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Y. Giga and N. Mizoguchi

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On time periodic solutions of the Dirichlet problem for degenerate parabolic equations of nondivergence type

YOSHIKAZU GIGA

Department of Mathematics, Hokkaido University,

Sapporo 060, Japan

AND

NORIKO MIZOGUCHI

Department of Mathematics, Tokyo Gakugei University,

Nukuikita-machi, Koganei-shi, Tokyo 184, Japan

Abstract. In this paper, we are concerned with the existence of periodic solutions of a quasilinear parabolic equation

$$u_t = u^\gamma(\Delta u + u + f) \quad \text{in } \Omega \times \mathbf{R}$$

with the Dirichlet boundary condition, where Ω is a smoothly bounded domain in $\mathbf{R}^N$ and f is a given function periodic in time defined on $\bar{\Omega} \times \mathbf{R}$. Our results depend on the first eigenvalue λ_1 of $-\Delta$ in Ω with the Dirichlet boundary condition. If $\lambda_1 > 1$, then there exists a unique positive periodic solution for a positive f ($\gamma \in \mathbf{R}$). In the case of $\lambda_1 < 1$, we construct a nonnegative periodic solution for a negative f ($1 \leq \gamma < 3$).

1 Introduction.

We are concerned with the following quasilinear parabolic equation

$$\begin{cases} u_t = u^\gamma(\Delta u + u + f) & \text{in } \Omega \times \mathbf{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbf{R}, \\ u(t+T) = u(t) & \text{in } \Omega \times \mathbf{R}, \end{cases} \tag{1}$$

where Ω is a smoothly bounded domain in $\mathbf{R}^N$ and f is a given function on $\bar{\Omega} \times \mathbf{R}$ which is T-periodic in time.

Degenerate parabolic equations of the form (1) are often studied in the literature. When $N = 1$, the equation (1) with $\gamma = 2, f = 0$ describes the evolution of curvature of convex curves moved by curvature under the Gauss map parametrization. Since the space variable x is actually the argument of normals of curves, 2π -periodic boundary condition (instead of the Dirichlet condition) is imposed to describe closed curves. If initial data is positive, solution blows up in a finite time and its profile is studied in Gage [6] and Gage and Hamilton [8]. Geometrically speaking, they proved that a convex embedded curve moved by curvature shrinks to a "round point" in a finite time. Blow up profiles of convex immersed curves were classfied by Angenent [2] based on results of Abresch and Langer [1] and Epstein and Weinstein [4], where $2\pi m$ -periodic boundary condition is imposed ($m = 1, 2, \cdots$). The Dirichlet condition arises when the curve is complete but non-closed and its curvature equals to zero at the end. The initial value problem of general dimension N for (1) with $\gamma = 2$ and $f = 0$ was investigated in Friedman and Mcleod [5] and Gage [7] for positive initial data. These results contain the one dimensional case. For an open set $D \subset \mathbf{R}^N$, let $\lambda_1(D)$ denote the first eigenvalue of $-\Delta$ in D with the Dirichlet boundary condition. In particular, we write $\lambda_1 = \lambda_1(\Omega)$. It was proved in [5] that

i) If $\lambda_1 > 1$, then there exists a unique global solution which decays to zero as $t \to \infty$

ii) If $\lambda_1 < 1$, then there exists a unique solution which blows up in a finite time.

The estimate of the blow up set in [5] was improved in [7]. Recently, results of type i) and ii) are extended by Wiegner [12] for general γ. There are nonuniqueness results for $\gamma = 1$ and $f = 0$ by Ughi [11] (for one dimensional case) and Dal Passo and Luckhaus [3]. They showed that weak solutions may not be unique if initial data takes zero in some open set of the domain because of degeneracy of the equation and discussed a class of solutions so that uniqueness holds.

It seems that there is only one result for the periodic problem for (1) in [9], where the one-dimensional case of $2\pi-$ periodic boundary condition is considered. Since they treat motions of closed curves, their solution satisfies the certain constraint. The constraint yields a positive lower bound for the class of all positive solutions, so the equation in [9] is actually nondegenerate. However the equation (1) in this paper is degenerate. When the Dirichlet boundary condition is imposed, the relation between λ_1 and 1 also gives a serious effect on results. The purpose of this paper is to prove the following two theorems.

Theorem 1. Assume that $\gamma \in \mathbf{R}$ and that f is a continuous positive function on $\bar{\Omega} \times \mathbf{R}$ which is T-periodic in time. If $\lambda_1 > 1$, then there exists a unique positive solution u of (1) in $\bigcap_{p>1} W^{2,1}_{p,loc}(\Omega \times \mathbf{R}) \cap C(\bar{\Omega} \times \mathbf{R})$. Furthermore, if f is Hölder continuous, then the solution u belongs to $C^{2+\alpha,1+\alpha}(\Omega \times \mathbf{R}) \cap C(\bar{\Omega} \times \mathbf{R})$ with some $0 < \alpha < 1$.

Theorem 2. Assume that $1 \leq \gamma < 3$ and that f is a continuous positive function on $\bar{\Omega} \times \mathbf{R}$ with $f_t \in C(\bar{\Omega} \times \mathbf{R})$ which is T-periodic in time. Then there is a nonnegative solution $u \in C(\bar{\Omega} \times \mathbf{R})$ of (1) such that $u \in \bigcap_{p>1} W^{2,1}_{p,loc}(\Omega_+ \times \mathbf{R})$, $u > 0$ in $\Omega_+ \times \mathbf{R}$, $u \equiv 0$ on $(\Omega \backslash \Omega_+) \times \mathbf{R}$ and $\nabla u = 0$ on $(\partial\Omega_+ \backslash \partial\Omega) \times \mathbf{R}$, where $\Omega_+ = \bigcup_{i=1}^{k} \Omega_i$ with a class $\{\Omega_i : 1 \leq i \leq k\}$ of connected open subsets in Ω with $\lambda_1(\Omega_i) < 1$ for $1 \leq i \leq k$.

We prove Theorem 1 in Section 2 and Theorem 2 in Section 3. We seek for a solution of (1) as a limit of its approximate solutions. The Leray-Schauder degree theory is adapted to solve the approximate equations. We need an a priori upper bound for positive approximate solutions. To do that, inequalities of Harnack type play an important part. We refer to the reference in [9] for literatures on applications of the Leray-Schauder degree theory and Harnack's inequality.

