EXISTENCE OF FAST DECAYING SOLUTIONS TO A HARAUX-WEISSLER EQUATION WITH A PRESCRIBED NUMBER OF ZEROES

Claus Dohmen

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Existence of Fast Decaying Solutions to a Haraux-Weissler Equation With a Prescribed Number of Zeroes

Claus Dohmen[1]
Institut für Angewandte Mathematik,
Wegelerstr. 6, 53115 Bonn, Germany
email: cdohmen@ sfb256.iam.uni-bonn.de

0. Introduction

Within this paper we consider the qualitative behaviour of solutions of a generalized Haraux-Weissler equation with $m > \frac{(N-2)_+}{N}$, $p > 1$ and $\gamma \in \{-1, 0, 1\}$:

$$(P_\gamma) \qquad \begin{aligned} &V'' + \frac{N-1}{\eta} V' + \beta\eta U' + \alpha U + \gamma |U|^{p-1} U = 0 \qquad \text{in } I\!R^+, \\ &V = |U|^{m-1} U, \qquad \alpha, \beta > 0, \qquad \alpha(m-1) + 2\beta := 1, \\ &\qquad\qquad V'(0) = 0, \qquad U(0) = a > 0. \end{aligned}$$

This equation arises when radial, selfsimilar solutions

$$u(x,t) = t^{-\alpha} U(\eta), \qquad \eta = |x| t^{-\beta},$$

of the porous medium equation with reaction term –

$$(PME) \qquad u_t - \Delta(|u|^{m-1} u) = \gamma\, |u|^{p-1} u \qquad \text{in } I\!R^N \times (0, T),$$

– are studied. Solutions of this special form are of interest, as they usually describe the large time behaviour of solutions to the Cauchy Problem with general initial data (see for instance [9]). In this case the parameters α, β are given by

$$\alpha := \frac{1}{p-1}, \qquad \beta := \frac{p-m}{2(p-1)}.$$

Here we want to consider them as parameters only subject to the condition stated in (P_γ) . From [4] the possible large time behaviour of those solutions is known to be as follows:

Theorem [4]:
Let $m > \frac{(N-2)_+}{N}$, $p > 1$ and $\gamma \in \{-1, 0, 1\}$. Moreover let $\alpha + \gamma |U(0)|^{p-1} > 0$.
Then there exists a unique solution V of (P_γ), i.e. a function U satisfying

$$U \in C^0_{loc}(I\!R^+) \cap H^{1,1}_{loc}(I\!R^+), \quad V \in C^1_{loc}(I\!R^+) \cap H^{2,1}_{loc}(I\!R^+),$$

which solves the differential equation in a weak sense with test functions in $L^\infty_{loc}(I\!R^+)$. The limit

$$L := \lim_{\eta \to \infty} \eta^k U(\eta), \qquad k := \frac{\alpha}{\beta},$$

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always exists and is finite. If $L = 0$, then the large time behaviour of V is given by

(a) If $m > 1$, then the solution has compact support.

(b) If $m = 1$, then the solution has exponential decay.

(c) If $m < 1$, then $\eta^{\frac{2m}{1-m}} V(\eta) \leq C(m, N) \neq 0$ and if V has only finitely many zeroes,

$$\eta^{\frac{2m}{1-m}} V(\eta) \rightarrow C(m, N) \neq 0 \text{ for } \eta \rightarrow \infty.$$

In the literature such behaviour is usually referred to as slow decay ($L \neq 0$) and fast decay ($L = 0$). Unfortunately the case of infinitely many sign changes can not be excluded for $m < 1$ yet. Motivated by this classification, we introduce the following notation:

Definition:
Let $U(., a, k)$ denote the solution of (P_γ) with initial value $U(0) = a$ and parameter values α, β, such that $k = \alpha/\beta$.
(i) The solution $U(., a, k)$ is of type $S(i)$, if and only if U has slow decay and exactly $(i-1)$ sign changes.
(ii) The solution $U(., a, k)$ is of type $R(i)$, if and only if U has fast decay and exactly $(i-1)$ sign changes.

