Note on H-separable Frobenius extensions

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Note on H-separable Frobenius extensions

Kozo Sugano

Throughout this paper A will be a ring with the identity element 1, B a subring of A containing 1, C the center of A and D will be the centralizer of B in A. We will use the same notation as the author's previous paper [8]. For any subset X of A and any A-A-module M we will write

$$V_A(X) = \{a \in A \mid ax = xa \text{ for any } x \in X\}, \text{ and}$$

$$M^A = \{m \in M \mid am = ma \text{ for any } a \in A\}$$

respectively. Thus we have $D = A^B = V_A(B)$ and $(A \otimes_B A)^A = \{\Sigma a_i \otimes b_i \in A \otimes_B A \mid \Sigma xa_i \otimes b_i = \Sigma a_i \otimes b_i x \text{ for any } x \in A\}$.

A is said to be a Frobenius extension of B in the case where A is left B-f.g. (finitely generated) projective and there exists a left A and right B-isomorphism of A to $\mathrm{Hom}({}_BA, {}_BB)$. This is the case if and only if there exist finite x_k, $y_k \in A$ and $h \in \mathrm{Hom}({}_BA_B, {}_BB_B)$ such that $x = \Sigma h(xx_k)y_k = \Sigma x_k h(y_k x)$ hold for each $x \in A$. In this case we call the set $\{x_k, y_k, h\}$ a Frobenius system, and the map h a Frobenius homomorphism, of $A|B$ respectively.

Now for any $h \in \mathrm{Hom}({}_BA_B, {}_BB_B)$ we can define a multiplication among the elements of $A \otimes_B A$ by $(a \otimes b)(c \otimes d) = ah(bc) \otimes d$ for any $a, b, c, d \in A$ (See Proposition 4.1 [2]). This multiplication is well defined, and by this definition we can make $A \otimes_B A$ an associative ring which does not always have the identity element. On the other hand we can define the following maps

$$\phi_r : A \otimes_B A \longrightarrow \mathrm{Hom}(A_B, A_B)$$

$$\phi_l : A \otimes_B A \longrightarrow \mathrm{Hom}({}_BA, {}_BA)$$

by $\phi_r(a \otimes b)(x) = ah(bx)$ and $\phi_l(a \otimes b)(x) = h(xa)b$ for any $a, b, x \in A$. Direct calculation shows that ϕ_r and ϕ_l are ring, and opposite ring, homomorphisms respectively. Now we have

Lemma 1. For an $h \in \mathrm{Hom}({}_BA_B, {}_BB_B)$ the following conditions are equivalent;

(i) A is a Frobenius extension of B with h a Frobenius homomorphism

(ii) ϕ_r defined as above is an isomorphism

(iii) ϕ_l defined as above is an isomorphism

(vi) $A\otimes_B A$ has the identity element as a ring defined as above

If $\Sigma x_k \otimes y_k$ is the identity of $A\otimes_B A$, then $\{x_k, y_k, h\}$ is a Frobenius system of $A|B$.

Proof. (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) are well known (See e.g. the proof of Theorem 1 on page 94 [4]), and (ii)   (vi) and (iii)   (vi) are obvious. Let $\Sigma x_k \otimes y_k$ be the identity of $A\otimes_B A$. Then for any $x \in A$ we have $1\otimes x = (1\otimes x)(\Sigma x_k\otimes y_k) = \Sigma h(xx_k)\otimes y_k$ and $x\otimes 1 = (\Sigma x_k\otimes y_k)(x\otimes 1) = \Sigma x_k h(y_k x)\otimes 1$. Then we have $x = \Sigma h(xx_k)y_k = \Sigma x_k h(y_k x)$. Thus we have (vi) $\Rightarrow$ (i) and the last assertion.