2 Proof of Theorem 1.

We first assume that f is smooth and solve an approximate equation which is nondegenerate. Take positive constants a_1, a_2 such that $a_1 \leq f \leq a_2$ on $\bar{\Omega} \times \mathbf{R}$. Let $\varphi_i \in C^\infty(\bar{\Omega})$ be the solution of

$$\begin{cases} \Delta\varphi_i + \varphi_i + a_i &= 0 \quad \text{in } \Omega \\ \varphi_i &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2}$$

Since $\lambda_1 > 1$, there is no point such that $\varphi_i(x) < -a_i$; otherwise there is a domain $D \subset \Omega$ such that $\lambda_1(\Omega) \leq \lambda_1(D) \leq 1$ which leads a contradiction. The maximum principle implies that φ_i cannot take a minimum of the value $\geq -a_i$ in Ω so $\varphi_i > 0$ in Ω. For $\varepsilon > 0$ and $\tau \in [0,1]$, we consider

$$\begin{cases} u_t = a(u)(\Delta u + u_+ + \tau f) & \text{in } \Omega \times \mathbf{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbf{R}, \\ u(t+T) = u(t) & \text{in } \Omega \times \mathbf{R}, \end{cases} \tag{3}$$

where $a : \mathbf{R} \to \mathbf{R}$ is a smooth function such that

$$\frac{\varepsilon^\gamma}{2} \leq a(r) \leq (2m+\varepsilon)^\gamma \quad \text{for } r \in \mathbf{R},$$

$$a(r) = (r+\varepsilon)^\gamma \quad \text{for } 0 \leq r \leq m$$

with $m = \max \varphi_2$. Then each solution of (3) is smooth and positive by standard regularity theory and maximum principle (see [10]). Letting u be a solution of (3) and $v = \varphi_2 - u$, we have

$$v_t = a(u)(\Delta v + v + a_2 - \tau f) \quad \text{in } \Omega \times \mathbf{R}.$$

Since $a_2 - \tau f \geq 0$, we observe that $v \geq 0$ in $\Omega \times \mathbf{R}$. Indeed, solution of the linear equation

$$w_t = a(u)(\Delta w + w + g) \tag{4}$$

is unique for given g. To see this we may assume $g = 0$. Multiplying $w_t/a(u)$ and integrating over Q by parts yields $w_t = 0$. Since $\lambda_1 > 1$ and $\Delta w + w = 0$, we see $w \equiv 0$. By the maximum principle solution v of

$$v_t = a(u)(\Delta v + v_+ + g)$$

is positive provided that $g \geq 0$. The existence of solution is not difficult to show so this implies $v \geq 0$ for

$$v_t = a(u)(\Delta v + v + a_2 - \tau f) \quad \text{in } \Omega \times \mathbf{R}.$$

In other words, $u \leq \varphi_2$ in $\Omega \times \mathbf{R}$. Therefore it follows from [10] that there is $C = C(\varepsilon, \|f\|_\infty, p) > 0$ such that $\|u\|_{W_p^{2,1}(Q)} \leq C$ for each solution u of (3) with $p > N + 1$, where $Q = \Omega \times (0, T)$. Choose $\alpha > 0$ such that $W_p^{2,1}(Q)$ is compactly embedded into $C^{\alpha,\alpha/2}(\bar{Q})$. Put $X = C^{\alpha,\alpha/2}(\bar{Q})$ and define $S(v, \tau)$ by the unique solution of

$$\begin{cases} u_t = a(v)(\Delta u + u_+ + \tau f) & \text{in } \Omega \times \mathbf{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbf{R}, \\ u(t+T) = u(t) & \text{in } \Omega \times \mathbf{R}. \end{cases} \tag{5}$$

It is easily seen that S is a compact continuous operator from $X \times [0, 1]$ into X and $S(v, 0) = 0$ for all $v \in X$. According to the Leray-Schauder fixed point thoerem, there exists $u \in X$ with $S(u, 1) = u$. By $u \leq \varphi_2$, u is a smooth positive solution of

$$\begin{cases} u_t = (u + \varepsilon)^\gamma (\Delta u + u + f) & \text{in } \Omega \times \mathbf{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbf{R}, \\ u(t+T) = u(t) & \text{in } \Omega \times \mathbf{R}. \end{cases} \tag{6}$$

It is immediate that $u \geq \varphi_1$ in $\bar{\Omega} \times \mathbf{R}$ in the same way as the proof of $u \leq \varphi_2$.

For continuous function f not necessarily smooth, take $\tilde{a}_1, \tilde{a}_2 > 0$ such that $\tilde{a}_1 < f < \tilde{a}_2$ in $\bar{\Omega} \times \mathbf{R}$ and a positive smooth solution $\tilde{\varphi}_i$ $(i = 1, 2)$ of (2) with $a_i = \tilde{a}_i$. Choose a sequence $\{f_\varepsilon\} \subset C^\infty(\bar{\Omega} \times \mathbf{R})$ of T-periodic-in-time functions such that

$$f_\varepsilon \to f \quad \text{in } C(\bar{\Omega} \times \mathbf{R}) \quad \text{as } \varepsilon \to 0,$$

$$\tilde{a}_1 \le f_\varepsilon \le \tilde{a}_2 \quad \text{in } \bar{\Omega} \times \mathbf{R} \quad \text{for each } \varepsilon > 0.$$

By the first paragraph of this proof, for each $\varepsilon > 0$, there exists a positive smooth solution u_ε of (6) with $f = f_\varepsilon$ such that

$$\tilde{\varphi}_1 \le u_\varepsilon \le \tilde{\varphi}_2 \quad \text{in } \bar{\Omega} \times \mathbf{R}.$$

Then we may assume that $\{u_\varepsilon\}$ converges weakly to some u in $\bigcap_{p>1} W^{2,1}_{p,loc}(\Omega \times \mathbf{R})$ as $\varepsilon \to 0$. It is clear that u satisfies

$$u_t = u^\gamma(\Delta u + u + f) \quad \text{in } \Omega \times \mathbf{R},$$

$$u(t + T) = u(t) \quad \text{in } \Omega \times \mathbf{R}.$$

Furthermore, we see that u is positive in $\Omega \times \mathbf{R}$, zero on $\partial\Omega \times \mathbf{R}$ and continuous in $\bar{\Omega} \times \mathbf{R}$ since $\tilde{\varphi}_1 \le u \le \tilde{\varphi}_2$.

It remains to show the uniqueness of positive solution of (1) in $\bigcap_{p>1} W^{2,1}_{p,loc}(\Omega \times \mathbf{R}) \cap C(\bar{\Omega}$
Put

$$\Omega_\varepsilon = \{x \in \Omega : \ \text{dist}\ (x, \partial\Omega) > \varepsilon\}$$

for $\varepsilon > 0$. Let $\varphi_\varepsilon > 0$ be an eigenfunction corresponding to $\lambda(\Omega_\varepsilon)$. For $\delta > 0$, denote by sgn_δ the piecewise linear function on $\mathbf{R}$ with $\text{sgn}_\delta(s) = -1$ for $s \le -\delta$ and $\text{sgn}_\delta(s) = 1$ for $s \ge \delta$ and define $\varphi_\delta : \mathbf{R} \to \mathbf{R}$ by

$$\varphi_\delta(s) = \int_0^s \text{sgn}_\delta(r)dr \quad \text{for } s \in \mathbf{R}.$$

