsilon^2)^2} \left( w_+ + \frac{\epsilon a}{\xi_\epsilon(x, aw + \epsilon^2)} - f \right) \geq 0$$

for all  $w \in \mathbf{R}, (x, t) \in K$  and  $\phi > 0$  if  $w > 0$ . For this  $b$  let  $S$  be the solution operator of

$$av_t = (v_+ + \epsilon^2)^2 (v_{xx} - bv + h) \quad \text{in } K,$$

which is well-defined by Lemma 8. Lemma 8 also yields;

- i)  $S$  is a continuous compact operator from  $C(K)$  into itself,
- ii)  $S(h)$  is Hölder continuous on  $K$  for  $h \in C(K)$ .

By standard regularity theory and maximum principle, we see that each fixed point of  $S \circ \phi$  in  $C(K)$  is a positive smooth solution of (17).

We can calculate values of the Leray-Schauder degree in a large and a small ball in  $C(K)$  in the same way as in Lemmas 6.3, 6.4 in [2].

**Lemma 9.** There is  $r_0 > 0$  such that the degree of  $I - S \circ \phi$  of the value zero in  $B_r(0)$  equals one, i.e.,

$$\deg(I - S \circ \phi, B_r(0), 0) = 1$$

for  $0 < r < r_0$ .

**Lemma 10.** There is  $R_0 > 0$  such that

$$\deg(I - S \circ \phi, B_R(0), 0) = 0 \quad \text{for } R > R_0.$$

We sketch proof of Theorem 1.

**Proof of Theorem 1.** Choose an approximate sequence  $\{a_\varepsilon\} \in C^\infty(\mathbb{T})$  and  $\{f_\varepsilon\} \in C^\infty(K)$  satisfying (7) such that

$$a_\varepsilon \rightarrow a \text{ in } C(\mathbb{T}), f_\varepsilon \rightarrow f, f_{\varepsilon t} \rightarrow f_t \text{ in } C(K) \text{ as } \varepsilon \rightarrow 0.$$

From Lemmas 9, 10, for each  $\varepsilon > 0$  there exists a positive solution  $v_\varepsilon \in C^\infty(K)$  of (17) with  $a = a_\varepsilon$  and  $f = f_\varepsilon$  for each  $\varepsilon > 0$ . Putting  $u_\varepsilon = v_\varepsilon + \varepsilon^2$ ,  $u_\varepsilon$  satisfies

$$a_\varepsilon u_t = u^2 \left\{ u_{xx} + \frac{(u - \varepsilon^2)^2}{u^2} \left( u + \frac{\varepsilon a_\varepsilon}{\xi_\varepsilon(x, u + \varepsilon^2)} - f_\varepsilon - \varepsilon^2 \right) \right\} \quad \text{in } K. \quad (18)$$

Setting

$$\varphi_\varepsilon(v) = \frac{(v - \varepsilon^2)^2}{v^2}, \psi_\varepsilon(x, v) = \frac{\varepsilon a_\varepsilon}{\xi_\varepsilon(x, v + \varepsilon^2)},$$

$\varphi_\varepsilon, \psi_\varepsilon$  satisfy the assumptions of Theorems 2, 3, so there are positive constants  $M_0, \delta_0, \varepsilon_0$  such that  $\delta_0 \leq u_\varepsilon \leq M_0$  on  $K$  for  $0 < \varepsilon < \varepsilon_0$ . Then we obtain a positive solution  $u$  of

$$a u_t = u^2 (u_{xx} + u - f) \quad (19)$$

as the limit of a subsequence of  $\{v_\varepsilon\}$  in  $W_p^{2,1}(K)$  with  $p > 2$ . It remains to prove the constraint (9) for  $w = u/a$ . Multiplying  $\sin(x - \zeta)/u^2$  with (19) and integrating over  $(0, 2\pi)$  yields

$$-\frac{d}{dt} \int_0^{2\pi} \frac{a}{u(x,t)} \sin(x - \zeta) dx = - \int_0^{2\pi} f \sin(x - \zeta) dx = 0$$

for all  $t, \zeta \in \mathbf{R}$ . Letting  $\varepsilon \rightarrow 0$  in (16), it follows that

$$\int \int_K \frac{a}{u} \sin(x - \zeta) dx dt = 0 \quad \text{for all } \zeta \in \mathbf{R}.$$

These imply that  $w = u/a$  satisfies the constraint (9). Therefore  $u$  is our desired solution of (19).  $\square$

## References

- [1] M. Gage, On the size of the blow-up set for a quasilinear parabolic equation, Contemporary Math. 127 (1992), 41-58.
- [2] Y. Giga and N. Mizoguchi, Existence of periodic solutions for equations of evolving curves, preprint.
- [3] M. Gurtin, Thermomechanics of evolving phase boundaries in the plane, Oxford Press, United Kingdom (1993).