There are some special maps as follows

η : $A\otimes_B A$ $\mathrm{Hom}({}_C D, {}_C A)$ $\eta(a\otimes b)(x) = axb$

η_l : $D\otimes_C A$ $\mathrm{Hom}({}_B A, {}_B A)$ $\eta_l(d\otimes a)(x) = dxa$

η_t : $D\otimes_C D$ $\mathrm{Hom}({}_B A_B, {}_B A_B)$ $\eta_t(d\otimes e)(x) = dxe$

for a, b, $x \in A$ and d, $e \in D$. η_r: $A\otimes_C D \longrightarrow \mathrm{Hom}(A_B, A_B)$ is defined similarly. η, η_l and η_r are A-A-maps and η_t is a D-D-map. A is an H-separable extension of B if and only if η is an isomorphism and D is C-f.g. projective. This is the case if and only if $1\otimes 1 = \Sigma d_i \Sigma x_{ij}\otimes y_{ij}$ for some $d_i \in D$ and $\Sigma x_{ij}\otimes y_{ij} \in (A\otimes_B A)^A$. We call such set $\{d_i, \Sigma x_{ij}\otimes y_{ij}\}$ an H-system of $A|B$. In the case where A is an H-separable extension of B all the above maps are isomorphisms. The next lemma is an immediate consequence of Corollary 3 [6].

Lemma 2. In the case where A is left B-f.g. projective the following conditions are equivalent:

(i) A is an H-separable extension of B

(ii) η_l is an isomorphism and D is C-f.g. projective.

(iii) There exists an A-A-split epimorphism of finite direct sum of copies of A to $\mathrm{Hom}({}_B A, {}_B A)$.

Let $\{x_k, y_k, h\}$ be a Frobenius system of $A|B$ and $\Sigma a_j \otimes b_j$ an arbitrary in $(A\otimes_B A)^A$. For any $x \varepsilon A$ we have $\Sigma xa_j h(b_j) = \Sigma a_j h(b_j x)$ and for any $b \varepsilon B$ $\Sigma ba_j h(b_j) = \Sigma a_j h(b_j b) = \Sigma a_j h(b_j)b$. Thus $\Sigma a_j h(b_j) \varepsilon D$, and $\Sigma a_j \otimes b_j = \Sigma x_k h(y_k a_j) \otimes h(b_j x_l) y_l = \Sigma x_k h(y_k a_j) h(b_j x_l) \otimes y_l = \Sigma x_k h(y_k a_j h(b_j x_l)) \otimes y_l = \Sigma a_j h(b_j x_l) \otimes y_l = \Sigma x_l a_j h(b_j) \otimes y_l \varepsilon \Sigma x_l D \otimes y_l$. Thus $(A\otimes_B A)^A \subset \Sigma x_k D \otimes y_k$. Since $\Sigma x_k \otimes y_k \varepsilon (A\otimes_B A)^A$, the converse inclusion is clear. Thus we have $(A\otimes_B A)^A = \Sigma x_k D \otimes y_k$ (See page 370 [1]). By this equality we have

Theorem 1. Let A be a Frobenius extension of B with a Frobenius system $\{x_k, y_k, \tilde{h}\}$. Then the following conditions are equivalent:

(i) A is an H-separable extension of B

(ii) There exist finite $d_i, e_i \varepsilon D$ such that $1\otimes 1 = \Sigma d_i x_k e_i \otimes y_k$ in $A\otimes_B A$

If there exist $d_i, e_i \varepsilon D$ which satisfy the condition of (ii), then we have the following assertions

(1) $\Sigma d_i \otimes e_i \varepsilon (D\otimes_C D)^D$, and we have $\tilde{h}(x) = \Sigma d_i x e_i$ for each $x \varepsilon A$

(2) We can obtain a Frobenius system $\{d_i, e_i, h\}$ of $D|C$, where h is defined by $h(d) = \Sigma x_k d y_k$ for each d in D.

Proof. Since A is H-separable extension of B if and only if $1\otimes 1 \varepsilon D(A\otimes_B A)^A$, we have the equivalence of (i) and (ii) immediately by $(A\otimes_B A)^A = \Sigma x_k D \otimes y_k$. Assume (ii). Then $\Sigma x_k e_i \otimes y_k \varepsilon (A\otimes_B A)^A$, and for each x in A we have $1\otimes x = \Sigma d_i x_k e_i \otimes y_k x = \Sigma d_i x x_k e_i \otimes y_k$, and $1\otimes \tilde{h}(x) = \Sigma d_i x x_k e_i \otimes \tilde{h}(y_k)$. But $\tilde{h}(y_k) \varepsilon B$ and $d_i \varepsilon D$. Hence we have $\tilde{h}(x) = \Sigma d_i x x_k e_i \tilde{h}(y_k) = \Sigma d_i x x_k \tilde{h}(y_k) e_i = \Sigma d_i x e_i \varepsilon B$. Now for each $d \varepsilon D$ we have $\Sigma dd_i x e_i = \Sigma d_i x e_i d$. Then since η_t is an isomorphism by (i), we have $\Sigma dd_i \otimes e_i = \Sigma d_i \otimes e_i d$ in $D\otimes_C D$. Thus we have proved (1). Next we will prove (2). The map h defined in (2) is in $\mathrm{Hom}({}_C D, {}_C C)$, since $\Sigma x_k D y_k \subset C$. Then since $\Sigma d_i x e_i \varepsilon V_A(D)$ for each $x \varepsilon A$, we have $\Sigma d_i h(e_i d) = \Sigma d_i x_k e_i d y_k = \Sigma dd_i x_k e_i y_k = d \Sigma h(x_k) y_k = d$. Similarly we have $\Sigma h(dd_i) e_i = d$. Thus $\{d_i, e_i, h\}$ is a Frobenius system of $D|C$.