Suppose that u, v are positive solutions of (1) in $\bigcap_{p>1} W^{2,1}_{p,loc}(\Omega \times \mathbf{R}) \cap C(\bar{\Omega} \times \mathbf{R})$. Then it holds that

$$(F(u))_t = \Delta u + u + f \tag{7}$$

and

$$(F(v))_t = \Delta v + v + f, \tag{8}$$

where $F(u)$ is a primitive of $u^{-\gamma}$ such that

$$F(u) = \begin{cases} \dfrac{u^{1-\gamma}}{1-\gamma} & \text{for } \gamma \neq 1 \\ \log u & \text{for } \gamma = 1. \end{cases}$$

Subtracting (8) from (7), multiplying $\mathrm{sgn}_\delta(u-v)\cdot\phi_\varepsilon$ and integrating over Q yields

$$\int_0^T \int_\Omega \{(F(u))_t - (F(v))_t\}\ \mathrm{sgn}_\delta(u-v)\cdot\phi_\varepsilon dx dt = \int_0^T \int_\Omega \{\Delta(u-v)+(u-v)\}\ \mathrm{sgn}_\delta(u-v)\cdot\phi_\varepsilon dx \tag{9}$$

where we define $\phi_\varepsilon = 0$ outside Ω_ε. Now integrating by parts, it follows that

$$\begin{aligned} \int_\Omega \Delta(u-v)\cdot\ \mathrm{sgn}_\delta(u-v)\cdot\phi_\varepsilon dx &\leq -\int_\Omega \mathrm{sgn}_\delta(u-v)\nabla(u-v)\nabla\phi_\varepsilon dx \\ &= \int_\Omega \varphi_\delta(u-v)\Delta\phi_\varepsilon dx \\ &= -\lambda_1(\Omega_\varepsilon)\int_\Omega \varphi_\delta(u-v)\phi_\varepsilon dx. \end{aligned}$$

Since $\varphi_\delta(u-v) \to |u-v|$ in Ω as $\delta \to 0$,

$$\int_\Omega \varphi_\delta(u-v)\phi_\varepsilon dx \to \int_\Omega |u-v|\phi_\varepsilon dx \quad \text{as } \delta \to 0.$$

On the other hand, since u and v are T-periodic, we have

$$\int_0^T \int_\Omega \{(F(u))_t - (F(v))_t\}\ \mathrm{sgn}_\delta(u-v)\phi_\varepsilon dx dt \to \int_0^T \int_\Omega |F(u)-F(v)|_t \phi_\varepsilon dt dx = 0$$

as $\delta \to 0$. Therefore letting $\delta \to 0$ in (9), we get

$$0 \leq -\lambda_1(\Omega_\varepsilon)\int_0^T \int_\Omega |u-v|\phi_\varepsilon dx dt + \int_0^T \int_\Omega |u-v|\phi_\varepsilon dx dt.$$

From $\lambda_1(\Omega_\varepsilon) \geq \lambda_1 > 1$, it follows that $u = v$ in $\Omega_\varepsilon \times \mathbf{R}$. This implies $u = v$ in $\Omega \times \mathbf{R}$ since $\varepsilon > 0$ is arbitrary. □

Remark 1. If f is nonpositive, then the problem (1) admits no positive solutions in $\bigcap_{p>1} W^{2,1}_{p,loc}(\Omega \times \mathbf{R}) \cap C(\bar{\Omega} \times \mathbf{R})$. In fact, for any $\varepsilon > 0$ there is $\rho_\varepsilon > 0$ such that $u \leq \varepsilon$ on $\partial\Omega_\rho \times \mathbf{R}$ for each $0 < \rho \leq \rho_\varepsilon$, where $\Omega_\rho = \{x \in \Omega : \text{ dist } (x, \partial\Omega) > \rho\}$. Let ϕ_1, $\phi_\rho > 0$ be the eigenfunctions corresponding to λ_1 and $\lambda_1(\Omega_\rho)$ normalized in $L^1(\Omega)$, respectively. Then $\{\phi_\rho\}$ converges to ϕ_1 in $L^1(\Omega)$ as $\rho \to 0$. Multiplying $\frac{u_t}{u^\gamma}$ with (1) and integrating over $\Omega_\rho \times \mathbf{R}$ yields

$$\int_0^T \int_{\Omega_\rho} \frac{u_t}{u^\gamma} \phi_\rho dxdt = \int_0^T \int_{\Omega_\rho} \{(1 - \lambda_1(\Omega_\rho))u\phi_\rho + f\phi_\rho\} dxdt - \int_0^T \int_{\partial\Omega_\rho} u \frac{\partial \phi_\rho}{\partial \nu} dsdt.$$

Since there is $C > 0$ satisfying

$$\int_{\partial\Omega_\rho} |\frac{\partial \phi_\rho}{\partial \nu}| ds \leq C$$

for sufficiently small ρ, we have

$$\int_0^T \int_{\Omega_\rho} \{(1 - \lambda_1(\Omega_\rho))u\phi_\rho + f\phi_\rho\} dxdt \geq -C\varepsilon$$

for sufficiently small ρ. Letting $\rho \to 0$, it follows that

$$\int_0^T \int_\Omega \{(1 - \lambda_1)u\phi_1 + f\phi_1\} dxdt \geq -C\varepsilon$$

and hence

$$\int_0^T \int_\Omega \{(1 - \lambda_1)u\phi_1 + f\phi_1\} dxdt \geq 0.$$

This implies that (1) cannot possess positive solutions in the above class.

3 Proof of Theorem 2.

This section is devoted to proof of Theorem 2, so we assume that $\lambda_1 < 1$ throughout this section. We suppose that $f \in C^\infty(\bar{\Omega} \times \mathbf{R})$ except for the proof of Theorem 2. We start with inequalities of Harnack type in time and in space direction for

$$\begin{cases} v_t = v^\gamma(\Delta v + g(v, x, t)) & \text{in } \Omega \times \mathbf{R}, \\ v = \varepsilon & \text{on } \partial\Omega \times \mathbf{R}, \\ v(t + T) = v(t) & \text{in } \Omega \times \mathbf{R}, \end{cases} \tag{10}$$

where g is a smooth function on $(0,\infty)\times\bar\Omega\times\mathbf{R}$ and $\gamma\geq 1$. Let $v\in C^\infty(\bar\Omega\times\mathbf{R})$ be a solution of (10) with $v>\varepsilon$ in $\Omega\times\mathbf{R}$. Putting $z=(\log v)_t=v_t/v$, we observe, in the same way as in [9], that

$$z\geq-\frac{(g_v v-g)_+}{\gamma}v^{\gamma-1}-(\frac{|g_t|}{\gamma})^{1/2}v^{(\gamma-1)/2} \tag{11}$$

at a minimizer of z over $\Omega\times\mathbf{R}$, where f_+ means the positive part of f. Since $z=0$ on $\partial\Omega$, it turns out that (11) holds on $\bar\Omega\times\mathbf{R}$. In the same way as in [9] we have Harnack type inequalities.

