Invertible Toeplitz Operators
and Uniform Algebras

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Invertible Toeplitz Operators

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Uniform Algebras

by

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Abstract. Toeplitz operators T_ϕ^M are defined on invariant subspaces M of an arbitrary uniform algebra A. We give a necessary and sufficient condition of uniformly invertible T_ϕ^M with respect to some family $\mathcal{F}$ of invariant subspaces M. This condition is the same to a classical one in case A is the disc algebra. In special uniform algebras, we can choose a small family, in fact, if A is a disc algebra then $\mathcal{F}$ can be a single set. Then this generalizes the Widom-Devinatz theorem. As an application, we study a $\mathcal{F}$-union of spectrums of $\{T_\phi^M \ ; \ M \in \mathcal{F}\}$.

§1. Introduction

Let X be a compact Hausdorff space, let $C(X)$ be the algebra of complex-valued continuous functions on X, and let A be a uniform algebra on X. Let τ be a nonzero complex homomorphism of A and let N_τ be the set of representing measures for τ whose support is contained in X and $m \in N_\tau$. The abstract Hardy space $H^p = H^p(m)$, $1 \leq p \leq \infty$, determined by A is defined to be the closure of A in $L^p = L^p(m)$ when p is finite and to be the weak* $-$ *closure of* A in $L^\infty = L^\infty(m)$ when p is infinite. Put $H_0^p = \{f \in H^p \ ; \ \int f dm = 0\}, K_0^p = \{f \in L^p \ ; \ \int fgdm = 0$ for all $g \in \ A\}$ and $K^p = K_0^p + C$. Then $H_0^p \subset K_0^p$ and $H_0^p \subset K^p$. The abstract Hardy spaces somtimes coincide with the concrete Hardy spaces or the concrete Bergman spaces.

A closed subspace M for A of $L^2 = L^2(m)$ is said to be invariant if fM is contained in M for all f in A. lat A denotes the set of all invariant subspaces of A in L^2. For ϕ in L^∞, the Toeplitz operator T_ϕ^M is the operator on M defined by

$$T_\phi^M(f) = P_M(\phi f)$$

where $M \in$ lat A and P_M is the orthogonal projection onto M. If $M = H^2$ then we will write $T_\phi = T_\phi^M$. In this paper we are interested in the equivalence of the following four statements.

(1) T_ϕ is invertible.

(2) For each $M \in$ lat A, T_ϕ^M is invertible.

(3) For each $M \in$ lat A, T_ϕ^M is invertible and sup $\{\|(T_\phi^M)^{-1}\| \ ; \ M \ \in$ lat $A\}$ is finite.

(4) ess.inf $|\phi| > 0$ and there exists a function g in $(H^\infty)^{-1}$ such that $\mathrm{Re}(\phi g) \geq \delta$ a.e. for some constant $\delta > 0$.

In the four statements, (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1) are clear. If (4) is valid, then there exist positive constants ε and ε_0 such that $\|\varepsilon\phi g - 1\|_\infty \leq 1 - \varepsilon_0$. Hence $T_{\varepsilon\phi g}^M$ is invertible for any $M \in$ lat A and

$$\|(T_{\varepsilon\phi g}^M)^{-1}\| \leq \frac{1}{1-\|T_{\varepsilon\phi g-1}^M\|} \leq \frac{1}{\varepsilon_0}.$$

Thus sup $\{\|(T_\phi^M)^{-1}\| \ ; \ M \ \in$ lat$A\} < \infty$. This implies (3). In this paper we will show (3) $\Rightarrow$ (4) under a condition : $H^\infty = H^1 \cap L^\infty$. This condition is satisfied by five natural examples in §2.

When A is the disc algebra of Example 1 in §2, Devinatz and Widom (see [4]) showed (1) $\Leftrightarrow$ (4). Hence the four statements (1) $\sim$ (4) are equivalent. When A is the rational function algebra of Example 3 in §2, Abrahamse [1] showed (2) $\Leftrightarrow$(4) in case the symbols of Toeplitz operators are unimodular. In §4 of this paper, we will show (2) $\Leftrightarrow$(4) without the condition on the symbols. Hence our result solves the problem which was proposed by Abrahamse [1, p294]. When A is the subalgebra of the disc algebra of Example 4 in §2, Anderson and Rochberg [2] gave a generalization of the theorem of

Devinatz and Widom. Their definition of Toeplitz operators is different from ours but their results are closed to (2) $\Leftrightarrow$(4). In §4, we will study their result in a more general setting. By Corollary 1 in §3, (3) $\Leftrightarrow$ (4) is still true for Toeplitz operators in the Bergman space (see Example 2 in §2) and in the Hardy space of the polydisc (see Example 5 in §2). In Examples 2, 3 and 4, it is known that (1) does not imply (4).