While the theorem above only lists the possible large time behaviours of solutions of (P_γ) , in this paper we want to determine, what types of solutions really occur. If there is no reaction term γu^p, this was done in [2], [3] and [6] by transforming (P_0) into an autonomous system of first order differential equations. As this seems no longer possible for (P_γ) , we choose a different approach here, similar to the one introduced by Weissler [W] for the semilinear case.
Other related papers also only deal with the semilinear case $m = 1$ ([1], [12]) or are restricted to non-negative solutions ([11], [13]). Moreover either $\gamma = 1$ or $\gamma = -1$ is considered. The results obtained here make rigorous some of what could be conjectured from the known results for these special cases and give a more unified approach to (P_γ) .
To present the results, we follow [1] or [15] and consider p, m and N as fixed parameters and vary k and the initial value $U(0)$. The general idea is to examine the behaviour of solutions for (P_γ) at the boundary of the $(k, U(0))$-parameter plane and to prove the existence of 'intermediate' types of solutions in the interiour. We distinguish three qualitatively different cases:

Theorem A: (Absorption)
Let $\gamma = -1$ and $m \geq 1$, $p > 1$.
(i) Let $U(0) < \alpha^{1/(p-1)}$ be fixed. Then for all $j \in I\!N$ there is a $k_j(m, p, N) > N$, such that the solution $U(., U(0), k_j)$ of (P_{-1}) is of type $R(j)$.
(ii) Let $k > N$ be fixed and let the solution $U(., 1, k)$ of (P_0) be of type $S(i_0)$ or $R(i_0)$, which is known to be finite and not equal zero. Then for all $i < i_0$ there is a $a_i > 0$, such that the solution $U(., a_i, k)$ of (P_{-1}) is of type $R(i)$.

Theorem B: (Supercritical source term)
Let $\gamma = 1$, $m \geq 1$, $p > 1$ and $\frac{N}{2} \geq \frac{p+m}{p-m}$. Then for all $j \in I\!N$ there is a $k_j(m, p, N) > \frac{N}{m+1}$, such that the solution $U(., U(0), k_j)$ of (P_1) is of type $R(j)$.

The corresponding result for fixed k can not be given yet, as the behaviour of solutions with large initial value is not known precisely enough.

Theorem C: (Subcritical source term)
Let $\gamma = 1$, $m \geq 1$, $p > 1$ and $\frac{N}{2} < \frac{p+m}{p-m}$.
(i) The solution $U(.,U(0),0)$ of (P_1) has at most a finite number of sign changes.
(ii) Let $U(0)$ be fixed and let the solution $U(.,U(0),0)$ of (P_1) be of type $S(j_0)$ or $R(j_0)$. Then for all $j > j_0$ there is a $k_j(m,p,N) > 0$, such that the solution $U(.,U(0),k_j)$ of (P_1) is of type $R(j)$.
(iii) Let $k > 0$ be fixed and let the solution $U(.,1,k)$ of (P_1) be of type $S(i_0)$. Then for all $i \geq i_0$ there is a $a_i > 0$,such that the solution $U(.,a_i,k)$ of (P_1) is of type $R(i)$. If the solution $U(.,1,k)$ of (P_1) is of type $R(i_0)$, then the same is true for all $i > i_0$.

The theorems above are stated only for the nonsingular case $m \geq 1$. If $m < 1$, most of the results remain valid, but there are three important problems which are not solved yet:
- Are the solutions for (P_1) with supercritical p and $k \leq N/(m+1)$, which are known to be positive, of type $S(1)$ or are solutions of type $R(1)$ possible in this parameter range?
- To prove theorem C (i), we have to assume $p + m > 2$. Is this a technical condition or not?
- And, what is most important, does in case $m < 1$ the number of sign changes increase in steps of one like in the nonsingular case?

