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Existence of Selfsimilar Shrinking Curves for Anisotropic Curvature Flow Equations

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1 Introduction.

We consider a simple looking ordinary differential equation of the form

$$u_{xx} + u - \frac{a(x)}{u} = 0 \quad \text{in } \mathrm{R} \tag{1.1}$$

with a given positive function a. This equation arises in describing a selfsimilar solution of anisotropic curvature flow equations. Since x is the argument of the normal of the curve it is natural to impose 2π-periodicity for a in (1.1) and to ask for existence of 2π-periodic solutions. To simplify the notation we notice that a 2π-periodic function can be regarded as a function on the flat torus $\mathrm{T} := \mathrm{R}/2\pi\mathrm{Z}$. For example the space $C^m(\mathrm{T})$ is the space of all 2π-periodic C^m-functions on R. Let $C^m_+(\mathrm{T})$ denote the set of all positive functions in $C^m(\mathrm{T})$. In particular

$$C^2_+(\mathrm{T}) = \{u \in C^2(\mathrm{R}) \,|\, u(x+2\pi) = u(x) \text{ for all } x \in \mathrm{R}, u > 0\}. \tag{1.2}$$

Using this notation, we want to investigate the existence of solutions of (1.1) in $C^2_+(\mathrm{T})$. As to this, we have the following

1.1 Main Existence Theorem. Assume that a is a positive, continuous function on T. Then there is a function $u \in C^2_+(\mathrm{T})$ solving (1.1).

The key step to prove this result is to derive a-priori bounds for solutions of (1.1):

1.2 Theorem on A-Priori Bounds. Let $0 < A_1 < A_2$ be two constants. Then there are two positive constants m and M, depending only on A_1 and A_2, such that if $u \in C^2_+(\mathrm{T})$ solves (1.1) on T with

$$A_1 \le a \le A_2 \tag{1.3}$$

then

$$m \le u \le M \quad \text{on T}. \tag{1.4}$$

The proof of this a-priori estimate actually shows that the continuity of a is not needed.

1.3 Corollary on existence. Let $a \in L^\infty(\mathrm{T})$ and satisfy (1.3). Then there is a function $u \in C^{1,1}_+(\mathrm{T})$ solving (1.1).

Here $C^{1,1}_+(\mathrm{T})$ denotes the space of all positive, 2π–periodic functions whose derivative is Lipschitz-continuous. The differential equation is solved in the sense of distributions and almost everywhere.
To prove Corollary 1.3, we approximate a by continuous functions a_j, keeping the bounds (1.3) and $a_j \to a$ in L^p_{loc}-sense for $p > 1$ as $j \to \infty$.
Let u_j be the solution of (1.1) taking a_j instead of a. By the a-priori bounds (1.4) and the equation (1.1) the sequence u_j is bounded in L^∞ along with u_{jx} and u_{jxx}. Thus a subsequence of the u_j converges to some function u in $C^1_+(\mathrm{T})$; it is not difficult to show $u \in C^{1,1}_+(\mathrm{T})$ and that u solves (1.1).

1.4 Generalizations. To get a better understanding of the mechanisms we will carry out the proof of the a-priori bounds considering the slightly more general equation

$$u_{xx} + u - a(x)\,g(u) = 0 \quad \text{in } I \subset \mathrm{R} \tag{1.5}$$

instead of (1.1). Here again a satisfies (1.3) on the interval I and g is assumed to be a positive, continuous, nonincreasing function on $(0, \infty)$. Defining

$$G(p) = \int_1^p g(s)\,ds, \tag{1.6}$$

we moreover impose the following conditions on g:

$$\lim_{p\to 0} G(p) = -\infty, \qquad \lim_{p\to\infty} G(p)p^{-2} = 0, \tag{1.7}$$

$$\lim_{p\to 0,\, q\to\infty} \frac{G(p)p}{g(q)q^2} = 0, \tag{1.8}$$

$$\lim_{p\to\infty} g(p) = 0. \tag{1.9}$$

(Note that condition $(1.7)_2$ is automatically satisfied by (1.6) and the nonincreasing property of g.) Examples for functions satisfying these conditions are given by

$$g(p) = p^{-\sigma}, \qquad 1 \le \sigma < 2. \tag{1.10}$$

1.5 Selfsimilar Shrinking Our main existence theorem has an application for evolution equations for embedded closed curves $\{\Gamma_t\}_{t>0}$ in R^2 derived in [Gu]:
Let V be the inward velocity of Γ_t in the direction of its unit inward normal vector

$$n(\theta) = (\cos\theta, \sin\theta).$$

Let k be the inward curvature of Γ_t and let f and β be positive functions on R, which are 2π-periodic. We consider an equation for Γ_t of the form

$$V = a(\theta)k, \qquad a(\theta) = \beta(\theta)^{-1}(f''(\theta) + f(\theta)).$$