Theorem 2. Assume $B = V_A(D)$, and let D be a Frobenius C-algebra with a Frobenius system $\{d_i, e_i, h\}$. Then the following three conditions are equivalent

(i) A is an H-separable extension of B.

(ii) There exist finite x_k, $y_k \varepsilon A$ such that $1\otimes 1 = \Sigma d_i \otimes x_k e_i y_k$ in $D\otimes_C A$.

(iii) There exists $\Sigma x_k \otimes y_k \varepsilon (A\otimes_B A)^A$ such that $1\otimes 1 = \Sigma d_i x_k e_i \otimes y_k$ in $A\otimes_B A$.

In the case where there exist x_k, $y_k \varepsilon A$ which satisfy the condition of (ii), we have the following assertions:

(1) $\Sigma x_k \otimes y_k \varepsilon (A\otimes_B A)^A$ and $h(d) = \Sigma x_k d y_k$ holds for any $d \varepsilon D$.

(2) A is a Frobenius extension of B with a Frobenius system $\{x_k, y_k, \tilde{h}\}$, where $\tilde{h}$ is defined by $\tilde{h}(x) = \Sigma d_i x e_i$ for each $x \varepsilon A$.

Proof. (ii)$\Longrightarrow$(i) is obvious, since $\Sigma x_k e_i \otimes y_k \varepsilon (A\otimes_B A)^A$. Assume A is an H-separable extension of B, and consider the isomorphism η introduced above. η induces the isomorphism $(A\otimes_B A)^A \cong \mathrm{Hom}(_CD, _CC)$. Therefore there exists $\Sigma x_k \otimes y_k \varepsilon (A\otimes_B A)^A$ such that $\eta(\Sigma x_k \otimes y_k) = h$. Then since $1 = \Sigma d_i h(e_i) = \Sigma d_i x_k e_i y_k$ and $\Sigma d_i x_k e_i \varepsilon V_A(D) = B$, we have in $A\otimes_B A$ that $1\otimes 1 = 1\otimes \Sigma d_i x_k e_i y_k = \Sigma d_i x_k e_i \otimes y_k$, while in $D\otimes_C A$ we have $1\otimes 1 = \Sigma d_i x_k e_i y_k \otimes 1 = \Sigma d_i \otimes x_k e_i y_k$, since $\Sigma x_k e_i y_k \varepsilon C$. Thus we have (i)$\Longrightarrow$(ii) and (i)$\Longrightarrow$(iii). Now assume (ii). Since $\Sigma d_i \otimes e_i \varepsilon (D\otimes_C D)^D$, we have also $\Sigma d_i a x_k e_i \varepsilon B$ for each $a \varepsilon A$ and k. Hence we can obtain left B-homomorphisms f_k of A to B defined by $f_k(a) = \Sigma d_i a x_k e_i$ for each $a \varepsilon A$, which satisfy $a = \Sigma d_i a x_k e_i y_k = \Sigma f_k(a) y_k$ for each $a \varepsilon A$. Thus $\{y_k, f_k\}$ forms a dual basis for $_BA$. Now consider the map η_1 of $D\otimes_C A$ to $\mathrm{Hom}(_BA, _BA)$ introduced above. For any $f \varepsilon \mathrm{Hom}(_BA, _BA)$ and $a \varepsilon A$ we have $\eta_1(\Sigma d_i \otimes x_k e_i f(y_k))(a) = \Sigma d_i a x_k e_i f(y_k) = \Sigma f_k(a) f(y_k) = f(\Sigma f_k(a) y_k) = f(a)$. Hence we have $\eta_1(\Sigma d_i \otimes x_k e_i f(y_k)) = f$, which means that η_1 is an epimorphism. Next suppose $\Sigma c_j \otimes a_j \varepsilon \mathrm{Ker}\,\eta_1$. Then $\Sigma c_j y_k a_j = 0$ and $\Sigma d_i \otimes x_k e_i \varepsilon (D\otimes_C D)^D$ for each k, and we have $\Sigma c_j \otimes a_j = \Sigma c_j d_i \otimes x_k e_i y_k a_j = \Sigma d_i \otimes x_k e_i c_j y_k a_j = 0$. Thus η_1 is a monomorphism, and we see that η_1 is an isomorphism. But D is C-f.g. projective, and A is left B-f.g. projective. Therefore A is an H-separable extension of B by Lemma 2. Thus we have proved (ii)$\Longrightarrow$(i). Now we will prove (1) and (2) of the second assertion under the condition of (ii). For any $d \varepsilon D$ we have $d\otimes 1 = \Sigma d d_i \otimes x_k e_i y_k$, and $h(d) = \Sigma h(d d_i) x_k e_i y_k = \Sigma x_k h(d d_i) e_i y_k = \Sigma x_k d y_k \varepsilon$