In §5, as an application of the equivalence of (2) $\sim$ (4) we will study $\cup\ \{\sigma(T_\phi^M)$; M $\in$ lat $A\}$ where $\sigma(T_\phi^M)$ denotes the spectrum of T_ϕ^M. In this paper, we use the theory of abstract Hardy spaces (see [3] and [6]). It is powerful to study the problem above.

§2. Concrete examples

(1) Let Δ be the unit disc of C and let A' be an algebra consisting of all functions with continuous extensions to the closure $\bar{\Delta}$ of Δ which are analytic in Δ. Put $A = A'|X$ and $X = \partial\Delta$. A is called the disc algebra which is a uniform algebra on X. Suppose $\tau(f) = \tilde{f}(0)$, where $\tilde{f}$ denotes the holomorphic extension of f in A, then τ is a nonzero complex homorphism. The normalized Lebesgue measure m on the unit circle is a representing measure for τ and $N_\tau = \{m\}$. H^2 is the classical Hardy space and $H_0^2 = K_0^2$, sometimes we will write $H^p = H^p(\Delta)$.

(2) In (1), put $A = A'$ and $X = \bar{\Delta}$. Suppose $\tau(f) = f(0)$ and m is the normalized area measure on Δ, then $m \in N_\tau$ and dim $N_\tau = \infty$. H^2 is the Bergman space.

(3) Let D be a bounded connected open subset of C whose boundary consists of $n+1$ non-intersecting, analytic Jordan curves and let A' be an algebra consisting of functions with continuous extensions to the closure $\bar{D}$ of D which are analytic in D. Put $A = A'|X$ and $X = \partial D$. A is uniform algebra on X and it is called an annulus algebra when D is an annulus. Suppose $\tau(f) = \tilde{f}(t)$, where $\tilde{f}$ denotes the holomorphic exension of f in A and $t \in D$, then τ is a nonzero complex homomorphism of A. If m is a harmonic measure of t then m is the unique logmodular measure of N_τ and dim $N_\tau = n < \infty$ [6, p116]. Sometimes we will write $H^p = H^p(D)$.

(4) Let $\mathcal{A}$ be the disc algebra and A be a subalgebra of $\mathcal{A}$ which contains the the constants and which has finite codimension in $\mathcal{A}$. If $\tau(f) = \tilde{f}(0)$ for $f \in A$ and m is the normalized Lebesgue measure on the unit circle $\partial\Delta$, then it is easy to check that m is a core point of N_τ, dim $N_\tau < \infty$ and N_τ has a lot of logmodular measures (see [7, p154]).

(5) The unit polydisc Δ^n and the torus $(\partial\Delta)^n$ are cartesian products of n copies of Δ and of $\partial\Delta$, respectively. A' denotes the class of all continuous functions on the closure $\bar{\Delta}^n$ of Δ^n with holomorphic restrictions to Δ^n. Let $A = A'|X$ and $X = (\partial\Delta)^n$. This is the so-called polydisc algebra. Let m be the normalized Lebesgue measure, then m is a representing measure for τ on X where $\tau(f) = \tilde{f}(0)$ and $0 \in \Delta^n$. Then dim $N_\tau = \infty$.

§3. The Inversion Theorem

Put $\mathcal{L} = \{v \in L^\infty ; v^{-1} \in L^\infty \text{ and } v \geq 0\}$ and $\mathcal{F} = \{vH^2 ; v \in \mathcal{L}\}$. Then $\mathcal{F}$ is a subfamily of lat A. In [9, Proposition 7], the following theorem was proved when the symbols are unimodular. In this section, we show it for arbitrary symbols using a lifting theorem in a uniform algebra due to the first author and Yamamoto [11, Theorem 2′]. In fact we use Theorem 2′ in case $\mu = (\mu_{ij})$ is absolutely continuous with respect to m.