Finally, let us indicate what can be said about the case that α, β originate from (PME), i.e.

$$k := \frac{2}{p-m}.$$

First, we have to admit that the condition $\beta > 0$ forces $p > \max\{1, m\}$ although we can allow $p < m$ in the more general equation, where k is independent of p and m.
However, in case of supercritical p only nonnegative solutions are possible for (P_γ) . Concerning smaller p, we have existence of fast decaying solutions with any number of sign changes by varying $\gamma U(0)$ over $\mathbb{R}$, i.e. considering (P_γ) with all three possible values of γ. If $p > m + 2/N$, then all these fast decaying solutions occur for $\gamma = 1$, whereas in case of smaller p there are also fast decaying solutions for $\gamma = -1$. What number of sign changes are at least possible in the absorption case is determined by the number of sign changes of the solution of (P_0).

The remaining chapters of this paper are devoted to the proof of theorem A–C. As it usually requires only minor changes in the arguments, we include the results for the singular case whenever possible.

1. Varying the initial value $U(0)$

In order to examine the behaviour of solutions of (P_γ) in case that the initial value is very small or very large, we make use of a blow-up technique:
Let

$$U_\lambda(\eta) = \lambda^q U(\lambda^r \eta, \lambda^{-q}),$$

where $U(.,a)$ denotes the solution of (P_γ) with initial value $U(0) = a$. Straightforward calculations yield the following initial value problem for U_λ:

$$(P_{\gamma,\lambda}) \qquad \begin{aligned} &V_\lambda'' + \frac{N-1}{\eta} V_\lambda' + \lambda^{q(m-1)+2r}(\beta\eta U_\lambda' + \alpha U_\lambda) \\ &\qquad + \lambda^{q(m-p)+2r}\gamma |U_\lambda|^{p-1} U_\lambda = 0 \qquad \text{in } \mathbb{R}^+ \\ &V_\lambda = |U_\lambda|^{m-1} U_\lambda, \qquad V_\lambda(0) = 1, \qquad V_\lambda'(0) = 0. \end{aligned}$$

Proceeding as usual we get a scaled energy for $(P_{\gamma,\lambda})$ by

$$E_\lambda(\eta) = \frac{1}{2}|V_\lambda'|^2 + mF_\lambda(U_\lambda)$$
$$F_\lambda(U) = \int_0^U (\lambda^{q(m-1)+2r}\alpha s + \lambda^{q(m-p)+2r}\gamma|s|^{p-1}s)|s|^{m-1}ds.$$

E_λ is monotone decreasing, if $\beta > 0$, hence

$$E_\lambda(\eta) \leq mF_\lambda(1)$$
$$\leq C\left\{\lambda^{q(m-1)+2r} + \lambda^{q(m-p)+2r}\right\}$$

These calculations suggest to consider the two scalings, that give one λ-exponent the value zero. It turns out that $q(m-1)+2r = 0$ is suitable to consider the case $U(0) \to 0$ and $q(m-p)+2r = 0$ can be used to decide what happens in the case $U(0) \to \infty$.

Theorem 1.1:
Let $m > 0$, $p > 1$ and U_λ, defined as above with

$$q = -2 \qquad \text{and} \qquad r = -(p-m),$$

be a solution of $(P_{\gamma,\lambda})$ with $\gamma = 1$.
Then $V_\lambda \to \bar{V}$ in $C^1_{loc}(\mathbb{R}^+)$ as $\lambda \to \infty$ and $\bar{V}$ solves the initial value problem

$$\bar{V}'' + \frac{N-1}{\eta}\bar{V}' + \gamma|\bar{U}|^{p-1}\bar{U} = 0 \qquad \text{in } \mathbb{R}^+$$
$$\bar{V} = |\bar{U}|^{m-1}\bar{U}, \qquad \bar{V}(0) = 1, \qquad \bar{V}'(0) = 0.$$

Remark 1.2:
(i) As a consequence of a result of Weissler ([15]) the function $\bar{V}$ has infinitely many zeroes, if and only if $N \leq 2$ or $N > 2$ and

$$\frac{p}{m} < \frac{N+2}{N-2}.$$

(Although originally proved only for $p > m$, it can easily be seen by elementary ODE-theory that it is also true if $p \leq m$, see for instance [3], proof of lemma 3.3.)
Otherwise $\bar{V}$ is positve and slowly decaying and in case that equality holds, the solution is given explicitely by

$$\bar{V}(\eta) = a\left(1 + \frac{a^{p/m-1}}{N(N-2)}\eta^2\right)^{\frac{2-N}{2}}.$$

