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A POHOZAEV-TYPE INEQUALITY FOR A QUASILINEAR HARAUX-WEISSLER EQUATION

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1. Introduction.

Since their first application in the work of Pohozaev [P], Pohozaev-Identities have become a common tool to determine, whether solutions of certain classes of equation can possess zeroes or not. In this paper we want to present an application to a quasilinear porous medium equation with a source term

$$(PME) \qquad u_t - \Delta(|u|^{m-1}u) = |u|^{p-1}u \qquad \text{in } R^N \times (0,T),$$

where the parameters are taken as $m > 0$ and $p > 1$. In studying this equation, the so-called selfsimilar solutions

$$u(x,t) = t^{-\alpha}U(r), \qquad r = |x|t^{-\beta},$$

are of interest, as they usually describe the large time behaviour of solutions to the Cauchy Problem with general initial data (see for instance [KV]). Inserting this ansatz, we arrive at the following initial value problem, which was first considered by Haraux & Weissler [HW] for the semilinear case $m = 1$:

$$(P) \qquad \begin{aligned} &V'' + \frac{N-1}{r}V' + \beta rU' + \alpha U + |U|^{p-1}U = 0 \qquad \text{in } R^+, \\ &V = |U|^{m-1}U, \qquad \alpha, \beta > 0, \qquad \alpha(m-1) + 2\beta = 1, \\ &V'(0) = 0, \qquad U(0) = a > 0. \end{aligned}$$

In fact, if the equation originates from (PME), the parameters α, β have to fulfill the addititional condition $\alpha p = p + 1$ and are thus given by

$$\alpha = \frac{1}{p-1}, \qquad \beta = \frac{p-m}{2(p-1)}.$$

Here, as also done in [HW] and [AP], we want to consider them as parameters only subject to the condition stated in (P) .

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To our knowledge the structure of solutions of equation (P) above was studied only in the semilinear case $m = 1$: In the works of Haraux & Weissler [HW] and Atkinson & Peletier [AP], among other results it was proved, that if p is supercritical, i.e.

$$N > 2 \qquad \text{and} \qquad p \geq \frac{N+2}{N-2},$$

then solutions of (P) with $\alpha \leq \max\{2, N/2\}$ can not possess zeroes. In sharp contrast to this result solutions of (P) with subcritical p may have any number of zeroes, provided the initial value is large enough (see [W]).

Another aspect of problem (P) is that it can be regarded as the equation satisfied by radial solutions of a quasilinear elliptic equation with a gradient term

$$\Delta(|u|^{m-1}u) + \beta x \cdot \nabla u + \alpha u + |u|^{p-1}u = 0 \qquad \text{in } \boldsymbol{R}^N.$$

This equation was studied by Hirose [H] in the particular case $m = 1$ and $\alpha = 0$ (thus $\beta = 1/2$). Among other results he shows the nonexistence of zeroes for solutions of (P) in this setting for all supercritical p.

A similar equation was studied by Chipot & Weissler [CW] and Serrin & Zou [SZ]. They study the influence of another gradient term on the solutions of

$$\Delta u + |u|^{p-1}u - |\nabla u|^q = 0 \qquad \text{in } \boldsymbol{R}^N.$$

They found that again for supercritical p the solutions can not possess zeroes, provided q is small enough. The same equation without gradient term

$$\Delta u + |u|^{p-1}u = 0 \qquad \text{in } \boldsymbol{R}^N,$$

was also studied by various authors. Again the behaviour of solutions strongly depends on the relation of p to its critical value. For an overview on results on the latter equation we would like to refer to [KNY] and the literature cited therein.

A very general approach to Pohozaev-identities for quasilinear elliptic equations was made in Kawano, Ni & Yotsutani [KNY]. However, their equation –

$$div(A(|\nabla u|)|\nabla u|) + f(|x|, u) = 0 \qquad \text{in } \boldsymbol{R}^N$$

– does not include gradient terms of a form we consider here. Moreover their results are applicable to the generalized Laplace equation

$$A(v) = v^{m-2}$$

only if $1 < m < N$, thus not allowing arbitrary degeneracies in the leading term.

In order to state our results, let us introduce

$$k := \frac{\alpha}{\beta}.$$

Due to the condition on these two parameters given in (P) , there is a bijection between k and (α, β), so that the parameter values are easily recovered from k. Please observe that in the singular case $m < 1$ the value of k can only range from 0 to $\frac{2}{1-m}$, whereas it can be any positive number in case that $m \geq 1$.