Theorem 1. Let ϕ be a nonzero function in L^∞. For each M in $\mathcal{F}$ there exists a nonzero positive constant $\varepsilon(M)$ such that

$$\|T_\phi^M f\|_2 \geq \varepsilon(M)\|f\|_2, f \in M$$

and $\inf\{\varepsilon(M) ; M \in \mathcal{F}\} = \varepsilon > 0$ if and only if there exists a function g in $H^1 \cap L^\infty$ such that

$$|\phi|^2 \geq \varepsilon^2 + |\phi + g|^2 \quad \text{a.e..}$$

Proof. If $\|T_\phi^M f\|_2 \geq \varepsilon\|f\|_2$ for all $M \in \mathcal{F}$ then it is easy to see that $|\phi| \geq \varepsilon$ a.e. and for all $M \in \mathcal{F}$

$$(T_\phi^M)^*(T_\phi^M) \geq \varepsilon^2.$$

Put $H_\phi^M f = (I - P_M)(\phi f)$ for $f \in M$. Since $(T_\phi^M)^* T_\phi^M + (H_\phi^M)^* H_\phi^M = T_{|\phi|^2}^M$

$$\|H_\phi^M f\|_2 \leq \|(|\phi|^2 - \varepsilon^2)^{1/2} f\|_2, \quad f \in M$$

and hence

$$\sup\{|(H_\phi^M f, g)|^2 ; g \in M^\perp \text{ and } \int |g|^2 dm \leq 1\} \leq \int (|\phi|^2 - \varepsilon^2)|f|^2 dm$$

where (,) is an inner product with respect to m. Thus for $f \in M$ and $g \in M^\perp$

$$|\int \phi f \bar{g} dm|^2 \leq \int (|\phi|^2 - \varepsilon^2)|f|^2 dm \int |g|^2 dm. \qquad \text{(a)}$$

Since $M = vH^2$ and $M^\perp = v^{-1}\bar{K}_0^2$ for some $v \in \mathcal{L}$, for $F \in H^2$ and $G \in \bar{K}_0^2$

$$|\int \phi F \bar{G} dm|^2 \leq \int (|\phi|^2 - \varepsilon^2)|F|^2 v^2 dm \int |G|^2 v^{-2} dm.$$

Hence for $F \in H^\infty$ and $G \in \bar{K}_0^\infty$

$$-2Re \int \phi F \bar{G} dm \leq 2|\int \phi F \bar{G} dm|$$

$$\leq 2\{\int (|\phi|^2 - \varepsilon^2)|F|^2 v^2 dm\}^{1/2}\{\int |G|^2 v^{-2} dm\}^{1/2}$$

$$\leq \int (|\phi|^2 - \varepsilon^2)|F|^2 v^2 dm + \int |G|^2 v^{-2} dm. \qquad \text{(b)}$$

Let $\tilde{X}$ be the maximal ideal space of L^∞, then $\tilde{X}$ is a compact Hausdorff space and L^∞ is isometrically isomorphic to $C(\tilde{X})$ by the Gelfand fransform. Put B be the image of H^∞ by the transform and let $\tilde{m}$ be the Radonization of m. Then the measure $\tilde{m}$ on $\tilde{X}$ is multiplicative on B and $H^P(\tilde{m})$ or $L^p(\tilde{m})$ is isometrically isomorphic to $H^p(m)$ or $L^p(m)$, respectively, where $H^p(\tilde{m})$ is the abstract Hardy space determined by B. The inequality (b) implies that the measure matrices for all $v \in \mathcal{L}$

$$\begin{pmatrix} v^2(|\phi|^2 - \varepsilon^2) & \phi \\ \bar{\phi} & v^{-2} \end{pmatrix}$$

are positive on $H^\infty \times \bar{K}_0^\infty$ as $A = H^\infty$ and $K_0 = K_0^\infty$ in [11, p93]. By the absolutely continuous case of the lifting theorem [11, Theorem 2′] and the isomorphism above, there exists a nonzero function g in $(K_0^\infty)^\perp \cap L^1 = H^1$ such that the measure martix

$$\begin{pmatrix} |\phi|^2 - \varepsilon^2 & \phi + g \\ \bar{\phi} + \bar{g} & 1 \end{pmatrix}$$

is positive on $L^\infty \times L^\infty$. Therefore

$$|\phi + g|^2 + \varepsilon^2 \leq |\phi|^2 \qquad \text{a.e..}$$

Conversely suppose that for some $\varepsilon > 0$ there exists a function g in $H^1 \cap L^\infty$ such that $|\phi|^2 \geq \varepsilon^2 + |\phi + g|^2$ a.e.. Then for arbitrary $v \in \mathcal{L}$

$$\int \phi F \bar{G} dm = \int (\phi + g) F v \cdot \bar{G} v^{-1} dm$$

where $F \in H^\infty$ and $G \in \bar{K}_0^\infty$. Hence from the Schwarz's inequality, (b) and hence (a) follow. This implies $(T_\phi^M)^*(T_\phi^M) \geq \varepsilon^2 > 0$ for all $M \in \mathcal{F}$.