This last result is originally due to Joseph and Lundgren ([8]). (Please observe that $\bar{V}$ is neither a fast nor a slow decaying solution in the sense that it has the large time behaviour stated in the theorem from [4] stated in the introduction.)
(ii) Note that the case $k = 0$, i.e. $\alpha = 0$ and $\beta = 1/2$, can be included here. So in the subcritical case every number of sign changes is possible for U, even if k is very small. This is a sharp contrast to the case of supercritical p, where zeroes of U can be excluded for all $k < N/(m+1)$ (see [5] or Proposition 2.3 below).

Proof of theorem 1.1:
From the energy estimate we have

$$|U_\lambda| \leq C, \qquad |V_\lambda'| \leq C$$

uniformly in λ. Let us define

$$W_\lambda := V_\lambda' + \lambda^{-2(p-1)}\beta\eta U_\lambda;$$

this function is bounded uniformly on compact subsets of $\mathbb{R}^+$. Due to

$$W_\lambda' = -\frac{N-1}{\eta}W_\lambda - \lambda^{-2(p-1)}(\alpha - \beta N)U_\lambda + \gamma|U_\lambda|^{p-1}U_\lambda$$

and

$$\lim_{\eta\to 0} W_\lambda'(\eta) = \frac{1}{N}\left\{\lambda^{-2(p-1)}(\alpha-\beta N) - \gamma\right\} < \infty$$

the first derivative of W_λ is also bounded. Hence

$$W_\lambda \to \bar{W} \qquad \text{in } C^0_{loc}(\mathbb{R}^+),$$

and consequently

$$V_\lambda' \to \bar{V}' \qquad \text{in } C^0_{loc}(\mathbb{R}^+)$$

from the definition of W_λ. As U_λ is an absolutely continuous function of V_λ, this implies

$$U_\lambda' \to \bar{U}' \qquad \text{in } L^1_{loc}(\mathbb{R}^+).$$

Now we can pass to the limit in the differential equation and get the L^1_{loc}-convergence for V_λ'' and thereby the result. q.e.d.

The same procedure works in case that $V(0) \to 0$ and the scaling satisfying $q(m-1)+2r = 0$. To be precise, let us formulate the result; as the proof is similar to those of Theorem 1.1, we omit it here.

Theorem 1.3:
Let $m > 0$, $p > 1$ and U_λ, defined as above with

$$q = -2 \qquad \text{and} \qquad r = m-1,$$

if $m \neq 1$ – if $m = 1$, set $r = 0$ and $q < 0$ arbitrary – be a solution of $(P_{\gamma,\lambda})$.
Then $V_\lambda \to \bar{V}$ in $C^1_{loc}(\mathbb{R}^+)$ as $\lambda \to 0$ and $\bar{V}$ solves the initial value problem

$$\bar{V}'' + \frac{N-1}{\eta}\bar{V}' + \beta\eta\bar{U}' + \alpha\bar{U} = 0 \qquad \text{in } \mathbb{R}^+$$
$$\bar{V} = |\bar{U}|^{m-1}\bar{U}, \qquad \bar{V}(0) = 1, \qquad \bar{V}'(0) = 0.$$

Remark 1.4:
(i) We want to stress the fact that the result is valid for both $\gamma = -1$ and $\gamma = +1$, which means that we could make rigorous the heuristically obvious argument, that the superlinear term u^p has no qualitative influence on the structure of the solution provided the initial value is small.
(ii) As shown in [2], [3] and [6] Theorem 1.3 implies that for parameter values satisfying $k > N$ the function $\bar{V}$ has sign changes, and that the number of sign changes increases monotonically in k. This result is independent of the initial value chosen for V_λ, as (P_0) possesses a scaling invariance: If U is a solution, all U_λ, defined as in Theorem 1.3, are also solutions.

2. Varying the parameter k

Within this chapter we consider (P_γ) with fixed initial value and investigate the behaviour of solutions depending on k. First we consider values of k near zero.