Here $f'' + f$ is assumed to be positive, so that the equation is parabolic. Such an equation arises in a model describing the motion of phase boundaries in an anisotropic medium (see [Gu]). The

function f is called the surface energy density and β is called the cinetic coefficient.
If $a(\theta)$ is constant, the equation becomes the curvature flow equation and the evolution of Γ_t is well studied: No matter what initial curve is given, the solution stays smooth and embedded and eventually becomes convex ([Gr]). It then stays convex and shrinks to a point in finite time ([GH]). The type of shrinking is asymptotically similar to that of a shrinking circle $\{C_t\}$ ([Ga1], [Ga2], [GH]), which is self-similar in the sense that

$$C_t = (t_* - t)^{1/2}\, C,$$

where C denotes the unit circle centered at the origin, the time t_* is the extinction time and λC denotes the dilatation of C with multiplier λ. Selfsimilar solutions are classified even for immersed curves ([AL]) and the asymptotic shape of singularities of this type is classified ([A]).
We are interested in finding such selfsimilar solutions

$$\Gamma_t = (t_* - t)^{1/2}\, \Gamma$$

for general $a(\theta)$. Such solutions exist in the case that $\beta(\theta)^{-1}$ equals a constant multiple of $f(\theta)$. Then Γ is the boundary of the so-called Wulff-shape W of f, i.e.

$$W := \left\{x \in \mathbb{R}^2 \,|\, x \cdot n(\theta) \leq f(\theta) \text{ for all } \theta \in \mathbb{R}\right\}.$$

This is explicitely stated in [Son], including the multidimensional case where β and the second differential f'' are assumed continuous, so also a is continuous. It is not difficult to see that such results extend to $f \in C^{1,1}$, provided that $f'' + f$ is still bounded away from zero and if the definition of a solution is given in some appropriate sense.
Our main existence theorem yields the existence of selfsimilar solutions for arbitrary bounded a (see 1.3). Indeed, every equation $V = a(\theta)k$ can be rewritten as

$$V = u(u'' + u)\, k,$$

where u is a solution of (1.1) with θ replacing x.

We would like to note that our existence theorem is not included in the available theory of singular Lagrangian systems developed in [AC], [C], [Sol] and [T]:
Equation (1.1) can be written as

$$u_{xx} + \frac{\partial V}{\partial u}(x, u) = 0$$

with potential

$$V(x, u) := \frac{1}{2}u^2 - a(x)\log\, u.$$

Although the existing theory includes existence of periodic solutions for potentials having a singularity near $u = 0$, our V here violates the assumptions the theory requires: The logarithmic singularity near $u = 0$ is too weak – a singularity like u^{-2} or stronger is required – and the growth near infinity is also not included.
We thank Professor Kazunaga Tanaka for informing us about the above references concerning the variational approach. According to him the method used in [T] may lead a way to weaken the strong singularity condition. In any case it would be interesting to study (1.1) from the variational point of view.

After this work was completed, we were informed of a recent work of Matano and Taniyama closely related to ours. They claimed that if $a(\theta)$ is a smooth, even function, i.e. $a(\theta) = a(-\theta)$, a general solution Γ_t of $V = a(\theta)k$ becomes convex after some finite time t_0 and shrinks to a point at some time $t_* > t_0$ in a way that the curvature of

$$S_t = (t_* - t)^{1/2}\,\Gamma_t$$

is bounded from above and away from zero for all $t > t_0$. This yields the existence of selfsimilar solutions by applying the theory of dynamical systems for parabolic equations. However, our proof is more direct and does not assume any symmetry or regularity of a.

2 A-Priori-Estimates

Our main estimate (1.4) follows from the results we will prove in section 3 and section 4. To simplify the terminology let us define the following terms:

Definition: A solution $u \in C^2_+(\mathrm{T})$ of (1.1) or (1.5) is called a singlepeak-solution, if the set of points not being local extrema consists of two connected components in T. Otherwise u is called a multipeak-solution.