Lemma 1. Assume that there are positive constants c_i $(i=0,1,2)$ such that

$$g_\rho\rho-g\leq c_0 \quad \text{and} \quad \|g_t\|_\infty^{1/2}\leq c_1 \tag{12}$$

for all $(\rho,x,t)\in(0,\infty)\times\bar\Omega\times\mathbf{R}$ and $\max_{\bar\Omega\times\mathbf{R}} v\geq c_2$ for each solution v of (10) with $v>\varepsilon$ in $\Omega\times\mathbf{R}$. Then there exists $C_0=C_0(c_0,c_1,c_2)>0$ such that if v solves (10) with $v>\varepsilon$ in $\Omega\times\mathbf{R}$, then

$$v(x,t)\leq v(x,s)\exp(-C_0M^{\gamma-1}(t-s)) \tag{13}$$

for all $(x,t),(x,s)\in\Omega\times\mathbf{R}$ with $s-T\leq t\leq s$, where $M=\max_{\bar\Omega\times\mathbf{R}} v$.

In a high dimensional domain, an inequality of Harnack type in space direction is described in terms of means in balls centered at a maximizer of solution instead of Lemma 2.4 in [9]. The difficulty of the proof in this case is that maximizers may approach to the boundary of Ω, while this difficulty does not happen in the case periodic boundary condition. To overcome that, we need the following proposition.

Proposition 1. Assume that $w\in C^2(\bar\Omega)$ equals some constant α on the boundary $\partial\Omega$, where $\partial\Omega$ is at least C^2. If $\tilde w$ is the extension of w to $\mathbf{R}^N$ such that $w=\alpha$ outside Ω. then $g(\tilde w)\in C^2(\mathbf{R}^N)$ provided that $g\in C^2(\mathbf{R})$ and that $g'(0)=g''(0)=0$.

Proof. We may assume $\alpha = 0$ and $g(0) = 0$. Let x_0 be a point in $\partial\Omega$. It suffices to prove $g(\tilde{w})$ is C^2 at $x_0 \in \mathbf{R}^N$. We may assume that Ω is a half space

$$\Omega = \{(x_1, x_2, \cdots, x_N) \in \mathbf{R}^N : x_N > 0\}$$

and that $x_0 = 0$ without the loss of generality. It now suffices to prove that

$$\lim_{h \downarrow 0} \frac{1}{h} v(he_N) = 0, \quad \lim_{h \downarrow 0} \frac{1}{h} \frac{\partial v}{\partial x_N}(he_N) = 0$$

for $v = g(\tilde{w})$, where e_N is the unit vector $(0, \cdots, 0, 1) \in \mathbf{R}^N$.

By the mean value theorem

$$v(he_N) = g(w(he_N)) - g(0) = g'(\eta) w(he_N)$$

for some η between 0 and $w(he_N)$, so we see

$$\frac{1}{h} v(he_N) \to g'(0) \frac{\partial w}{\partial x_N}(0) \quad \text{as } h \downarrow 0.$$

Since $g'(0) = 0$, this yields

$$\frac{1}{h} v(he_N) \to 0 \quad \text{as } h \downarrow 0 \quad \text{and} \quad \frac{\partial v}{\partial x_N}(0) = 0.$$

Similarly, since $g'(0) = 0$, we observe that

$$\begin{aligned} \frac{\partial v}{\partial x_N}(he_N) &= g'(w(he_N)) \frac{\partial w}{\partial x_N}(he_N) \\ &= g''(\zeta) w(he_N) \frac{\partial w}{\partial x_N}(he_N) \end{aligned}$$

for some ζ between 0 and $w(he_N)$. Thus, since $g''(0) = 0$, we see

$$\frac{1}{h} \frac{\partial v}{\partial x_N}(he_N) \to g''(0) \left(\frac{\partial w}{\partial x_N}(0)\right)^2 = 0 \quad \text{as } h \downarrow 0. \ \square$$

For any $w \in C^2(\bar{\Omega})$ with a constant value on $\partial\Omega$, we suppose that $\tilde{w}$ means the extension of w defined in Proposition 1 in the rest of this section.

Lemma 2. Assume $\gamma \geq 1$ and (12) on g. Then if $v \in C^\infty(\bar{\Omega} \times \mathbf{R})$ solves (10) with $v > \varepsilon$ in $\Omega \times \mathbf{R}$, then

$$(M^\gamma - \varepsilon^\gamma)^3 \leq \frac{1}{\omega_N \rho^N} \int_{B_\rho(x_0)} \{\tilde{v}(x, t_0)^\gamma - \varepsilon^\gamma\}^3 dx + \frac{3\gamma C_M (M^\gamma - \varepsilon^\gamma)^2}{2(N+2)} \rho^2, \tag{14}$$

where

$$C_M = \frac{c_0}{\gamma} M^{\gamma-1} + \frac{c_1}{\gamma^{1/2}} M^{(\gamma-1)/2} + M^{\gamma-1} g_M,$$

$$g_M = \max\{(g(v, x, t))_+ : \varepsilon < v \leq M, (x, t) \in \bar{\Omega} \times \mathbf{R}\},$$

$$\max_{\bar{\Omega} \times \mathbf{R}} v = v(x_0, t_0)$$

and ω_N means the volume of unit balls in $\mathbf{R}^N$.

Proof. Let $v \in C^\infty(\bar{\Omega} \times \mathbf{R})$ be a solution of (10) with $v > \varepsilon$ in $\Omega \times \mathbf{R}$. Since (11) implies

$$v^{\gamma-1} \Delta v = z - v^{\gamma-1} g \geq -C_M \quad \text{in } \Omega \times \mathbf{R}, \tag{15}$$

we have

$$\Delta(v^\gamma) = \gamma(v^{\gamma-1} \Delta v + (\gamma - 1) v^{\gamma-2} |\nabla v|^2) \geq -\gamma C_M \quad \text{in } \Omega \times \mathbf{R}.$$

Putting $w = v^\gamma - \varepsilon^\gamma$, it follows that $\tilde{w}^3 \equiv (\tilde{v}^\gamma - \varepsilon^\gamma)^3 \in C^2(\mathbf{R}^N \times \mathbf{R})$ and

$$\Delta(\tilde{w}^3) = 0 \quad \text{on } (\mathbf{R}^N \backslash \Omega) \times \mathbf{R}$$

from Proposition 1. We also see

$$\begin{aligned} \Delta(\tilde{w}^3) &= 3(2\tilde{w}|\nabla \tilde{w}|^2 + \tilde{w}^2 \Delta \tilde{w}) \\ &\geq 3\tilde{w}^2 \Delta \tilde{w} \\ &\geq -3\gamma C_M (M^\gamma - \varepsilon^\gamma)^2 \quad \text{in } \Omega \times \mathbf{R}. \end{aligned}$$

Therefore we get

$$\Delta(\tilde{w}^3) \geq -3\gamma C_M (M^\gamma - \varepsilon^\gamma)^2 \quad \text{in } \mathbf{R}^N \times \mathbf{R},$$

that is,

$$\Delta\{\tilde{w}^3 + \frac{3\gamma C_M(M^\gamma - \varepsilon^\gamma)^2}{2N}|x - x_0|^2\} \geq 0 \quad \text{in } \mathbf{R}^N \times \mathbf{R}.$$