Lemma 2.1:
Let $k \leq N$ and in case that $\gamma = 1$ let additionally $|U(0)|^{p-1} \leq \beta N - \alpha$.
Then the solution V of (P_γ) is positive in $\mathbb{R}^+$ and has slow decay, i.e.

$$\lim_{\eta\to\infty} \eta^k U(\eta) =: L > 0.$$

Proof:
We rewrite the differential equation as

$$\eta^{N-1}V'(\eta) + \beta\eta^N U(\eta) = \int_0^\eta t^{N-1}(\beta N - \alpha - \gamma|U(t)|^{p-1})U(t)dt.$$

Now suppose there exists a first zero η_0 of V. At this point $V'(\eta_0) \leq 0$ must hold. On the other hand the conditions on the parameter values imply $\beta N - \alpha \geq 0$ and in case that $\gamma = 1$ also

$$\beta N - \alpha - |U(0)|^{p-1} \geq 0.$$

In order to get a negative value for the integral, the integrand must have been negative somewhere in $(0, \eta_0)$. But the solution is bounded by its initial value – this can be seen by a simple energy argument – so it is impossible to get a negative integrand under the above assumptions. Thus the solution is positive in $\mathbb{R}^+$.
If $m > 1$, this argument excludes a nonnegative solution with fast decay as well, because it would have compact support and consequently a finite zero.
If $m < 1$, we have to exploit the fact that a positive solution satisfies $V' \leq 0$ on $\mathbb{R}^+$ the above equality can only hold in case that for large η

$$\eta^N U(\eta) \geq c_0 > 0.$$

Suppose $L = 0$, i.e. U is bounded qualitatively by $\eta^{\frac{2}{1-m}}$ if η is large enough. This implies

$$\eta^N U(\eta) \leq C\eta^{N-\frac{2}{1-m}} \to 0 \qquad \text{for } \eta \to \infty;$$

contradiction, if $m > \frac{(N-2)_+}{N}$.
The case $m = 1$ can be treated similarily, due to the exponential decay of fast decaying solutions (see for instance [15]). q.e.d.

Lemma 2.1 gives no information about the case $\gamma = 1$ combined with large initial values. The behaviour of solutions then depends on whether p is subcritical or supercritical. In general we have

Lemma 2.2:
Let $\{k_j\}_{j\in N}$ be a sequence converging to zero as $j \to \infty$. Then the corresponding solutions V_j of (P_1) with fixed initial value $a > 0$ converge uniformly on compact subsets of $\mathbb{R}^+$ to a function $\bar{V}$ with finitely many zeroes, provided $p + m \geq 2$.

Remark:
Please note that in contrast to an analogous result for the equation (P_1) proved in [3], we do not have to restrict the argument here to the nonsingular case $m \geq 1$, because the term αU is absent. The restriction

here comes into effect only for $m < 1$ and subcritical p. However, it may be essential and not merely technical, as this is the case for the degenerate case with strong absorption $p < 1$, see [10] and the references therein.

Proof:
Similar to the above proofs we can derive that the limit equation satisfied by $\bar{V}$ reads

$$\bar{V}'' + \frac{N-1}{\eta}\bar{V}' + \frac{\eta}{2}\bar{U}' + |\bar{U}|^{p-1}\bar{U} = 0 \qquad \text{in } \mathbb{R}^+.$$

To prove that this equation has always a solution with at most finitely many zeroes, we constuct a positively invariant region in the phase plane:
Consider the equivalent system – replacing $\bar{V}$ by V to simplify the notation –

$$\begin{aligned} V' &= W \\ W' &= -\frac{N-1}{\eta}W - \frac{\eta}{2m}|V|^{\frac{1}{m}-1}W - |U|^{p-1}U. \end{aligned}$$

For this system one can prove in a way completely analogous to Proposition 3.5 of [D1], that the sets defined by

$$\begin{aligned} I^+ &:= \{(V,W)\,|\,0 < V < V(0),\, 0 > W > -\lambda V^{1/m}\}, \\ I^- &:= \{(V,W)\,|\,(-V,-W) \in I^+\} \end{aligned}$$

are positively invariant provided

$$\eta > 2\left(\lambda + \frac{m}{\lambda}|U|^{p+m-2}\right).$$

The exponent of U on the right hand side is positive; we can thus estimate U by its initial value. Choosing the optimal λ yields

$$\eta > 4\sqrt{m}|V(0)|^{\frac{p+m-2}{2m}}.$$

As zeroes can not accumulate at one point, the lemma is proved.

q.e.d.