To prove the a-priori bounds these two types of solutions need essentially different techniques. Thus let us state the results separately:

2.1 Peak-Bound Lemma for Multipeak-Solutions. Let $u \in C^2_+(I)$ be a solution of (1.5) on some open interval I and let (1.3) be satisfied. If u attains local minima in $\alpha, \beta \in I$, $\alpha < \beta$ and u_x changes its sign only once in (α, β), then there is a positive constant M_0, depending only on A_1, A_2 and g, such that

$$u \le M_0 \quad \text{in } (\alpha, \beta) \tag{2.1}$$

provided that $\beta - \alpha \le \pi$.

2.2 Peak-Bound Lemma for Singlepeak-Solutions. Let $u \in C^2_+(\mathrm{T})$ be a singlepeak-solution of (1.5) and let (1.3) be satisfied. Then there is a positive constant M_1, depending only on A_1, A_2 and g, such that

$$u \le M_1 \quad \text{in T}. \tag{2.2}$$

2.3 Proposition on Equivalence of Upper and Lower Bounds. Let$u \in C^2_+(\mathrm{T})$ be a solution of (1.5) and let (1.3) be satisfied.
(i) If there is a constant $\tilde{M}$, depending only on A_1, A_2 and g, such that one local maximum $u(\gamma)$ is estimated by $u(\gamma) \le \tilde{M}$, then there are two other constants $0 < m < M$, also depending only on A_1, A_2 and g, such that

$$m \le u \le M \quad \text{on T}.$$

(ii) The conclusion in (i) also holds, if there is a constant $\tilde{m} > 0$, depending only on A_1, A_2 and g, such that one local minimum $u(\alpha)$ is estimated by $u(\alpha) \ge \tilde{m}$.

Theorem 1.2 is an immediate consequence of Lemma 2.1, 2.2 and Proposition 2.3, as can be seen as follows: If u is a multipeak solution, there exists at least one pair of local mimima with a distance less or equal π. On these intervals Lemma 2.1 can be applied and due to Proposition 2.3 all extrema are estimated in terms of one extremum. The situation needed to apply Lemma 2.1 fails to exist only if u has exactly one local minimum, i.e. is a singlepeak solution. But in this case Lemma 2.2 yields the upper bound and due to Proposition 2.3 we again have a lower bound; thus the theorem is proved.

The results above also show, that the set of all 2π-periodic solutions of (1.1) or (1.5) is bounded uniformly in the set of all a that satisfy (1.3).

3 Nonexistence of Large Multipeak-Solutions

This chapter is devoted to the proof of Lemma 2.1 and Proposition 2.3. For later use let us define

$$B_1 := \min\{v | v - A_1 g(v) \geq 0\} > 0, \tag{3.1}$$

$$B_2 := \max\{v | v - A_2 g(v) \leq 0\} < \infty. \tag{3.2}$$

3.1 Lemma on Estimates of Local Extrema. Let $u \in C^2_+(I)$ solve (1.5) with (1.3) on I, and assume that u takes a local maximum at $\gamma \in I$, a local minimum at $\alpha \in I$ and is monotone between α and γ. Then

$$\begin{aligned} u(\alpha) &\leq B_2, & (3.3)\\ u(\gamma) &\geq B_1, & (3.4)\\ u(\gamma)^2 - 2A_2G(u(\gamma)) &\leq u(\alpha)^2 - 2A_2G(u(\alpha)) & (3.5)\\ u(\alpha)^2 - 2A_1G(u(\alpha)) &\leq u(\gamma)^2 - 2A_1G(u(\gamma)) & (3.6) \end{aligned}$$

Proof: We may assume $\alpha < \gamma$. Since

$$u_{xx} + u - A_2 g(u) \leq 0 \quad \text{in } I,$$

multiplying u_x yields

$$\frac{d}{dx}\left\{\frac{1}{2}u_x^2 + \frac{1}{2}u^2 - A_2 G(u)\right\} \leq 0 \quad \text{in } I.$$

Integrating this inequality on (α, γ) yields (3.5), since $u_x(\alpha) = u_x(\gamma) = 0$. If we start with

$$u_{xx} + u - A_1 g(u) \geq 0 \quad \text{in } I$$

and proceed as above, we obtain (3.6).
The other two inequalities are simple consequences from the maximum principle and the monotonicity of g. □

3.2 Proof of Proposition 2.3. We consider the local situation given in Lemma 3.1. So if u has a maximum γ satisfying $u(\gamma) \leq M$, then the neighboured minimum α is estimated using (3.6) by

$$-A_1 G(u(\alpha)) \leq M^2 - 2A_1 G(u(\gamma)) \leq M^2.$$

This gives a lower bound $u(\alpha) \geq m$ due to the unboundedness of G in a neighbourhood of zero stated in (1.7). Note that m only depends on A_1, g and M.
Conversely, if $u(\alpha) \geq m$ then (3.5) and (3.3) lead to

$$u(\gamma)^2 - 2A_2G(u(\gamma)) \leq B_2^2 - 2A_2G(m)$$

As the conditions on g also include that $G(p)p^{-2}$ tends to zero as p tends to infinity, this inequality gives a bound $u(\gamma) \leq M$, where M only depends on A_2, g and m. □

Remark: Please note that in the proof of Proposition 2.3 only the conditions (1.7) are used. The stronger conditions (1.8) will only be needed below in the proof of Lemma 2.1.