Corollary 1. Suppose $H^\infty = H^1 \cap L^\infty$. Let ϕ be a nonzero function in L^∞. T_ϕ^M is invertible for every M in $\mathcal{F}$ and sup $\{\|(T_\phi^M)^{-1}\| \ ; \ M \in \mathcal{F}\} < \infty$ if and only if both

(1) ess.inf $|\phi| > 0$ and

(2) there exists a function g in $(H^\infty)^{-1}$ such that $Re(\phi g) \geq \delta$ a.e. for some constant $\delta > 0$.

Proof. We may assume $\|\phi\|_\infty = 1$. Suppose $sup\{\|(T_\phi^M)^{-1}\| \ ; \ M \in \mathcal{F}\} < \infty$, then

$$\|T_\phi^M(f)\|_2 \geq \varepsilon \|f\|_2, \ f \in M$$

where $\varepsilon = \{sup\|(T_\phi^M)^{-1}\|\}^{-1}$. By Theorem 1, there exists a function g in $H^1 \cap L^\infty = H^\infty$ such that $|\bar{\phi}|^2 \geq \varepsilon^2 + |\bar{\phi} - g|^2$ a.e.. Hence $ess.inf|\phi| > 0$, and

$$Re\phi g \geq \delta > 0 \text{ a.e. and } \delta = \varepsilon^2/2.$$

Therefore there exist positive constants a and ε_0 such that

$$\|a\phi g - 1\|_\infty \leq 1 - \varepsilon_0.$$

This implies $T_{\phi g}$ is invertible. By hypothesis on T_ϕ, T_g is invertible. Hence $gH^2 = H^2$ and $g^{-1} \in L^\infty$ because $Re\phi g \geq \delta > 0$ a.e.. Thus $g^{-1} \in H^2 \cap L^\infty = H^\infty$.

In Corollary 1, if ϕ is unimodular then (1) and (2) hold if and only if to the distance from ϕ to the invertible elements in H^∞ is less than one. Results in this section apply to Toeplitz operators in Examples 1, 2 and 5.

§4. Some special cases

In this section, assuming that the set of representing measures for τ is finite dimensional, we give the inversion theorems. These appear to be more useful than Theorem 1 and Corollary 1. They apply to Toeplitz operators in Examples 3 and 4.

Suppose $n = dimN_\tau < \infty$. Let m be a core point of N_τ and let N^∞ be the real annihilator of A in L^∞_R. Then $dimN^\infty = n(cf.[6, p109])$. Set $\mathcal{E} = expN^\infty$, then $\mathcal{E}$ is a subgroup of $\mathcal{L}$. Put $\mathcal{F}_1 = \{vH^2 : v \in \mathcal{E}\}$ then $\mathcal{F}_1$ is a subfamily of $\mathcal{F}$. Then it can be shown that Theorem 1 and Corollary 1 are true for $\mathcal{F}_1$ instead of $\mathcal{F}$. These give inversion theorems in Example 4 which are related to [2, Theorems 2 and 3]. If $n = 0$ then $\mathcal{E} = \{1\}$ and hence $\mathcal{F}_1 = \{H^2\}$. Therefore if $n = 0$ then Corollary 1 shows the equivalence of (1) $\sim$ (4) in Introduction and gives the theorem of Widom and Devinatz (see [4]).

For any v in $\mathcal{L}$. let P_v be the projection operator which takes L^2 onto H^2 and is self-adjoint as an operator on the weighted Lebesgue space $L^2(v^2dm)$. Define the associated Toeplitz operator R^v_ϕ mapping H^2 to itself by $R^v_\phi(f) = P_v(\phi f)$. This definition is due to Anderson and Rochberg [2]. $(\ ,\ )_v$ and $(\ ,\ )$ denote the inner products in $L^2(v^2dm)$ and L^2, respectively. Put $M = vH^2$. For any f and g in H^2,

$$(R^v_\phi(f), g)_v = (P^v(\phi f), g)_v = (\phi f, g)_v = (\phi vf, vg) = (P^M(\phi vf), vg) = (T^M_\phi(vf), vg)$$

It is clear that R^v_ϕ is (left) invertible if and only if T^M_ϕ is (left) invertible. Hence Theorem 2 and the remark above Theorem 3 in [2] show the equivalence of (2) $\sim$ (4) in Introduction when A is a subalgebra of the disc algebra in Example 4.

If we assume that m is the unique logmodular measure for τ then the linear span of $N^\infty \cap log|(H^\infty)^{-1}|$ is N^∞ (cf. [6, p114]). Choose $h_1, ---, h_n \in (H^\infty)^{-1}$ so that $\{log|h_j|\}^n_{j=1}$ is a basis in N^∞. Put $u_j = log|h_j| (1 \leq j \leq n)$ and $\mathcal{E}_0 = \{exp(\Sigma_{j=1} s_j u_j) : 0 \leq s_j \leq 1\}$. Then $\mathcal{E}_0 \subset \mathcal{E}$. Put $\mathcal{F}_0 = \{vH^2 : v \in \mathcal{E}_0\}$ then $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F} \subseteq$ lat A. Note that the unique logmodular measure is a core point of N_τ when N_τ is finite dimensional.