C. Then since $\sum x_k D y_k \subset C$, we have $h = \eta(\sum x_k \otimes y_k) \varepsilon \mathrm{Hom}(_C D, _C C) \cong (A \otimes_B A)^A$. Therefore we have $\sum x_k \otimes y_k \varepsilon (A \otimes_B A)^A$. Since $\sum d_i \otimes e_i \varepsilon (D \otimes_C D)^D$, we can define the map $\tilde{h}$ of A to B $(= V_A(D))$ by $\tilde{h}(x) = \sum d_i x e_i$ for each $x \varepsilon A$. Then $\sum \tilde{h}(ax_k) y_k = \sum d_i a x_k e_i y_k = \sum d_i x_k e_i y_k a = a$. Similarly we have $\sum x_k \tilde{h}(y_k a) = a$. Thus $\{x_k, y_k, \tilde{h}\}$ is a Frobenius system of $A|B$.

Proposition 1. Let A be a Frobenius extension of B with a Frobenius system $\{x_k, y_k, h\}$. Consider the following two conditions;

(i) A is an H-separable extension of B.

(ii) There exists $\sum d_i \otimes e_i \varepsilon (D \otimes_C D)^D$ with $1 \otimes 1 = \sum d_i \otimes x_k e_i y_k$ in $D \otimes_C A$

(i) always implies (ii). If $V_A(D) = B$, (i) and (ii) are equivalent.

Proof. Assume (i). By Theorem 1 there exists $\sum d_i \otimes e_i \varepsilon (D \otimes_C D)^D$ such that $1 \otimes 1 = \sum d_i x_k e_i \otimes y_k$ holds in $A \otimes_B A$. Then since $\sum x_k e_i y_k \varepsilon C$, we have $\sum d_i \otimes x_k e_i y_k = \sum d_i x_k e_i y_k \otimes 1 = 1 \otimes 1$ in $D \otimes_C A$. Thus we have (i) $\Longrightarrow$ (ii). Next let $B = V_A(D)$, and assume (ii). Then for the completely same reason as the proof of (ii) $\Longrightarrow$ (i) of Theorem 2 we see that the map η_1 of $D \otimes_C A$ to $\mathrm{End}(_B A)$ is an isomorphism. On the other hand we have $d = \sum d d_i x_k e_i y_k = \sum d_i x_k e_i d y_k$ for each $d \varepsilon D$, since $1 = \sum d_i x_k e_i y_k$ and $\sum d_i x_k e_i \varepsilon V_A(D)$. But $\sum x_k e_i d y_k \varepsilon C$, since $\sum x_k \otimes y_k \varepsilon (A \otimes_B A)^A$. Therefore if we define maps g_i by $g_i(d) = \sum x_k e_i d y_k$ for each $d \varepsilon D$ and each i, we have $g_i \varepsilon \mathrm{Hom}(_C D, _C C)$ and $d = \sum d_i g_i(d)$ for each $d \varepsilon D$. Hence D is C-f.g. projective. Then by Lemma 2 we have (i).