Then for each $\rho > 0$, the mean value theorem yields

$$\begin{aligned}
&(M^\gamma - \varepsilon^\gamma)^3 \\
\leq\ & \frac{1}{\omega_N \rho^N} \int_{B_\rho(x_0)} \{\tilde{w}^3(x, t_0) + \frac{3\gamma C_M(M^\gamma - \varepsilon^\gamma)^2}{2N}|x - x_0|^2\} dx \\
=\ & \frac{1}{\omega_N \rho^N} \int_{B_\rho(x_0)} (v(x, t_0)^\gamma - \varepsilon^\gamma)^3 dx + \frac{3\gamma C_M(M^\gamma - \varepsilon^\gamma)^2}{2(N+2)}\rho^2. \quad \square
\end{aligned}$$

For $\varepsilon > 0$, we consider the following approximate equation of (1)

$$\begin{cases} u_t = (u+\varepsilon)^\gamma\{\Delta u + \dfrac{u}{u+\varepsilon}(u+f)\} & \text{in } \Omega \times \mathbf{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbf{R}, \\ u(t+T) = u(t) & \text{in } \Omega \times \mathbf{R}. \end{cases} \tag{16}$$

We adapt the Leray-Schauder degree theory to get a positive solution of (16). To do that, an a priori bound of positive solutions for

$$\begin{cases} u_t = (u+\varepsilon)^\gamma\{\Delta u + \dfrac{u}{u+\varepsilon}(u+\tau f) + (1-\tau)(u+1)\} & \text{in } \Omega \times \mathbf{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbf{R}, \\ u(t+T) = u(t) & \text{in } \Omega \times \mathbf{R} \end{cases} \tag{17}$$

with $\tau \in [0, 1]$. Putting $v = u + \varepsilon$, v satisfies

$$v_t = v^\gamma\{\Delta v + \frac{v-\varepsilon}{v}(v - \varepsilon + \tau f) + (1-\tau)(v + 1 - \varepsilon)\} \quad \text{in } \Omega \times \mathbf{R}. \tag{18}$$

Lemma 3. There are positive constants C_1, C_2 depending only on $\|f\|_\infty$ and $\|f_t\|_\infty$ such that

$$\int_0^T \int_\Omega v\,dxdt \leq C_1 \quad \text{and} \quad \int_0^T \int_\Omega \frac{v_t^2}{v^\gamma} dxdt \leq C_2$$

for $0 < \varepsilon < 1, 0 \leq \tau \leq 1$ and solution v of (18) with $v > \varepsilon$ in $\Omega \times \mathbf{R}$.

Proof. Let v be a solution of (18) with $v > \varepsilon$ in $\Omega \times \mathbf{R}$ and put

$$V(x) = \int_0^T v(x,t)dt \quad \text{for } x \in \bar{\Omega}.$$

Multiplying $\frac{1}{v^\gamma}$ with (18) and integrating over $(0,T)$ yields

$$\Delta V + V = F \quad \text{in } \Omega, \tag{19}$$

where

$$F(x) = \int_0^T \{(1 - \frac{v-\varepsilon}{v})v + \frac{v-\varepsilon}{v}(-\tau f + \varepsilon) - (1-\tau)(v+1-\varepsilon)\} \quad \text{for } x \in \bar{\Omega}.$$

Multiplying a positive first eigenfunction ϕ_1 of the Laplacian with (19) and integrating over Ω, it follows that

$$(1-\lambda_1)\int_\Omega V\phi_1 dx = \int_\Omega F\phi_1 dx + \varepsilon T \int_{\partial\Omega} \frac{\partial \phi_1}{\partial \nu} ds$$

and hence

$$\int_\Omega V dx \leq (\int_\Omega V\phi_1 dx)^{1/3} (\int_\Omega V\phi_1^{-1/2} dx)^{2/3} \leq KM^{2/3} \tag{20}$$

for some $K > 0$ independent of V, where $M = \max V = V(x_0)$. On the other hand, it follows that

$$\Delta V \geq -M - 1 \quad \text{in } \Omega$$

from (19) and $F \geq -1$. We proceed as in the proof of Lemma 2. Setting $W = V - \varepsilon T$, Proposition 1 implies $\tilde{W}^3 \equiv (\tilde{V} - \varepsilon T)^3 \in C^2(\mathbf{R}^N)$ and

$$\Delta(\tilde{W}^3) = 0 \quad \text{on } \mathbf{R}^N \backslash \Omega.$$

We also see

$$\Delta(\tilde{W}^3) \geq -3(M+1)(M-\varepsilon T)^2 \quad \text{in } \Omega,$$

so

$$\Delta(\tilde{W}^3 + \frac{3(M+1)(M-\varepsilon T)^2}{2N}|x - x_0|^2) \geq 0 \quad \text{in } \mathbf{R}^N.$$

Therefore for each $\rho > 0$, the mean value theorem yields

$$\begin{aligned}
&(M - \varepsilon T)^3 \\
\leq\ & \frac{1}{\omega_N \rho^N} \int_{B_\rho(x_0)} \{\tilde{W}^3(x) + \frac{3(M+1)(M-\varepsilon T)^2}{2N} \|x - x_0\|^2\} dx \\
=\ & \frac{1}{\omega_N \rho^N} \int_{B_\rho(x_0)} (\tilde{V}(x) - \varepsilon T)^3 dx + \frac{3(M+1)(M-\varepsilon T)^2}{2(N+2)} \rho^2.
\end{aligned}$$

From this inequality and (20), it follows that

$$(M - \varepsilon T)^3 \leq \frac{K}{\omega_N \rho^N} M^{8/3} + \frac{3(M+1)(M-\varepsilon T)^2}{2(N+2)} \rho^2.$$

Choosing $0 < \rho < (\frac{N+2}{3})^{1/2}$, we obtain

$$(M - \varepsilon T)^3 \leq \frac{K}{\omega_N \rho^N} M^{8/3} + \frac{1}{2}(M+1)(M - \varepsilon T)^2.$$

This yields some $M_0 > 0$ such that

$$\int_0^T v(x,t) dt \leq M_0 \quad \text{for } x \in \bar{\Omega}.$$

Therefore the first inequality in this lemma holds with $C_1 = M_0 |\Omega|$, where $|\Omega|$ denotes the volume of Ω. Define

$$\Phi(s) = \int_\varepsilon^s \frac{r - \varepsilon}{r} dr \quad \text{for } s > \varepsilon.$$

Multiplying $\frac{v_t}{v^\gamma}$ with (18) and integrating over Q yields

$$\int_0^T \int_\Omega \frac{v_t^2}{v^\gamma} dx dt = -\tau \int_0^T \int_\Omega \frac{v - \varepsilon}{v} v_t f dx dt = \tau \int_0^T \int_\Omega \Phi(v) f_t dx dt \leq C_1 \|f_t\|_\infty. \ \square$$

Conbining Lemmas 2-3, we derive an a priori upper bound for solutions v of (18) with $v > \varepsilon$ in $\Omega \times \mathrm{R}$.