3.3 A Technical Lemma. Let $(1.7)_2$ and (1.8) hold. Then for every $\lambda > 0$ there exist positive constants $r_0 = r_0(\lambda, A_2, g)$ and $R_0 = R_0(\lambda, A_2, g)$, such that if $0 < r < R$ satisfy

$$R^2 - 2A_2G(R) \leq r^2 - 2A_2G(r), \tag{3.7}$$

$$\frac{r}{g(R)} \geq \lambda, \tag{3.8}$$

then either $r \geq r_0 > 0$ or $R \leq R_0$.

Proof: Suppose that there exist sequences $r_j \to 0$ and $R_j \to \infty$, if $j \to \infty$, satisfying $r_j/g(R_j) \geq \lambda$ and

$$R_j^2 - 2A_2G(R_j) \leq r_j^2 - 2A_2G(r_j).$$

Multiplying this inequality by $\frac{r_j}{g(R_j)R_j^2}$ we obtain

$$\frac{r_j}{g(R_j)}\left(1 - 2A_2G(R_j)R_j^{-2} - \frac{r_j^2}{R_j^2}\right) \leq -2A_2\frac{G(r_j)r_j}{g(R_j)R_j^2}$$

Invoking the conditions $(1.7)_2$, (1.8) on g this implies

$$\begin{aligned} \frac{r_j}{g(R_j)} &\leq -2A_2\frac{G(r_j)r_j}{g(R_j)R_j^2} \\ &< \lambda \end{aligned}$$

for sufficiently large j, which contradicts the assumption. □

3.4 Lemma on the Distance of Critical Points. Let $u \in C_+^2(I)$ solve (1.5) with (1.3) on I and let $\alpha, \beta \in I$, $\alpha < \beta$ be critical points of u, i.e. $u_x(\alpha) = u_x(\beta) = 0$. Then

$$\frac{A_1 \int_\alpha^\beta g(u)u}{\int_\alpha^\beta u} > 2\max\{u(\alpha), u(\beta)\} \tag{3.9}$$

implies $\beta - \alpha > \pi$.

The easy, but fundamental observation needed to prove this lemma is the validity of the following integral identity, which is obtained from (1.5) by multyplying with u and integrating by parts:

3.5 Lemma. Let $u \in C^2_+(I)$ solve (1.5) on I and let $\alpha, \beta \in I$, $\alpha < \beta$ and $u_x(\alpha) = u_x(\beta) = 0$. Then

$$\int_\alpha^\beta u_x^2 = \int_\alpha^\beta u^2 - \int_\alpha^\beta a(x)ug(u). \tag{3.10}$$

Proof of Lemma 3.4. Let $l(x)$ be an affine function defined by $u(\alpha) = l(\alpha)$ and $u(\beta) = l(\beta)$, i.e.

$$l(x) = \mu x + u(\alpha), \qquad \mu := \frac{u(\beta) - u(\alpha)}{\beta - \alpha}.$$

Applying Lemma 3.5 and setting $v(x) = u(x) - l(x)$, we have

$$\int_\alpha^\beta (v_x + \mu)^2 = \int_\alpha^\beta (v + l)^2 - \int_\alpha^\beta a(x)ug(u).$$

Since $v(\alpha) = v(\beta) = 0$,

$$\int_\alpha^\beta (v_x + \mu)^2 \geq \int_\alpha^\beta v_x^2$$

holds. We thus derive

$$\int_\alpha^\beta v_x^2 \leq \int_\alpha^\beta v^2 - J, \tag{3.11}$$

where J is given by

$$J := \int_\alpha^\beta a(x)ug(u) - 2\int_\alpha^\beta vl - \int_\alpha^\beta l^2.$$