In case of subcritical p this result is optimal, as can be seen by Theorem 1.1 and Remark 1.2 above. It shows that solutions of the limit problem can have an arbitrary large number of sign changes, provided the initial value is large enough.
In case of supercritical p the existence of zeroes is excluded by a result in [5] obtained via a Pohozaev-type inequality:

Proposition 2.3:
Let $N > 2$, $m \geq \frac{N-2}{N}$ and $\frac{p}{m} \geq \frac{N+2}{N-2}$. Then the solution U of (P_1) can not possess a zero, regardless of the initial value $U(0) > 0$, provided

$$k < \frac{N}{m+1}.$$

From this result we can also exclude solutions of type $R(1)$ for $m > 1$, as these solutions would have compact support and thus a zero.
As to large values of k, we can state the following

Lemma 2.4:
Let $\{k_j\}_{j\in N}$ be a sequence converging to $\frac{2}{1-m}$, if $m<1$, or to infinity, if $m\geq 1$. Then the corresponding solutions V_j of (P_γ) with fixed initial value $a>0$ satisfying $a^{m-1}<\frac{1}{m-1}$ converge uniformly on compact subsets of $\mathbb{R}^+$ to a function $\bar{V}$ with infinitely many zeroes.

Proof:
We consider the differential equation of (P_γ) in a scaled version

$$U_j(\eta) := U(\alpha_j^{1/2}\eta).$$

The function $\bar{U}$ then satisfies

$$V_j'' + \frac{N-1}{\eta}V_j' + \frac{\beta_j}{\alpha_j}\eta U_j' + U_j + \frac{\gamma}{\alpha_j}|U_j|^{p-1}U_j = 0 \qquad \text{in } \mathbb{R}^+.$$

Due to the condition $\alpha(m-1)+2\beta=1$ we have $\alpha_j,\beta_j\to\infty$, if $m<1$, $\alpha\to\infty$, $\beta\to\frac{1}{2}$, if $m=1$ and $\alpha\to\frac{1}{m-1}$, $\beta\to 0$, if $m>1$. Again we have an energy estimate, such that $|U_j|$ and $|V_j'|$ are uniformly bounded in j and the arguments of the proof of Theorem 1.1 can be used to show that

$$V_j\to\bar{V} \qquad \text{in } C^1_{loc}(\mathbb{R}^+)$$

and

$$\begin{aligned} \bar{V}'' + \frac{N-1}{\eta}\bar{V}' + \bar{U} + \gamma(m-1)|\bar{U}|^{p-1}\bar{U} &= 0 \qquad \text{in } \mathbb{R}^+ \text{ if } m\geq 1, \\ \bar{V}'' + \frac{N-1}{\eta}\bar{V}' + \frac{1-m}{2}\eta\bar{U}' + \bar{U} &= 0 \qquad \text{in } \mathbb{R}^+ \text{ if } m<1. \end{aligned}$$

The solution of the first equation under the above assumptions on the data, however, has infinitely many zeroes. This can be shown by elementary calculations similar to those in [D1], Lemma 3.3.
To the latter equation, rewritten in terms of

$$t = \ln\eta, \qquad f(t) = \eta^{\frac{2m}{1-m}}V(\eta),$$

a theorem from [SC], p. 330 is applicable, which says that this equation has a periodic solution, provided $m>\frac{(N-2)_+}{N}$. As this argument has already been used in [2] and [3], we omit the details.

q.e.d.

As a last result, we give a range of k, where all solutions necessarily possess sign changes.