1.1 Theorem. Let $N > 2$, $m \geq 0$ and

$$\frac{p}{m} \geq \frac{N+2}{N-2}.$$

Then the solution U of (P) has no finite zero, regardless of the initial value $U(0) > 0$, provided

$$k < \frac{N}{m+1}.$$

1.2 Theorem. Let $N > 2$, $m \in (0, \frac{N-2}{N})$ and define $p_0(m, N)$ by

$$p_0(m,N) := \max\left\{m\frac{(N+2)m-(N-4)}{N(1-m)-2}, 1\right\}.$$

Then, if $p \geq p_0(m, N)$, the solution U of (P) has no finite zeroes, regardless of the value of $k \in [0, \frac{2}{1-m})$ and the initial value $U(0) > 0$.

1.3 Corollary. If $N > 2$ and $\frac{p}{m} \geq \frac{N+2}{N-2}$, then a selfsimilar solution of (PME) satisfying (P) with some $a \in \mathbb{R}^+$ has to be positive.

The corollary is easily proved by observing that such a solution must satisfy (P) and that

$$k = \frac{2}{p-m} \leq \frac{N}{m+p} < \frac{N}{m+1},$$

due to the conditions imposed on p, m and N. Thus theorem 1.1 can be applied, which proves the corollary.

Before turning to the proof of the other results, we would like to give some comments.

1.4 Remarks. (i) Theorem 1.2 can not be extended to some $m > \frac{N-2}{N}$. In this range of m the space dimension N is a possible value for k, and in [D] it is proved that every solution of (P) with $k > N$ possesses a zero, regardless of p. (Note that $p_0(m, N) \to \infty$ if $m \to \frac{N-2}{N}$.)

(ii) Theorem 1.1 can not be extended to $\frac{p}{m} < \frac{N+2}{N-2}$. In this case it is shown in [D], that for sufficiently large initial value the solution U of (P) can possess an arbitrary large number of zeroes.

(iii) Concerning the bound on k in theorem 1.1, we improve the result of [AP] for the semilinear case $m = 1$ and $N \geq 4$. For general $m > 0$ this result may not be sharpened, as $\frac{N}{m+1} \to N$ if $m \to 0$, and – as mentioned already – any solution of (P) with $k > N$ possesses a zero. However, k ranges only up to $\frac{2}{1-m}$, it is not clear whether this result is optimalor not.

2. Proof of the main results.

In this chapter we will derive a Pohozaev-type inequality involving two additional parameters. By carefully choosing these parameters, we are able to prove theorems 1.1 and 1.2.

2.1 Lemma. Let $m > 0$, $q > 2$, $r_0 > 0$ and let V be a solution of (P) . Moreover let V be positive in $(0, r_0)$ and $V(r_0) = 0$. Then

$$\int_0^{r_0} r^q |U'| V \leq \frac{m+1}{mq} \int_0^{r_0} r^{q+1} U' V'.$$

Proof:
We just apply the Cauchy-Schwartz Inequality and integration by parts:

$$\left(\int_0^{r_0} r^q|U'|V\right)^2 \le \frac{1}{m^2}\left(\int_0^{r_0} r^{q-1}|U|^{m+1}\right)\left(\int_0^{r_0} r^{q+1}|U|^{1-m}|V'|^2\right)$$
$$\le \frac{m+1}{mq}\left(\int_0^{r_0} r^q|U'|V\right)\left(\int_0^{r_0} r^{q+1}\frac{1}{m}|U|^{1-m}V'V'\right),$$

which is equivalent to (i), since $U' = \frac{1}{m}|U|^{1-m}V'$.

q.e.d.

After this preparation we can now turn to the Pohozaev-type inequality.

2.2 Proposition. Let $m > 0$, $p > 1$, $r_0 > 0$ and let V be a solution of (P) . Moreover let V be positive in $(0, r_0)$ and $V(r_0) = 0$. Then

$$\frac{1}{2}r_0^q\,|V'(r_0)|^2 \le \left\{q\left(\frac{1}{2}+\frac{m}{m+p}\right)+1-N\right\}\int_0^{r_0} r^{q-1}|V'|^2$$
$$\frac{p-1}{m+p}\left\{\alpha\frac{m+1}{q}-\beta\right\}\int_0^{r_0} r^{q+1}U'V', \qquad \text{if } q \ge N.$$

Proof:
Extending an idea introduced in [PS] we define

$$G(r) := \frac{1}{2}|V'(r)|^2 + \frac{\alpha m}{m+1}|V(r)|^{1+\frac{1}{m}} + \frac{m}{m+p}|V(r)|^{1+\frac{p}{m}} + \delta\frac{V(r)V'(r)}{r}.$$