Theorem 3. If $1 \leq \gamma < 3$ and $f \in C^\infty(\bar{\Omega} \times \mathrm{R})$ is negative on $\bar{\Omega} \times \mathrm{R}$ and T-periodic in time, then there exists $M_0 = M_0(\|f\|_\infty, \|f_t\|_\infty) > 0$ such that $\max v \leq M_0$ for each $\varepsilon \in (0,1)$, $\tau \in [0,1]$ and each solution $v \in C^\infty(\bar{\Omega} \times \mathrm{R})$ of (18) with $v > \varepsilon$ in $\Omega \times \mathrm{R}$.

Proof. For each solution $v \in C^\infty(\bar\Omega \times \mathbf{R})$ with $v > \varepsilon$ in $\Omega \times \mathbf{R}$, let $M = \max v = v(x_0, t_0)$. Putting $g(v,x,t) = \dfrac{(v-\varepsilon)}{v}(v - \varepsilon + \tau f) + (1-\tau)(v + 1 - \varepsilon)$, we easily see

$$g_v v - g \le 2 + \|f\|_\infty \equiv c_0 \quad \text{and} \quad \|g_t\|_\infty^{1/2} \le \|f_t\|_\infty^{1/2} \equiv c_1$$

and hence

$$C_M \le \frac{c_0}{\gamma} M^{\gamma-1} + \frac{c_1}{\gamma^{1/2}} M^{(\gamma-1)/2} + M^{\gamma-1}(M+1) \le KM^\gamma$$

for some $K = K(c_0, c_1) > 0$. Taking $0 < \rho < (\dfrac{N+2}{3\gamma K})^{1/2}$, it follows that

$$\int_\Omega \{v(x,t_0)^\gamma - \varepsilon^\gamma\}^3 dx \ge \omega_N \rho^N (M^\gamma - \varepsilon^\gamma)^2 (\frac{1}{2} M^\gamma - \varepsilon^\gamma) \tag{21}$$

from Lemma 2. On the other hand, choose $s_0 \in [0, T)$ such that

$$\int_\Omega v(x, s_0) dx \le \frac{1}{T} \int_0^T \int_\Omega v(x,t) dx dt.$$

Lemma 3 and Hölder's inequality yield

$$\begin{aligned}
\int_\Omega v(x,t_0) dx &= \int_\Omega v(x,s_0) dx + \int_\Omega \int_{s_0}^{t_0} v_t dx dt \\
&\le \int_\Omega v(x,s_0) dx + (\int_0^T \int_\Omega v^\gamma dx dt)^{1/2} (\int_0^T \int_\Omega \frac{v_t^2}{v^\gamma} dx dt)^{1/2} \\
&\le \frac{C_1}{T} + (C_1 C_2)^{1/2} M^{(\gamma-1)/2}
\end{aligned}$$

and hence

$$\int_\Omega v(x,t_0)^{3\gamma} dx \le M^{3\gamma-1}(\frac{C_1}{T} + (C_1C_2)^{1/2} M^{(\gamma-1)/2}). \tag{22}$$

By (21) and (22), we obtain

$$\omega_N \rho^N (M^\gamma - \varepsilon^\gamma)^2 (\frac{1}{2} M^\gamma - \varepsilon^\gamma) \le M^{3\gamma-1}(\frac{C_1}{T} + (C_1C_2)^{1/2} M^{(\gamma-1)/2}).$$

This yields our desired upper bound. □

Lemma 4. Assume that a is a positive continuous function on $\mathbf{R}$. Assume that there are constants $\lambda, \Lambda > 0$ such that

$$\lambda \le a(r) \le \Lambda \quad \text{on } \mathbf{R}.$$

Let $u \in W_p^{2,1}(Q), p > n+1$ be a solution of

$$\begin{cases} u_t = a(u)(\Delta u + c(x)u + h(x)) & \text{in } \Omega \times (0,T) = Q, \Omega \subset \mathbf{R}^N \\ u = 0 & \text{on } \partial\Omega \times (0,T) \end{cases} \tag{23}$$

with continuous functions c and h. Then there are constants $0 < \alpha < 1$ (depending only on $N, \lambda, \Lambda, \max|c|, \max|h|, M$) and $C = C(t_0, N, \lambda, \Lambda, \max|c|, \max|h|, M)$ such that

$$\|u\|_{C^{\alpha,\alpha/2}(Q')} \leq C, \quad Q' = \Omega \times (t_0, T)$$

provided that $\max_Q |u| \leq M$.

Remark 2. One can prove that this lemma along the same line of the proof of the Hölder estimate for quasilinear parabolic equation of divergence type [22 , p.419, Chap.V, Theorem 1.1]. The main ingredient is an energy type identity obtained by multiplying $\zeta^2 u/a(u)$ with the equation (23) and integrating by parts on $\Omega \times (t_1, t)$ with use of $u = 0$ on $\partial\Omega$; here $\zeta \in C_0^\infty(\bar{\Omega} \times (0,T])$. The resulting identity is

$$\begin{aligned} &\int_\Omega \zeta^2 B(u(x,t))dx + \int_{t_1}^t \int_\Omega |\nabla u(x,\tau)|^2 \zeta^2 dx d\tau \\ = &\int_\Omega \zeta^2 B(u(x,t_1))dx + \int_{t_1}^t \int_\Omega 2\zeta\zeta_t B(u) dx d\tau - \int_{t_1}^t \int_\Omega 2\zeta u \nabla\zeta \cdot \nabla u dx d\tau \\ &+ \int_{t_1}^t \int_\Omega c\zeta^2 u^2 dx d\tau + \int_{t_1}^t \int_\Omega \zeta^2 h u dx d\tau. \end{aligned}$$

Here $B(v) = \int_0^v \frac{\sigma}{a(\sigma)} d\sigma$. Since $\lambda \leq a \leq \Lambda$, we see

$$\frac{v^2}{2\Lambda} \leq B(v) \leq \frac{v^2}{2\lambda}.$$

If $B(v) = v^2$, from this identity we apply the embedding lemma [22, Chap.II, §7] to get the Hölder estimate. Since $B(v)$ is comparable with v^2, one can modify the proof to get the desired Hölder estimate.