But estimating the integrals on the right hand side of this definition we see

$$\begin{aligned} 2\int_\alpha^\beta vl + \int_\alpha^\beta l^2 &\leq 2\int_\alpha^\beta ul \\ &\leq 2\max\{\, u(\alpha), u(\beta)\}\int_\alpha^\beta u, \end{aligned} \tag{3.12}$$

which implies $J > 0$ by using (3.9). But then, assuming $\beta - \alpha \leq \pi$ and estimating the left hand side of (3.11) from below by the Wirtinger Inequality gives

$$\int_\alpha^\beta v^2 \leq \int_\alpha^\beta v_x^2 \leq \int_\alpha^\beta v^2 - J,$$

which is clearly a contradiction. Thus $\beta - \alpha > \pi$. □

3.6 Proof of Lemma 2.1. Since u is a multipeak solution, there exist local minima α, β of u satisfying $\beta - \alpha \leq \pi$, thus Lemma 3.4 implies

$$L := \frac{A_1 \int_\alpha^\beta g(u)u}{\int_\alpha^\beta u} \leq 2\max\{\, u(\alpha), u(\beta)\,\}$$

Let $u(\gamma)$ be a maximum of u in (α, β). As g is monotone,

$$L \geq A_1 g(u(\gamma)),$$

so that

$$\frac{\max\{\, u(\alpha), u(\beta)\,\}}{g(u(\gamma))} \geq \frac{1}{2}A_1.$$

As the estimate on the local maximum (3.5) holds, applying the technical lemma 3.3 yields either

$$\max\{ u(\alpha), u(\beta) \} \geq r_0 > 0 \quad \text{or} \quad u(\gamma) \leq R_0.$$

By Proposition 2.3, proved earlier in this section, we conclude that if one of the above bounds hold, we automatically have an upper bound for u, eventually replacing R_0 by another constant M_0.

This completes the proof of Lemma 2.1. □

4 Nonexistence of Large Singlepeak-Solutions

In this chapter we will consider singlepeak solutions of (1.5), i.e. solutions that without loss of generality can be assumed to have exactly one local maximum and one local minimum in $\mathbb{T}$, as nonstrict extrema can only appear in a finite range of positive values of u.

The starting point is again a local property of the solution; here we will examine the distance of inflection points next to the local maximum and the scaled limit of u if this maximum becomes large.

4.1 Lemma on Inflection Points. Let $\{u_n\}_{n\in\mathbb{N}} \subset C^2_+([p_n, q_n])$ with $q_n - p_n \leq 2\pi$ be a sequence of solutions for

$$u_{nxx} + u_n - a_n(x) g(u_n) = 0 \quad \text{in } [p_n, q_n], \tag{4.1}$$

a_n satisfying (1.3) and u_n satisfying $u_{nxx}(p_n) = u_{nxx}(q_n) = 0$ and $u_{nxx} \leq 0$ on $[p_n, q_n]$. Moreover suppose

$$M_n := \max_{x\in[p_n,q_n]} u_n(x) \longrightarrow \infty \quad \text{if } n \to \infty,$$

and the maximum is attained in (p_n, q_n). Then $q_n - p_n \longrightarrow \pi$ and $u_n(x + p_n)/M_n \longrightarrow \sin x$ pointwise in $(0, \pi)$, if $n \to \infty$.

Proof: By translation we may assume $p_n = 0$. Rescaling

$$v_n(x) := \frac{1}{M_n} u_n(\lambda_n x) \quad \text{with } \lambda_n := \frac{q_n}{\pi},$$

yields

$$v_{nxx} + \lambda_n^2 v_n - \frac{\lambda_n^2 a_n}{M_n} g(u_n) = 0 \quad \text{in } [0, \pi]. \tag{4.2}$$

By the estimates $0 \leq v_n \leq 1$, $v_{nxx} \leq 0$ and $v_{nxx} + \lambda_n^2 v_n \geq 0$ we conclude that $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $C^2([0, \pi])$. Therefore the Arzela-Ascoli Theorem guarantuees the existence of a subsequence – still denoted by v_n – such that $v_n \longrightarrow v \in C^1([0, \pi])$. Taking another subsequence if necessary we have $\lambda_n^2 \to k \in [0, 2]$. Since $u_n \geq B_1$ on $[p_n, q_n]$ by the concavity of u_n, passing to the limit in (4.2) yields

$$v_{xx} + k\, v = 0 \quad \text{in } [0, \pi]$$

along with $v(0) = v(\pi) = 0$, $v \geq 0$ and $\max v = 1$. But the only solution to this problem is given by

$$k = 1 \quad \text{and} \quad v(x) = \sin x.$$

Thus $q_n/\pi \to 1$ and using this we conclude

$$\frac{1}{M_n}u_n(x) \to \sin x.$$

As the limit is independent of the choice of the subsequence, the proof is complete. □