Abrahamse [1, Part 4] studied T_ϕ^M for simply invariant subspaces M. $\mathcal{F}$ is a proper subset of the set of simply invariant subspaces.

Theorem 2. Suppose N_τ is finite dimensional and m is a core point of N_τ. Let ϕ be a nonzero function in L^∞.

(1) For each M in $\mathcal{F}_1$ there exists a nonzero positive constant $\varepsilon(M)$ such that

$$\|T_\phi^M f\|_2 \geq \varepsilon(M)\|f\|_2, \qquad f \in M$$

and $inf\{\varepsilon(M); M \in \mathcal{F}_1\} = \varepsilon > 0$ if and only if there exists a function g in H^∞ such that

$$|\phi|^2 \geq \varepsilon^2 + |\phi + g|^2 \qquad \text{a.e..}$$

(2) When m is the unique logmodular measure for τ, T_ϕ^M is left invertible for any M in $\mathcal{F}_0$ if and only if there exists a positive constant ε and a function g in H^∞ such that

$$|\phi|^2 \geq \varepsilon^2 + |\phi + g|^2 \qquad \text{a.e.}$$

Proof. It is known that $H^\infty = H^1 \cap L^\infty$ (cf. [6, p109]). (1) By Theorem 1 it is sufficient to show that if $inf\{\varepsilon(M)\ ; M \in \mathcal{F}_1\} > 0$ then $inf\{\varepsilon(M)\ ; M \in \mathcal{F}\} > 0$. In order to prove it we will show that

$$inf\{\varepsilon(M)\ ; M \in \mathcal{F}\} = inf\{\varepsilon(M)\ ; M \in \mathcal{F}_1\}. \qquad \text{(c)}$$

If $v \in \mathcal{L}$ then $logv = u + u_1$ where $u_1 \in N^\infty$ and u is in the weak$*-$*closure of* ReA (cf. [6, p109]). *Then* u $=$ log $|h|$ for h in $(H^\infty)^{-1}$ and so $v = |h|v_1$ where $v_1 = e^{u_1}$ and $v_1 \in \mathcal{E}$. Hence $M = vH^2 = b(v_1H^2) = bN, M \in \mathcal{F}$ and $N \in \mathcal{F}_1$, where $b = |h|/h$. Then $T_\phi^M = M_b T_\phi^N M_{\bar{b}}$ where M_b is a multiplication operator on L^2. This implies (c) because M_b is unitary.

(2) By (1) it is sufficient to show that if T_ϕ^M is left invertible for every M in $\mathcal{F}_0$, then for any M in $\mathcal{F}_1$

$$\|T_\phi^M f\|_2 \geq \varepsilon(M)\|f\|_2 \qquad f \in M$$

and $inf\{\varepsilon(M); M \in \mathcal{F}_1\} > 0$. We will show that

$$inf\{\varepsilon(M); M \in \mathcal{F}_1\} = inf\{\varepsilon(M)\ ; M \in \mathcal{F}_0\} \qquad \text{(d)}$$

and

$$inf\{\varepsilon(M); M \in \mathcal{F}_0\} > 0 \qquad \text{(e)}$$

If $v \in \mathcal{E}$ then $v = |h|v_0$ for some h in $(H^\infty)^{-1}$ and some $v_0 \in \mathcal{E}_0$ by the remark above Theorem 2. By the same argument as in the proof of (1), we can show (d). Suppose $v_\ell \in \mathcal{E}_0$ and $M_\ell = v_\ell H^2 \in \mathcal{F}_0$ with $\varepsilon(M_\ell) \to 0$. Since $v_\ell \in \mathcal{E}_0, v_\ell = exp(\Sigma_{j=1}^n s_{j\ell}u_j)$ and $0 \le s_{j\ell} \le 1 (1 \le j \le n)$. By passing to a subsequence, if necessary, we can assume that $s_{j\ell}$ converges to a constant s_j for each j, and $|s_j| \le 1 (1 \le j \le n)$. Put $t = exp(\Sigma_{j=1}^n s_j u_j)$ then $t \in \mathcal{E}_0$ and $\varepsilon(t) = 0$. This contradicts that $T_\phi^{M_t}$ is left invertible. Thus (e) follows.

Corollary 2. Suppose N_τ is finite dimensional and m is a core point of N_τ. Let ϕ be a nonzero function in L^∞.