As is introduced in [8] when we write $\{A/B, S/T\}$, we mean that S is a ring containing A as subring with the common identity, and T is a subring of S containing B. In this case we will always write $\tilde{D} = V_S(T)$ and $\tilde{C} = V_S(S)$, the center of S. There exists also the canonical homomorphism $\tilde{\eta}$ of $S \otimes_T S$ to $\mathrm{Hom}(_{\tilde{C}}\tilde{D}, _{\tilde{C}}S)$ defined by $\tilde{\eta}(s \otimes t)(\tilde{d}) = s\tilde{d}t$ for $s, t \varepsilon S$ and $\tilde{d} \varepsilon \tilde{D}$. $\{A/B, S/T\}$ is said to have the centralizer property in the case where $V_S(A) = \tilde{C}$, $V_S(B) = \tilde{D}$ and $T = V_S(\tilde{D})$ hold. By the same argument as is stated on page 602 [7] we have the next lemma

Lemma 3. Let $\{A/B, S/T\}$ have the centralizer property, and assume that A is an H-separable extension of B. Then we have

(1) The canonical map $i\otimes i$ of $A\otimes_B A$ to $S\otimes_T S$ is a monomorphism, where i is the inclusion map of A to S.

(2) If $\tilde{\eta}$ is a monomorphism, then S is an H-separable extension of T, and we have $(i\otimes i)[(A\otimes_B A)^A]\subset(S\otimes_T S)^S$.

Proof. We will give the proof very briefly following the same lines as page 602 [7]. Since $D\otimes_C\tilde{C}=\tilde{D}$ via $d\otimes\tilde{c}\to d\tilde{c}$ for $d\in D$ and $\tilde{c}\in\tilde{C}$ we have a natural isomorphism ϕ of $\mathrm{Hom}(\tilde{c}\tilde{D},\tilde{c}S)$ to $\mathrm{Hom}(cD,cS)$ such that $\phi(f)(d)=f(d)$ for $f\in\mathrm{Hom}(\tilde{c}\tilde{D},\tilde{c}S)$ and $d\in D$ and the following commutaive diagram;

$$\begin{array}{ccccc}
A\otimes_B A & & \xrightarrow{\quad\eta\quad} & & \mathrm{Hom}(cD,cA)\\
\downarrow i\otimes i & & & & \downarrow i_*=\mathrm{Hom}(D,i)\\
S\otimes_T S & \xrightarrow{\tilde{\eta}} & \mathrm{Hom}(\tilde{c}\tilde{D},\tilde{c}S) & \longrightarrow & \mathrm{Hom}(cD,cS)
\end{array}$$

Since $i_*\eta$ is a monomorphism, so is $i\otimes i$. On the other hand we have

$$i_*\eta[(A\otimes_B A)^A]=i_*[(\mathrm{Hom}(cD,cA))^A]\subset[\mathrm{Hom}(cD,cS)]^A=\mathrm{Hom}(cD,cV_S(A))$$
$$=\mathrm{Hom}(cD,c\tilde{C})=\phi[\mathrm{Hom}(\tilde{c}\tilde{D},\tilde{c}\tilde{C})]$$

Therefore if $\tilde{\eta}$ is a monomorphism, we have $(i\otimes i)[(A\otimes_B A)^A]\subset(S\otimes_T S)^S$. Then since $D\subset\tilde{D}$ each H-system of $A|B$ is an H-system of $S|T$. Thus we have (2).

Theorem 3. Let $\{A/B,\ S/T\}$ have the centralizer property, and assume that A is an H-separable Frobenius extension of B with a Frobenius system $\{x_k,\ y_k,\ \tilde{h}\}$. Then S is also an H-separable Frobenius extension of T with a Frobenius system $\{x_k,\ y_k,\ \tilde{h}^*\}$ such that $\tilde{h}^*|A=\tilde{h}$.