4.2 Lemma on the Concave Part. Let $u \in C^2_+([p,q])$ be a concave solution of (1.5) on $[p,q]$ and $u_{xx}(p) = u_{xx}(q) = 0$. Then for each $\epsilon > 0$ there is a constant M, depending only on A_1, A_2 and ϵ, such that

$$\left|\int_p^q \sin(x-p)a(x)g(u(x))dx\right| < \epsilon \quad \text{and} \quad |\pi - (q-p)| < \epsilon \tag{4.3}$$

holds provided

$$\max_{x\in[p,q]} u(x) \geq M.$$

Proof: If not, then there is a $\epsilon_0 > 0$ and a sequence $\{u_n, p_n, q_n, a_n\}_{n\in\mathbb{N}}$ satisfying the assumptions of Lemma 4.1 and

$$\left|\int_{p_n}^{q_n} \sin(x-p_n)a_n(x)g(u_n(x))dx\right| \geq \epsilon_0 \quad \text{or} \quad |\pi - (q_n - p_n)| \geq \epsilon_0. \tag{4.4}$$

As $(q_n - p_n) \to \pi$ by Lemma 4.1, the integral expression has to be valid. Since $u_n \geq B_1$ and g is nonincreasing, the integrand is bounded. Moreover, by (1.9) and Lemma 4.1 we observe that for each $x \in (0,\pi)$

$$\lim_{n\to\infty} |\sin x \; a_n(x+p_n)g(u_n(x+p_n))| = 0$$

holds. Thus applying the dominate convergence theorem yields a contradiction to the integral expression in (4.4). □

The next lemma quantifies the relation between the minimum value of u and the values of u at points distant from the minimum point.

4.3 Lemma on a Lower Bound away from the Local Minimum. Let $u \in C^2_+(I)$ solve (1.5) in an open interval I, where u_x changes its sign only once, and let $\alpha \in I$ be the point, where the minimum is attained. Moreover let $u(\alpha) < \zeta < B_1$ and define

$$\delta^2 := \frac{2\zeta}{A_1 g(\zeta) - \zeta}.$$

Then $|x - \alpha| \geq \delta$ implies $u(x) \geq \zeta$.

Proof: We may assume $x > \alpha$; the case $x < \alpha$ is similar. Supposing $u(x) < \zeta$, the monotonicity of u on (x,α) yields

$$u_{xx} = ag(u) - u \geq A_1 g(\zeta) - \zeta = \frac{2\zeta}{\delta^2}.$$

Integrating twice on (x,α) then gives

$$u(x) - u(\alpha) \geq \frac{\zeta}{\delta^2}(x-\alpha)^2$$

which implies $u(x) \geq \zeta$, if $|x-\alpha| \geq \delta$; a contradiction to our assumption. □

4.4 Proof of Lemma 2.2. Without loss of generality we may assume that

$$u(\gamma) := \max_{x\in\mathbf{T}} u(x) > B_2 \quad \text{and} \quad u(\alpha) := \min_{x\in\mathbf{T}} u(x) < B_1,$$

so that both maximum and minimum are strict and thus γ and α uniquely determined. Suppose p and q are two inflection points and $u_{xx} \leq 0$ in $[p,q]$. We distinguish two cases:

Case 1: $q - p < \pi$.
Multiplying the differential equation (1.5) by $\sin(x-p)$ and integrating yields

$$\begin{aligned} 0 &= \int_p^{p+2\pi} \sin(x-p)a(x)g(u(x))dx \\ &= \int_p^q + \int_q^{p+\pi} + \int_{p+\pi}^{p+2\pi} \sin(x-p)a(x)g(u(x))dx \\ &=: I_0 + I_1 + I_2. \end{aligned}$$

Now we know from Lemma 4.2, that for all $\epsilon > 0$ there exists an $M > 0$, such that if $u(\gamma) \geq M$, then

$$0 < I_0 \leq \epsilon \quad \text{and} \quad |\pi - (q-p)| \leq \epsilon$$

holds. Besides the estimate on I_0 this result also allows a statement about where to locate the minimumpoint α: Generally α has a distance greater or equal $\pi/2$ either from p or from q; here we can conclude, that the distance either to p or $p+\pi$ is larger than $\pi/4$, provided we choose $\epsilon < \pi/8$.
Suppose $|\alpha - (p+\pi)| > \pi/4$. Due to this minimal distance between α and $p+\pi$ we derive a lower bound for u on $[q, p+\pi]$: Choosing ζ in Lemma 4.3 in a way that $\delta \geq \pi/4$, the conclusion of the lemma gives