(1) T_ϕ^M is invertible for every M in $\mathcal{F}_1$ and $sup\{\|(T_\phi^M)^{-1}\| \,; M \in \mathcal{F}_1\} < \infty$ if and only if $ess.inf|\phi| > 0$ and there exists a function g in $(H^\infty)^{-1}$ such that $Re(\phi g) \ge \delta$ a.e. for some constant $\delta > 0$.

(2) When m is the unique logmodular measure for τ, T_ϕ^M is invertible for every M in $\mathcal{F}_0$ if and only if both $ess.inf|\phi| > 0$ and there exists a function g in $(H^\infty)^{-1}$ such that $Re(\phi g) \ge \delta$ a.e. for some constant $\delta > 0$.

Proof. Since for any subset S of lat A

$$(inf\{\varepsilon(M) \,; M \in S\})^{-1} = sup\{\|(T_\phi^M)^{-1}\| \,; M \in S\},$$

both (1) and (2) are clear by Theorem 2.

If ϕ is unimodular, then (2) of Theorem 2 shows Theorem 4.1 in [1] and (2) of Corollary 2 shows Theorem 4.6. in [1]. Our results apply to more general uniform algebras than that of [1] by [7, p157]. (1) of Corollary 2 does not show Theorem 2 in [2] for general functions ϕ in L^∞. However by the remark above Theorem 2 in this paper we can get the invertibility theorems about a family $\{R_\phi^v, v \in \mathcal{E}_1\}$.

§5. Spectrums of selfadjoint Toeplitz operators

$\sigma(T_\phi^M)$ denotes the spectrum of T_ϕ^M for each M in lat A. Hartman and Wintner [8] showed $\sigma(T_\phi) = [ess.inf\ \phi, ess.\ sup\ \phi]$ for a real valued function ϕ in L^∞ where A is the disc algebra. This theorem is not valid in general. For example, it is not true in Examples 2 and 3. In the case of Example 2, if ϕ is a real valued continuous function on $\bar{\Delta}$ and $\phi = 0$ on $\partial\Delta$ then T_ϕ is compact (cf. [14, p107]). Hence $\sigma(T_\phi) \ne [ess.inf\ \phi, ess.\ sup\ \phi]$. In the case of Example 3, see [1, p295].

For each M in $\mathcal{F}$, let $\varepsilon(M) = \varepsilon(M, \phi)$ be the maximum of non-negative constants δ such that

$$\|T_\phi^M f\|_2 \ge \delta \|f\|_2 \text{ for all } f \in M.$$

Put $\Sigma_\phi = \{s \in C : inf[\varepsilon(M, \phi - s), M \in \mathcal{F}] = 0\}$, then

$$\Sigma_\phi \cup \bar{\Sigma}_{\bar{\phi}} \supseteq \bigcup_{M\in\mathcal{F}} \sigma(T_\phi^M).$$

The above two sets sometimes coincide.

Proposition 3. Let ϕ be a function in L^∞.

(1) If A is a uniform algebra in Example 4, then

$$\bigcup_{M\in\mathcal{F}_1} \sigma(T_\phi^M) = \Sigma_\phi \cup \bar{\Sigma}_{\bar{\phi}}.$$

(2) If N_τ is finite dimensional and m is the unique logmodular measure, then

$$\bigcup_{M\in\mathcal{F}_0} \sigma(T_\phi^M) = \Sigma_\phi \cup \bar{\Sigma}_{\bar{\phi}}.$$

Proof. We have to prove that $\Sigma_\phi \cup \bar{\Sigma}_{\bar{\phi}} \subset \cup\{\sigma(T_\phi^M) \ ; M \in S\}$ for $S = \mathcal{F}_1$ in (1) and $S = \mathcal{F}_0$ in (2).

(1) If $s \notin \cup \ \{\sigma(T_\phi^M); M \in \mathcal{F}_1\}$, then for any $M \in \mathcal{F}_1$ $T_{\phi-s}^M$ is invertible. By the proof of (1) of Theorem 2, for any $M \in \mathcal{F}$ there exists $N \in \mathcal{F}_1$ such that $T_{\phi-s}^M$ is unitarily equivalent to $T_{\phi-s}^M$. Hence for any $M \in \mathcal{F}$ $T_{\phi-s}^M$ is invertible. By the remark above Theorem 2, for any $v \in \mathcal{L}$ $R_{\phi-s}^v$ is invertible. By Theorem 2 and the remark above Theorem 3 in [2], $inf|\phi - s| > 0$ a.e., and there exists a function g in $(H^\infty)^{-1}$ such that $Re(\phi - s)g \geq \delta$ a.e. for some constant $\delta > 0$. By Corollary 1, $sup\{\|(T_{\phi-s}^M)^{-1}\| \ ; M \in \mathcal{F}\} < \infty$ and hence $s \notin \Sigma_\phi \cup \bar{\Sigma}_{\bar{\phi}}$.