Proof. Since A is left (and right) B-f.g. projective, S is an H-separable extension of T by Theorem 1.1 [8]. By Theorem 1 there exists $\Sigma d_i\otimes e_i\in(D\otimes_C D)^D$ such that $1\otimes1=\Sigma d_ix_ke_i\otimes y_k$ in $A\otimes_B A$, and $\{d_i,\ e_i,\ h\}$ is a Frobenius system of $D|C$, where h is defined by $h(d)=\Sigma x_kdy_k$ for $d\in D$. Then since $\tilde{D}=D\tilde{C}$ and $\Sigma x_k\otimes y_k\in(A\otimes_B A)^A\subset(S\otimes_T S)^S$ by Lemma 3, if we define a map h^* by $h^*(\tilde{d})=\Sigma x_k\tilde{d}y_k$ for $\tilde{d}\in\tilde{D}$, we have $h^*\in\mathrm{Hom}(\tilde{c}\tilde{D},\tilde{c}\tilde{C})$ with $h^*|D=h$, and $\{d_i,\ e_i,\ h^*\}$ forms a Frobenius system of $\tilde{D}|\tilde{C}$. On the other hand we have $1\otimes1=\Sigma d_i\otimes x_ke_iy_k$ in $A\otimes_B A$ by the proof of (i)$\Rightarrow$(ii) Proposition 3. The same equality holds also in $S\otimes_T S$. Then by Theorem 2 we see that

$\{x_k, y_k, \tilde{h}^{\cdot}\}$ is a Frobenius system of $S|T$, where $\tilde{h}^{\cdot}(x) = \Sigma d_i x e_i$ for any $x \varepsilon S$. We have $\tilde{h}^{\cdot}|A = \tilde{h}$, since $\tilde{h}(x) = \Sigma d_i x e_i$ for $x \varepsilon A$ by Theorem 1 (2).

The next proposition which is the improvement of Theorems 4 and 5 [5] is a modification of Theorems 1 and 2 [3] which were proved by using H-system.

Proposition 2. Let A be an H-separable extension of B. Then we have

(1) Assume that A is a Frobenius extension of B with a Frobenius system $\{x_k, y_k, \tilde{h}\}$, and define a map h by $h(d) = \Sigma x_k d y_k$ for any $d \varepsilon D$. Then for any $d_i, e_i \varepsilon D$ such that $\eta_t(\Sigma d_i \otimes e_i) = \tilde{h}$ $\{d_i, e_i, h\}$ is a Frobenius system of $D|C$.

(2) Assume that D is a Frobenius C-algebra with a Frobenius system $\{d_i, e_i, h\}$, and define a map $\tilde{h}$ by $\tilde{h}(x) = \Sigma d_i x e_i$ for each $x \varepsilon A$. Then for any $x_k, y_k \varepsilon A$ such that $\eta(\Sigma x_k \otimes y_k) = h$ $\{x_k, y_k, \tilde{h}\}$ is a Frobenius system of $A|B'$, where $B' = V_A(V_A(B))$.

Proof. (1). Let $d \varepsilon D$. We have $\Sigma d_i h(e_i d) = \Sigma d_i x_k e_i d y_k = \Sigma d d_i x_k e_i y_k = \Sigma d h(x_k) y_k = d$. Similarly we have $d = \Sigma h(d d_i) e_i$. (2). Let $x \varepsilon A$. Then we have $\Sigma x_k h(y_k x) = \Sigma x_k d_i y_k x e_i = \Sigma x x_k d_i y_k e_i = \Sigma x h(d_i) e_i = x$. Similarly we have $x = \Sigma h(x x_k) y_k$. Ovbiously h is a B'-B'-map of A to B'.

References

[1] K. Hirata and K. Sugano; On semisimple extensions and separable extensions over non commutative rings, J. Math. Soc. Japan, 18 (1966), 360-373.

[2] L. Kadison; Global dimension, tower of algebras, and Jones index of split separable subalgebras with unitality condition, preprint.

[3] T. Nakamoto; On QF-extensions in an H-separable exension, Proc. Japan Accad., 50 (1974), 440-443.

[4] T. Onodera; Some studies on projective Frobenius extensions, J. Fac. Sci. Hokkaido University, 18 (1964), 89-107.

[5] K. Sugano; Separable extensions and Frobenius extensions; Osaka J. Math., 7 (1970), 291-299.

[6] K. Sugano; Note on separability of endomorphism rings, J. Fac. Sci. Hokkaido University, 21 (1971), 196-208.

[7] K. Sugano; On bicommutators of modules over H-separable extension rings II, Hokkaido Math. J., 20 (1991), 601-608.

[8] K. Sugano; On bicommutators of modules over H-separable extension rings III, Hokkaido Math. J., 23 (1994).

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