$$u \geq \zeta \quad \text{in } [q, p+\pi].$$

(Please note that ζ can be choosen independent of ϵ.) Thus I_1 is estimated by

$$0 < I_1 \leq A_2 g(\zeta)(\pi - (q-p)) < A_2 g(\zeta)\epsilon.$$

The case $|\alpha - \pi| > \pi/4$ is similar, so the above estimate on I_1 holds in general.
As to I_2, we have the estimate

$$I_2 \leq -2A_1 g(B_2),$$

due to the bound $u \leq B_2$ on the possible values of u at inflection points, given by the differential equation.
Collecting terms we arrive at

$$\epsilon + A_2 g(\zeta)\epsilon - 2A_1 g(B_2) \geq 0.$$

But this leads to a contradiction, if ϵ is choosen small enough, i.e.

$$\epsilon := \min\left\{\frac{A_1 g(B_2)}{1 + A_2 g(\zeta)}, \frac{\pi}{8}\right\}.$$

So $u(\gamma)$ must be bounded by the constant $M = M(A_1, A_2, \epsilon)$ in Lemma 4.2.
Case 2: $q - p \geq \pi$.
Using the notation from case 1, we see that I_1 does not occur here. Thus

$$0 = \int_p^q + \int_q^{p+2\pi} \sin(x-p)a(x)g(u(x))dx =: I_0 + I_2$$

must hold. Again we have an upper bound for I_2 by

$$I_2 \leq -A_1 g(B_2) \int_{p+9\pi/8}^{p+2\pi} |\sin(x-p)| dx \leq -A_1 g(B_2),$$

provided $\epsilon < \pi/8$. The contradiction now follows immediately from lemma 4.2.
This completes the proof of lemma 2.2.

5 Existence of Solutions

In this chapter we will prove the existence of a solution of (1.1) using the Leray-Schauder degree. Herein we make use of the uniform boundedness of solutions of (1.1) with respect to functions a satisfying (1.3) stated in Theorem 1.2. We define

$$E := \left\{ v \in C^0_+(\mathrm{T}) \,|\, \frac{m}{2} \leq v \leq 2M \text{ in } \mathrm{T} \right\}. \tag{5.1}$$

5.1 Mappings. Let F be a continuous mapping from $E \times [0,1]$ into $C^0_+(\mathrm{T})$, defined by

$$F(u,\tau) := 2u - \frac{\tau a(x) + (1-\tau)a_0}{u} \tag{5.2}$$

with a constant a_0 satisfying the bounds imposed on a in (1.3).
Let T denote a linear compact operator from $C^0_+(\mathrm{T})$ into itself, given by $w = T(f)$, where w is the unique solution of

$$-w_{xx} + w = f \quad \text{in } \mathrm{T}.$$

Setting $S_\tau := S(.,\tau) := T \circ F(.,\tau)$, we have a continuous, compact mapping from E into $C^0_+(\mathrm{T})$. Clearly u is a fixed point of S_τ, if and only if $u \in E$ solves

$$u_{xx} - u + 2u - \frac{\tau a(x) + (1-\tau)a_0}{u} = 0 \quad \text{in } \mathrm{T},$$

which is (1.1) in case of $\tau = 1$. The a-priori bounds in Theorem 1.2 now imply that S_τ has no fixed point on the boundary of E, in other words

$$(I - S_\tau)u \neq 0 \quad \text{on } \partial E, \quad 0 \leq \tau \leq 1.$$

Thus the homotopy invariance of the Leray-Schauder degree yields

5.2 Proposition.

$$\deg\,(I - S_1, E, 0) = \deg\,(I - S_0, E, 0). \tag{5.3}$$

To show the existence of a solution of (1.5) it now suffices to prove that this degree is not equal zero.

5.3 Lemma. The number

$$\deg\,(I - S_0, E, 0) \tag{5.4}$$

is not zero; in fact, it equals -1.