Lemma 2.5:
Let $k>N$ and in case $\gamma=-1$ let additionally hold $|U(0)|^{p-1}<\alpha-\beta N$.
Then any solution V of (P_γ) has a sign change.

Proof: Again consider the equation

$$\eta^{N-1}V'(\eta) + \beta\eta^N U(\eta) = -\int_0^\eta t^{N-1}(\alpha-\beta N+\gamma|U(t)|^{p-1})U(t)dt.$$

The assumptions on the data here imply

$$\alpha - \beta N + \gamma |U(t)|^{p-1} \geq c_0 > 0 \quad \text{for all } t \geq 0,$$

the constant c_0 depending only on the parameter values and $U(0)$.
Now suppose V is nonnegative. Inserting this information into the above equality, we get

$$(*) \qquad \begin{aligned} \eta^{N-1}V'(\eta) + \beta\eta^N U(\eta) &\leq -c_0 \int_0^\eta t^{N-1}U(t)dt \\ &\leq -C \end{aligned}$$

for all η away from zero. But due to the assumption $k > N$ the second term on the left hand side converges to zero as $\eta \to \infty$, and so does the first one, too:
In case of fast decaying solutions and $m \geq 1$ this is clear. Otherwise we have polynomial decay and

$$\frac{\eta V'(\eta)}{V(\eta)} \to -c < 0.$$

Consequently we have

$$\begin{aligned} |\eta^{N-1}V'(\eta)| &= |\frac{\eta V'(\eta)}{V(\eta)}|\eta^{N-2}|V(\eta)| \\ &\leq C\eta^{N-2-km} \\ &\leq C\eta^{N(1-m)-2} \to 0 \end{aligned}$$

as $\eta \to \infty$ due to $m > \frac{(N-2)_+}{N}$. Otherwise $k > N$ cannot be fulfilled.
Consequently the inequality $(*)$ can not hold for large η if we assume a nonnegative solution. q.e.d.

3. Existence of Rapidly Decaying Solutions

We will prove our Theorems A–C using a technique introduced by Weissler in [15] for the semilinear case. Roughly it can be described as follows:
Let all but one parameter be fixed. To prove existence of solutions of type $R(i)$, we will split a set of solutions being not of type $R(i)$ into two disjoint, nonempty, open sets in $\mathbb{R}^+$; thus the complement of the union of these sets is also nonempty, which means existence of the required type of solution.
We will apply this technique to the case that the initial value a of U is not fixed. The case of varying k is similar and thus omitted.

As k is fixed, in what follows we will skip the dependence of V on k and denote $V(.,a)$ the solution of (P_γ) with initial value $V(0,a) = a$. Then define the weighted limit $L(a)$ and the number of sign changes $Z(a)$ of $V(.,a)$ by

$$\begin{aligned} L(a) &:= \lim_{\eta \to \infty} \eta^{km} V(\eta, a) \\ Z(a) &:= H^0(\{\eta \in \mathbb{R}^+ | V(\eta, a) = 0, V'(\eta, a) \neq 0\}). \end{aligned}$$

We begin by stating the three lemmata, whose contents are what is basically needed in the existence proof.

Lemma 3.1:
Let the solution of (P_0) be of type $S(i_0)$.
(i) Let $\gamma = 1$ and $\frac{N}{2} < \frac{p+m}{p-m}$. Then for all $i \geq i_0 - 1$ there exists an $a(i) > 0$, such that for all $a > a(i)$ the number of sign changes is greater than i. (If the solution of (P_0) is of type $F(i_0)$, then $i_0 - 1$ has to be excluded).
(ii) Let $\gamma = -1$. Then for all $i < i_0 - 1$ there exists an $a(i) > 0$, such that for all $a < a(i)$ the number of sign changes is greater than i.

Proof:
We only prove (i). The proof of (ii) can be done in a similar way.
Suppose the statement is false. Then we can choose a sequence $\{a_j\}_{j\in N}$, $a_j \to \infty$ as $j \to \infty$, such that $Z(a_j) \leq i$. The corresponding solutions $V(.,a_j)$ then converge in $C^0_{loc}(\mathbb{R}^+)$ to some function $\bar{V}$. Due to the continuous dependence of V on the initial data $\bar{V}$ also has less or equal i sign changes. But this is a contradiction to Theorem 1.1 and Remark 1.2 (ii), where we proved that under the above assumptions $\bar{V}$ has infinitely many sign changes.

q.e.d.