(2) If $s \notin \cup \ \{\sigma(T_\phi^M \ ; M \in \mathcal{F}_0\}$ then Corollary 1 and (2) of Corollary 2 $s \notin \Sigma_\phi \cup \bar{\Sigma}_{\bar{\phi}}$.

If ϕ is a real valued function in L^∞ then by Theorem 1

$$[ess.inf \ \phi, ess.sup \ \phi] \supseteq \Sigma_\phi \supseteq \bigcup_{M\in\mathcal{F}} \sigma(T_\phi^M).$$

The following theorem shows the relations of the three sets above.

Theorem 4. Let ϕ be a real valued function in L^∞. Then the following are valid.

(1) $\Sigma_\phi = [ess.inf \ \phi, ess. \ sup \ \phi]$.

(2) If A is a uniform algebrs in example (4) then

$$\bigcup_{M\in\mathcal{F}_1} \sigma(T_\phi^M) = [ess.inf \ \phi, ess.sup \ \phi].$$

(3) If N_τ is finite dimensional and m is the unique logmodular measure for τ, then

$$\bigcup_{M \in \mathcal{F}_0} \sigma(T_\phi^M) = [ess.inf\ \phi, ess.sup\ \phi].$$

Proof. (1) Put $a = ess.inf\ \phi$ and $b = ess.sup\ \phi$. It is easy to see that $\Sigma_\phi = \Sigma_\phi \cup \bar{\Sigma}_{\bar{\phi}} \subset [a, b]$. In fact, if s is a non-real complex number, that is, $s = x + iy$ and $y \neq 0$, then

$$|\phi - s| \geq \varepsilon + |\phi - s + (s - x)| \qquad a.e.$$

where ε is a positive number with $\varepsilon^2 + 2\varepsilon\|\phi - x\| \leq y^2$. Hence $s \notin \Sigma_\phi$ by Theorem 1. If s is a real number with $s < a$ then $|\phi - s| = |a - s| + |\phi - s + (s - a)|$ a.e. and if $s > b$ then $|\phi - s| = |s - b| + |\phi - s + (s - b)|$ a.e.. Hence $s \notin \Sigma_\phi$ by Theorem 1.

$R(\phi) \subset \Sigma_\phi$ by Theorem 1 where $R(\phi)$ is the essential range of ϕ. We will prove that $[a, b] \subset \Sigma_\phi$. Suppose $a < s < b$ and $s \notin \Sigma$. Then we will get one contradition. Put $s_a = ess.sup\ min\ (\phi, s)$ and $s_b = ess.inf\ max\ (\phi, s)$, then $s_a < s < s_b$. Put

$$E_a = \{x \in X : a \leq \phi(x) \leq s_a\} \text{ and } E_b = \{x \in X : s_b \leq \phi(x) \leq b\}.$$

then $m(E_a \cup E_b) = 1$ and $m(E_a \cap E_b) = 0$. Since $s \in \Sigma_\phi$, by Theorem 1 there exist a nonzero function g in H^∞ and $\varepsilon > 0$ such that

$$|\phi - s| \geq \varepsilon + |\phi + g| \qquad a.e..$$

Put $S_a = \{t \in R : dist(g(E_a), t) \leq \varepsilon/2\}$ and $S_b = \{t \in R : dist(g(E_b), t) \leq \varepsilon/2\}$ where R is a real line and $g(E_\ell)$ is an essential range of g in E_ℓ with ℓ = a, b. Then S_a and S_b are compact subsets in R. If $S_a \cap S_b$ is essentially nonempty then there exist $x \in E_a$ and $x' \in E_b$ such that $|g(x) - g(x')| \leq \varepsilon$. Hence

$$\begin{aligned}
&\phi(x') - \phi(x) = |\phi(x) - \phi(x')| \\
&\leq |\phi(x) + g(x)| + |\phi(x') + g(x')| + |g(x) - g(x')| \\
&\leq |\phi(x) - s| - \varepsilon + |\phi(x') - s| - \varepsilon + \varepsilon \\
&= \phi(x') - \phi(x) - \varepsilon.
\end{aligned}$$

because $\phi(x) \leq s_a < s < s_b \leq \phi(x')$. This contradiction shows that $S_a \cap S_b$ is essentially empty. By a theorem of Runge we can show that $\chi_E \in H^\infty$. Thus $m(E_a) = 0$ or 1. This contradiction shows that [a, b] $\subset \Sigma_\phi$. (2) and (3) are results of (1) and (2) in Propsition 3 and (1) in this theorem.