Proof: As proved by Gage and Hamilton in [GH] (see also [AL], [EW]), there is a unique solution $u \in E$ of

$$u_{xx} + u - \frac{a_0}{u} = 0 \quad \text{in } \mathrm{T},$$

which is given by the constant $a_0^{1/2}$. (Actually in [GH] the setting is $a_0 = 1/2$, but our problem here reduces to theirs by changing from u to $(2a_0)^{1/2}u$.)
So $u_0 := a_0^{1/2}$ is the only zero of $I - S_0$ in E; thus

$$\deg\,(I - S_0, E, 0) \;=\; \deg\,(I - S_0, B_\delta(u_0), 0)$$

for some sufficiently small δ. At u_0 the mapping $I - S_0$ is nondegenerate in the sense that the derivative $I - S_0'(u_0)$ is injective. Indeed, suppose that

$$(I - S_0'(u_0))v \;=\; 0$$

Since $S_0'(u_0) = T \circ F'(u_0, 0)$, this implies

$$-v_{xx} + v \;=\; 2v + \frac{a_0}{u_0^2}v$$

or, using the definition of u_0

$$v_{xx} + 2v \;=\; 0.$$

But this problem has no nontrivial 2π-periodic solution. This nondegeneracy enables us to apply a standard degree theory result (see [N], theorem 2.8.1, p.66 or [D], example 2.8.3, p.65), which states

$$\deg\,(I - S_0, B_\delta(u_0), 0) \;=\; (-1)^\beta$$

where β is the number of eigenvalues of S_0' (counting algebraic multiplicity) greater than one.
We show the elementary computation of β:
A number λ is an eigenvalue of $S_0'(u_0)$, if and only if there is a nontrivial solution $v \in C_+^0(\mathrm{T})$ of

$$\lambda v \;=\; S_0'(u_0)v$$

or, equivalently

$$-v_{xx} \;=\; \frac{3-\lambda}{\lambda}v.$$

Thus β equals the number of $\lambda > 1$ (counted with multiplicity), that solve $\frac{3-\lambda}{\lambda} = n^2$ for some integer $n \geq 0$. As these λ are given by $\lambda = 3$ and $\lambda = 3/2$ with multiplicity 1 and 2, respectively, we have

$$\deg\,(I - S_0, B_\delta(u_0), 0) \;=\; (-1)^3 \;=\; -1.$$

$\square$

Remark: Concerning the uniqueness of solutions of (1.1) in $C_+^2(\mathrm{T})$, the implicit function theorem implies that the zero of $I - S_\tau$ is unique provided τ is small, since no bifurcation from $(u_0, 0)$ occurs due to the nondegeneracy of the unique zero u_0 of $I - S_0$.
We intend to discuss the general uniqueness problem in a forthcoming paper.

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References

[A] S. Angenent, *On the Formation of Singularities in the Curve Shortening Flow*, J. Diff. Geometry **33** (1991), pp.601-633.

[AC] A. Ambrosetti, V. Coti Zelati, *Periodic Solutions of Singular Lagrangian Systems*, Birkhäuser, Zürich (1993).

[AL] U. Abresch, J. Langer, *The Normalized Curve Shortening Flow and Homothetic Solutions*, J. Diff. Geometry **23** (1986), pp.175-196.

[C] V. Coti Zelati, *Dynamical Systems with Effektive-like Potentials*, Nonlin. Anal. Appl. **12** (1988), pp.209-222.

[D] K. Deimling, *Nonlinear Functional Analysis*, Springer, Heidelberg, Berlin, New York (1985).

[EW] C. Epstein, M. Weinstein, *A Stable Manifold Theorem for the Curve Shortening Equation*, Comm. Pure Appl. Math. **40** (1987), pp.119-139.

[Ga1] M. Gage, *An Isoperimetric Inequality With Application to Curve Shortening*, Duke M. J. **50** (1983), pp.1225-1229.

[Ga2] M. Gage, *Curve Shortening Makes Convex Curves Circular*, Inv. Math. **76** (1984), pp.357-364.

[GH] M. Gage, R.S. Hamilton, *The Heat Equation Shrinking Convex Plane Curves*, J. Diff. Geometry **23** (1986), pp.69-96.

[Gr] M.Grayson, *The Heat Equation Shrinks Embedded Plane Curves to Points*, J. Diff. Geometry **26** (1987), pp.285-314.

[Gu] M.E. Gurtin, *Thermodynamics of Evolving Phase Boundaries in the Plane*, Clarendon Press, Oxford (1993).

[N] L.Nirenberg, *Topics in Nonlinear Functional Analysis*, Lecture Notes 1973/74, Courant Inst. of Math. Sciences.

[Sol] S. Solimini, *On Forced Dynamical Systems With a Singularity of Repulsive Type*, Nonlin. Anal. Appl. 14 (1990), pp.485-500.

[Son] H.M. Soner, *Motion of a Set By the Curvature of Its Boundary*, J. Diff. Eq. **101** (1993), pp.313-392.

[T] S. Terracini, *Remarks on Periodic Orbits of Dynamical Systems With Repulsive Singularity*, J. Funct. Anal. 111 (1993), pp.213-238.