At this point let us introduce some further notation:

$$
\begin{aligned}
\gamma = -1: \quad a(-1) &:= \alpha^{\frac{1}{p-1}}, \\
a(i) &:= sup\{a \in (0, a(i-1)) | Z(b) > i \text{ for all } b < a\}, \\
A_i &:= \{a \in (0, a(i-1)) | Z(a) = i, L(a) \neq 0\}, \\
B_i &:= \{a \in (0, a(i-1)) | Z(a) > i\}, \qquad i = 0, \dots, i_0 - 1, \\
\gamma = 1: \quad a(i_0 - 1) &:= 0, \\
a(i) &:= inf\{a \in (a(i-1), \infty) | Z(b) > i \text{ for all } b > a\}, \\
A_i &:= \{a \in (a(i-1), \infty) | Z(a) = i, L(a) \neq 0\}, \\
B_i &:= \{a \in (a(i-1), \infty) | Z(a) > i\}, \qquad i \geq i_0.
\end{aligned}
$$

Lemma 3.2:
The sets A_i and B_i are open.

Proof:
The sets B_i are open due to the lower semicontinuity of Z as a function of a: If we assume an $a \in B_i$ and a sequence of a_j converging to a and $Z(a_j) < i$, then the continuous dependence of V on the initial data implies $Z(a) \leq Z(a_j) < i$; a contradiction.
To get the claimed property for the A_i, we have to transform the equation into a first order system, where the slowly decaying solutions can be identified with orbits entering a positively invariant region in the phase plane in finite time. For details we refer to [D2], chapter 2 and 3.

q.e.d.

The following lemma is the only place where we have to restrict the result to the nonsingular case:

Lemma 3.3:
Let $m \geq 1$ and $Z(a) = i$. Then there exists a neighbourhood of a in $\mathbb{R}$, such that for all b therein holds $Z(b) \leq i + 1$.

Proof:
Assume there is a sequence $a_j \longrightarrow a$ whose corresponding solutions satisfy $Z(a_j) \geq i+2$. Denoting the l.th zero by $z_l(a_j)$, we have by Proposition 3.5 of [4]

$$z_l(a_j) \leq C \qquad \text{for all } l \leq i+1$$

independently of j. (This result is only proved for the case $m \geq 1$.) As two zeroes can not accumulate at one point, we have

$$z_l(a_j) \longrightarrow z_l(a) \qquad \text{for all } l \leq i+1.$$

But this is a contradiction, as $V(.,a)$ possesses only i sign changes.

q.e.d.

Proof of Theorem A (ii), C (iii):
The disjoint sets A_i and B_i are defined in a way such that $A_i \cup B_i$ is exactly the complement of the set of solutions to (P_γ) of type $R(i)$ in the range $(a(i-1),\infty)$ for the initial value (in $(0,a(i-1))$, if $\gamma=-1$). Lemma 3.2 states that A_i and B_i are open, in Lemma 3.1 we proved that B_i is nonempty and Lemma 3.3 implies that there is a one-sided neighbourhood of $a(i-1)$, such that the corresponding solutions have at most i sign changes. If slowly decaying solutions do not exist in this neighbourhood, the solutions have to decay rapidly and we are done.
Otherwise A_i is nonempty, so we conclude that there is an $a \notin A_i \cup B_i$, i.e. a rapidly decaying solution of (P_γ) with exactly i sign changes.

q.e.d.

To extend the result to the singular case, we lack of a result similar to Lemma 3.3, which guarantees that A_i is not empty. Up to now this is only possible for nonnegative solutions due to the constant solution $U(\eta)=a(-1)$ and the continuous dependence of the solution on the initial value. So in general, if we assume the existence of a slow decaying solution, then we derive the existence of a fast decaying solution via the above technique.

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