Widom [13] proved that $\sigma(T_\phi)$ is connected for arbitrary symbol ϕ when A is the disc algebra. This theorem is not valid in Examples 2, 3 and 4. Here we assume that A is a uniform algebra in Example 3. Abrahamse [1, p295] announced without proof that if ϕ is unimodular then $\cup\{\sigma(T_\phi^M)\ ; M \in S\}$ is connected when A is an annulus algebra. S denotes the set of simply invariant subspaces.

The universal covering surface of D is conformally equivalent to the open unit disc Δ and an analytic projection map π from Δ onto D can be chosen so that $\pi(0) = t$. Abrahamse [1, p294], using his inversion theorem, when ϕ is unimodular and $\tilde{\phi} = \phi \circ \pi$, if T_ϕ^M is invertible for each $M \in S$ then $T_{\tilde{\phi}}$ is invertible and hence $\sigma(T_{\tilde{\phi}}) \subset \cup\{\sigma(T_\phi^M)\ ; M \in S\}$. The following proposition generalizes the Abrahamse's result to arbitrary symbols. Recall that $\mathcal{F}$ is a proper subset of S.

Proposition 5. Suppose A is a uniform algebra in Example 3. Let ϕ be a nonzero function in L^∞. If T_ϕ^M is invertible for each M in $\mathcal{F}_0$. then $T_{\tilde{\phi}}$ is invertible. Hence

$$\sigma(T_{\tilde{\phi}}) \subset \bigcup_{M \in \mathcal{F}_0} \sigma(T_\phi^M).$$

If $n = 1$ then $\sigma(T_{\tilde{\phi}}) = \cup\ \{\sigma(T_\phi^M)\ ; M \in \mathcal{F}_0\}$ and hence $\cup\ \sigma(T_\phi^M)$ is connected.

Proof. If T_ϕ^M is invertible for each $M \in \mathcal{F}_0$, then by Corollary 2 there is $g \in (H^\infty)^{-1}$ such that $Re(\phi g) \geq \delta$ a.e. for some constant $\delta > 0$. This implies that $Re(\tilde{\phi}\tilde{g}) \geq \delta$ a.e. and $\tilde{g} \in H^\infty(\Delta)^{-1}$. By Corollary 2 $T_{\tilde{\phi}}$ is invertible. If $n = 1$, we will show that the converse is valid. Suppose $T_{\tilde{\phi}}$ is invertible. By Theorem 2, there exists a positive constant ε and a function g in $H^\infty(\Delta)$ such that

$$|\tilde{\phi}|^2 \geq \varepsilon^2 + |\tilde{\phi} + g|^2 \quad \text{a.e..}$$

If we can find h in H^∞ such that

$$|\tilde{\phi}|^2 \geq \varepsilon^2 + |\tilde{\phi} + \tilde{h}|^2 \quad \text{a.e.}$$

then by Theorem 2 T_ϕ^M is left invertible for every $M \in \mathcal{F}_0$. The same proof of the complex conjugate $\bar{\phi}$ shows that T_ϕ^M is invertible for every $M \in \mathcal{F}_0$.

We will show the existence of such an h in H^∞. We can regard $\tilde{H}^\infty$ as the closed subalgebra of $H^\infty(\Delta)$ consisting of those $f \in H^\infty(\Delta)$ which are invariant under a certain group of conformal maps of Δ onto itself. This group, which we shall denote by G, is in an infinite cyclic group. Put $S = \{g \in H^\infty(\Delta) : |\tilde{\phi}|^2 \geq \varepsilon^2 + |\tilde{\phi} + g|^2 \quad a.e.\}$ then S is a convex subset of $H^\infty(\Delta)$. We define a group operators $\{\Phi_\gamma\}_{\gamma \in G}$ on $H^\infty(\Delta)$ by means of $\Phi_\gamma h = h \circ \gamma$ for all $\gamma \in G$. Let σ be the topology of almost uniform convergence in $H^\infty(\Delta)$. Then σ is metrizable, and a normal families argument shows that S is σ-compact. Since they commute, the fixed point theorem of Markov and Kakutani ([5, p456]), [12, Theorem 2.1]) affirms the existence of k in S which is invariant under the group$\{\Phi_\gamma\}_{\gamma \in G}$. Hence $k = \tilde{h}$ for some $h \in H^\infty$ and

$$|\tilde{\phi}|^2 \geq \varepsilon^2 + |\tilde{\phi} + \tilde{h}|^2 \qquad a.e..$$

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