An elementary computation yields

$$\{r^qG(r)\}' = \left(\frac{q}{2}-N+1+\delta\right)r^{q-1}|V'|^2 + (q-N)\delta r^{q-2}VV'$$
$$-\beta r^{q+1}U'V' - \delta\beta r^q VU'$$
$$+\alpha\left(\frac{mq}{m+1}-\delta\right)r^{q-1}|U|^{m+1} + \left(\frac{mq}{m+p}-\delta\right)r^{q-1}|U|^{p+m}.$$

Integrating from 0 to r_0, we see that in order to estimate the right hand side from above without integrals containing $|U|^p$, we have to choose δ greater or equal $\frac{mq}{m+p}$. Thus we set

$$\delta = \frac{mq}{m+p}$$

and therefore arrive at

$$\frac{1}{2}r_0^q\,|V'(r_0)|^2 = \left\{q\left(\frac{1}{2}+\frac{m}{m+p}\right)+1-N\right\}\int_0^{r_0} r^{q-1}|V'|^2 + \frac{mq}{m+p}(N-q)\int_0^{r_0} r^{q-2}|V'|V$$
$$-\beta\int_0^{r_0} r^{q+1}U'V' + \beta\frac{mq}{m+p}\int_0^{r_0} r^qV|U'| + \alpha\frac{mq}{m+p}\frac{p-1}{m+1}\int_0^{r_0} r^{q-1}|U|^{m+1},$$

using that $V' \le 0$ on $(0, r_0)$. We now integrate by parts in the last integral and use Lemma 1.1 to estimate the last two integrals in terms of $\int r^{q+1}U'V'$; this gives

$$\frac{1}{2}r_0^q\,|V'(r_0)|^2 \le \left\{q\left(\frac{1}{2}+\frac{m}{m+p}\right)+1-N\right\}\int_0^{r_0} r^{q-1}|V'|^2 + \frac{mq}{m+p}(N-q)\int_0^{r_0} r^{q-2}|V'|V$$
$$+\frac{p-1}{m+p}\left\{\alpha\frac{m+1}{q}-\beta\right\}\int_0^{r_0} r^{q+1}U'V'.$$

If we choose $q \geq N$, then we can just omit the second integral on the right hand side and the proposition is proved.

q.e.d.

We will now apply the Pohozaev inequality to determine parameter ranges, where solutions of (P) can not possess a finite zero and thereby prove our results.

2.3 Proof of theorem 1.1 and 1.2. First let us clarify the effect of q on the coefficients on the right hand side of the Pohozaev-inequality. As can be easily seen, they are nonpositive, if

$$k \leq \frac{q}{m+1}$$

and

$$\frac{p}{m} \geq 1 + \frac{4(q-(N-1))}{2(N-1)-q}, \qquad \text{if } N \leq q < 2(N-1).$$

So if these conditions are fulfilled by p, m, N and k, and at least one inequality is strict, the assumption that U has a finite zero r_0 immediately leads to a contradiction, since Proposition 2.2 then implies

$$\frac{1}{2} r_0^q \, |V'(r_0)|^2 < 0.$$

It remains to show, that this is the case for the parameter ranges described in theorem 1.1 and 1.2.

If we choose $q = N$ in Proposition 2.2, we immediately get theorem 1.1. Concerning theorem 1.2, we observe that in order to cover all possible values of k, we should set

$$q = \frac{2}{1-m}(m+1).$$

If $m \in \left[\frac{N-2}{N+2}, \frac{N-2}{N}\right)$, this q is larger or equal N and we apply Proposition 2.2 again. Elementary computations then yield the value of $p_0(m,N)$ in this range of m. If $m \in (0, \frac{N-2}{N+2})$, it suffices to take $q = N$, as in this range of m we have

$$k < \frac{2}{1-m} < \frac{N}{m+1};$$

so this case is already covered in theorem 1.1 and we are done.

2.4 Remark. One could ask the question whether the results can be sharpened by taking q less than N. In this case we have to estimate the integral

$$\frac{mq}{m+p}(N-q)\int_0^{r_0} r^{q-2}|V'|V$$

at the end of the proof of proposition 2.2, since now the coefficient is positive. This can be done analogously to lemma 2.1 by using the Cauchy-Schwartz inequality and integration by parts, which gives

$$\int_0^{r_0} r^{q-2}|V'|V \leq \frac{2}{q-2}\int_0^{r_0} r^{q-1}|V'|^2.$$

Proceeding as above we then obtain that the coefficients in the integrals on the right hand side of the Pohozaev-inequality are nonpositive, if

$$k \leq \frac{q}{m+1} \quad \text{and} \quad \frac{p}{m} \geq \frac{q+2}{q-2}, \quad \text{if } 2 < q \leq N.$$

Thus this range of possible q does not give sharper results.

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