Lemma 5. Assume that a is a positive continuous function satisfying

$$\lambda \leq a(r) \leq \Lambda \quad \text{for all } r \in \mathbf{R}$$

with some constant $\lambda, \Lambda > 0$. Let k be a positive number. Then for each $h \in C(\bar{Q})$ there is a unique solution $u \in \bigcap_{p>1} W_p^{2,1}(Q)$ of

$$\begin{cases} u_t = a(u)(\Delta u - ku + h) & \text{in } \Omega \times \mathbf{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbf{R}, \\ u(t+T) = u(t) & \text{in } \Omega \times \mathbf{R}. \end{cases}$$

Moreover for each $p > 1$ there is $C = C(\lambda, \Lambda, p)$ such that

$$\|u\|_{W_p^{2,1}(Q)} \leq C\|h\|_\infty.$$

Sketch of the proof of Lemma 5. We appeal to the continuity method by considering

$$\begin{cases} u_t = a_\tau(v)(\Delta u - ku + h) & \text{in } \Omega \times \mathbf{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbf{R}, \\ u(t+T) = u(t) & \text{in } \Omega \times \mathbf{R} \end{cases} \tag{24}$$

with $a_\tau(v) = \tau a(v) + (1-\tau)\lambda$. Let $X = W_q^{2,1}(Q)$ for some $q > n+1$ so that $X \subset C(\bar{Q})$ for $Q = \Omega \times (0, T)$. Let S be a mapping from $X \times [0,1]$ to X by setting a solution $u = S(v, \tau)$ of (24) for given $v \in X$. It is rather standard to see that S is compact and continuous by L^p-theory [10].

To apply the Leray-Schauder fixed point theory it suffices to get a estimate for $u = S(u, \tau)$, i.e., there is $K > 0$ such that

$$\|u\|_X \leq K \quad \text{for all } u = S(u,\tau), 0 \leq \tau \leq 1.$$

By the maximum principle we have

$$\|u\|_\infty \leq \frac{\|h\|_\infty}{k}.$$

By a priori Hölder estimate we have

$$\|u\|_{C^{\alpha,\alpha/2}} \leq C \quad \text{for } u = S(u,\tau), 0 \leq \tau \leq 1$$

with $C > 0, 0 < \alpha < 1$ depending only on $N, \lambda, \Lambda, k, \|h\|_\infty$. By L^p-theory for linear equations we have

$$\|u\|_{W_q^{2,1}(Q)} \leq C'$$

with C' independent of u and τ. Thus we find a solution u of the original problem is given by $u = S(u, 1)$.

The uniqueness is the same as in [9]. □

Now take $k_\varepsilon > 0$ such that

$$\Phi(s) = k_\varepsilon s + \frac{s}{s+\varepsilon}(s+f) > 0 \quad \text{for all } s > 0.$$

We suppose that $\Phi(s) = 0$ for $s \leq 0$.

By Lemma 5, for $h \in C(\bar{Q})$, we can define $S(h)$ by the unique solution of

$$\begin{cases} u_t = (u+\varepsilon)^\gamma(\Delta u - k_\varepsilon u + h) & \text{in } \Omega \times \mathbf{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbf{R}, \\ u(t+T) = u(t) & \text{in } \Omega \times \mathbf{R}. \end{cases} \tag{25}$$

Then S is a continuous compact operator from $C(\bar{Q})$ into itself by Lemma 5. Thus, the Leray-Schauder degree for $I - S \circ \Phi$ in $C(\bar{Q})$ is well-difined.

Lemma 6. If $f \in C^\infty(\bar{\Omega} \times \mathbf{R})$ is negative on $\bar{\Omega} \times \mathbf{R}$, then there is a sufficiently small $r > 0$ such that the Leray-Schauder degree of $I - S \circ \Phi$ of the value zero in $B_r(0)$ equals one, that is,

$$\deg\,(I - S \circ \Phi, B_r(0), 0) = 1,$$

where $B_r(0)$ denotes the ball with radius r centered at 0 in $C(\bar{Q})$.

Proof. For $\tau \in [0, 1]$, we consider

$$\begin{cases} u_t = (u+\varepsilon)^\gamma(\Delta u - k_\varepsilon u + \tau\Phi(u)) & \text{in } \Omega \times \mathbf{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbf{R}, \\ u(t+T) = u(t) & \text{in } \Omega \times \mathbf{R}. \end{cases} \tag{26}$$

Multiplying $\dfrac{\phi_1}{(u+\varepsilon)^\gamma}$ with (26) and integrating over Q yields

$$0 = \int_0^T \int_\Omega \{-\lambda_1 u + (\tau - 1)k_\varepsilon u + \tau \frac{u}{u+\varepsilon}(u + f)\}\phi_1 dxdt.$$

This implies $\max u \geq \min(-f)$ for any $\varepsilon > 0, \tau \in [0,1]$ and each positive solution u of (26). Choose $r < \min(-f)$ so that $S \circ (\tau\Phi)$ has no fixed points on $\partial B_r(0)$ for all $\tau \in [0,1]$. From the homotopy invariance of degree, it follows that

$$\deg\,(I - S \circ \Phi, B_r(0), 0) = \deg\,(I, B_r(0), 0) = 1. \quad \square$$

Lemma 7. If $1 \leq \gamma < 3$ and $f \in C^\infty(\bar{\Omega} \times \mathbf{R})$ is negative on $\bar{\Omega} \times \mathbf{R}$, then there is $R > r$ such that

$$\deg\,(I - S \circ \Phi, B_R(0), 0) = 0$$

for $0 < \varepsilon < 1$.

Proof. Define $\tilde{\Phi} : \mathbf{R} \to \mathbf{R}$ by

$$\tilde{\Phi}(s) = k_\varepsilon s_+ + \frac{s^2}{s_+ + \varepsilon} + s_+ + 1 \quad \text{for } s \in \mathbf{R}.$$

For $\tau \in [0,1]$, we consider

$$\begin{cases} u_t = (u+\varepsilon)^\gamma\{\Delta u - k_\varepsilon u + \tau\Phi(u) + (1-\tau)\tilde{\Phi}(u)\} & \text{in } \Omega \times \mathbf{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbf{R}, \\ u(t+T) = u(t) & \text{in } \Omega \times \mathbf{R}. \end{cases} \tag{27}$$

By Theorem 3, there is $M_0 > 0$ such that $\max u \leq M_0$ for any $\varepsilon \in (0,1), \tau \in [0,1]$ and each positive solution of (27). Take $R > \max(M_0, r)$ so that $S \circ (\tau\Phi + (1-\tau)\tilde{\Phi})$ has no fixed points on $\partial B_R(0)$ for all $\tau \in [0,1]$. It now follows that

$$\deg\,(I - S \circ \Phi, B_R(0), 0) = \deg\,(I - S \circ \tilde{\Phi}, B_R(0), 0).$$

Multiplying $\dfrac{\phi_1}{(u+\varepsilon)^\gamma}$ with (27) with $\tau = 0$ and integrating over Q yields

$$0 = \int_0^T \int_\Omega \{(1 - \lambda_1)u + \frac{u^2}{u+\varepsilon} + 1\}\phi_1 dxdt > 0.$$

This implies that there are no solutions for (27) with $\tau = 0$. We thus obtain

$$\deg\,(I - S \circ \tilde{\Phi}, B_R(0), 0) = 0.$$

This completes the proof. □

Lemma 8. If $1 \leq \gamma < 3$ and $f \in C^\infty(\bar{\Omega} \times \mathbf{R})$ is negative on $\bar{\Omega} \times \mathbf{R}$, then for each $\varepsilon \in (0, 1)$, there is a positive smooth solution of (16).

Proof. Let $0 < \varepsilon < 1$. From Lemmas 6, 7, it follows that

$$\deg\,(I - S \circ \Phi, B_R(0) \backslash B_r(0), 0) = -1,$$

which implies the existence of positive smooth solution of (16). □

We are now in a position to prove Theorem 2.

Proof of Theorem 2. Take a sequence $\{f_\varepsilon\} \subset C^\infty(\bar{\Omega} \times \mathbf{R})$ of negative functions such that each f_ε is T-periodic in t and

$$f_\varepsilon \to f \quad \text{and} \quad f_{\varepsilon t} \to f_t \quad \text{in} \quad C(\bar{Q}) \quad \text{as } \varepsilon \to 0.$$

Then it follows that $\|f_\varepsilon - f\|_\infty \leq 1, \|f_{\varepsilon t} - f_t\|_\infty \leq 1$ for sufficiently small ε. Let u_ε be a positive smooth solution of (16) with $f = f_\varepsilon$ for $\varepsilon \in (0, 1)$ obtained in Lemma 8. Define $U_\varepsilon, F_\varepsilon : \bar{\Omega} \to \mathbf{R}$ by

$$U_\varepsilon(x) = \int_0^T u_\varepsilon(x, t)dt$$

and

$$F_\varepsilon(x) = \int_0^T \{(1 - \frac{u_\varepsilon}{u_\varepsilon + \varepsilon})u_\varepsilon - \frac{u_\varepsilon}{u_\varepsilon + \varepsilon} f_\varepsilon\}dt.$$

Multiplying $\dfrac{1}{(u_\varepsilon + \varepsilon)^\gamma}$ with (16) and integrating over $(0, T)$ yields

$$\Delta U_\varepsilon + U_\varepsilon = F_\varepsilon \quad \text{in } \Omega. \tag{28}$$

Let $M_0 > 0$ be an upper bound for $\{u_\varepsilon\}$ by Theorem 3. Since $0 \leq F_\varepsilon \leq (2 + \|f\|_\infty)T$ and $\|U_\varepsilon\|_\infty \leq M_0 T$, there is $C_p > 0$ such that $\|U_\varepsilon\|_{W^{2,p}(\Omega)} \leq C_p$ for sufficiently small ε

with $p > \frac{N}{2}$. Then we may assume that $\{U_\varepsilon\}$ converges to some U weakly in $W^{2,p}(\Omega)$ and strongly in $C(\bar{\Omega})$ as $\varepsilon \to 0$. Integrating (13) in Lemma 1 over $(s-T, s)$ and $(t, t+T)$ respectively yields some positive constants λ, Λ independent of ε such that

$$\lambda u_\varepsilon(x,t) \le U_\varepsilon(x) \le \Lambda u_\varepsilon(x,t) \quad \text{in } Q. \tag{29}$$

Multiplying $\frac{u_\varepsilon}{(u_\varepsilon+\varepsilon)^\gamma}$ with (16) and integrating by parts over Q, we get

$$\int_0^T \int_\Omega |\nabla u_\varepsilon|^2 dxdt \le (M_0 + \|f\|_\infty + 1) M_0 T |\Omega|.$$

Multiplying $\frac{u_{\varepsilon t}}{(u_\varepsilon+\varepsilon)^\gamma}$ with (16) and integrating by parts over Q yields

$$\int_0^T \int_\Omega \frac{u_{\varepsilon t}^2}{(u_\varepsilon+\varepsilon)^\gamma} dxdt \le (\|f_t\|_\infty + 1) M_0 T |\Omega|$$

and hence

$$\int_0^T \int_\Omega u_{\varepsilon t}^2 \le (M_0+1)^\gamma (\|f_t\|_\infty + 1) M_0 T |\Omega|.$$

Therefore we may suppose that $\{u_\varepsilon\}$ converges to some u a.e. in Q. From (29), it follows that

$$\lambda u(x,t) \le U(x) \le \Lambda u(x,t) \quad \text{a.e. in } Q. \tag{30}$$

Since there is $\delta_0 > 0$ such that $\max u_\varepsilon \ge \delta_0$ by the same argument in the proof of Lemma 6, we see $\max U \ge \lambda\delta_0 > 0$ from (29). Put $\Omega_+ = \{x \in \Omega : U(x) > 0\}$ and decompose $\Omega_+ = \bigcup_{i=1}^k \Omega_i$ into its connected components $\Omega_i, i = 1, \cdots, k$ so that $U = 0$ on $\partial\Omega_i$. Setting

$$F(x) = -\int_0^T f(x,t)dt \quad \text{for } x \in \bar{\Omega}$$

and letting $\varepsilon \to 0$ in (28) yields

$$\Delta U + U = F > 0 \quad \text{in } \Omega_+. \tag{31}$$

Multiplying U with (31) and integratig over Ω_i, we have

$$\int_{\Omega_i} (-|\nabla U|^2 dx + U^2) dx = \int_{\Omega_i} FU dx$$

and hence

$$\lambda_1(\Omega_i) \le \frac{\int_{\Omega_i} |\nabla U|^2 dx}{\int_{\Omega_i} U^2 dx} < 1 \quad \text{for } i = 1, \cdots, k.$$

Since $U_\varepsilon \to U > 0$ uniformly in Ω_+ and $U_\varepsilon \le \Lambda u_\varepsilon$ in $\Omega \times \mathbf{R}$, we may assume that $u_\varepsilon \to u$ weakly in $W^{2,1}_{p,loc}(\Omega_+ \times \mathbf{R})$ and strongly in $C_{loc}(\Omega_+ \times \mathbf{R})$. Therefore $u \in C(\Omega_+ \times \mathbf{R})$,

$$u(t+T) = u(t) \quad \text{in } \Omega \times \mathbf{R}$$

and

$$u_t = u^\gamma(\Delta u + u + f) \quad \text{in } \Omega_+ \times \mathbf{R}.$$

By (30), we get $u \equiv 0$ on $(\Omega \backslash \Omega_+) \times \mathbf{R}$ and hence $u \in C(\bar{\Omega} \times \mathbf{R})$. We finally derive $\nabla u = 0$ on $(\partial\Omega_+ \backslash \partial\Omega) \times \mathbf{R}$ by (30) since $\nabla U = 0$ on $\partial\Omega_+ \backslash \partial\Omega$. This completes the proof. □

Remark 3. If f is nonnegative, there are no nonnegative solutions of (1) with properties stated in Theorem 2 by the same method as in Remark 1.

Remark 4. In the case of $\lambda_1 > 1$, (1) possesses the positive solution (Theorem 1), while in general there are no positive solutions when $\lambda_1 < 1$ as the following example

$$\begin{cases} u_{xx} + u - 1 = 0 & \text{in } (0, 3\pi), \\ u(0) = u(3\pi) = 0, \end{cases}$$

shows.

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