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DILATION METHOD AND SMOOTHING EFFECT OF THE SCHRÖDINGER EVOLUTION GROUP

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ABSTRACT. We reexamine the mechanism of smoothing effects of the Schrödinger evolution group in the weighted Sobolev spaces by using the generator of space-time dilations instead of Galilei transformations.

1. Introduction

In this paper we prove smoothing effects of the Schrödinger evolution groups in the weighted Sobolev spaces on the basis of the commutation relations for the generator of space-time dilations instead of Galilei transformations. Let $H = -(1/2)\Delta + V$ be a Schrödinger operator in $L^2 = L^2(\mathbf{R}^n)$ with potential V, the multiplication operator by the real-valued function V on $\mathbf{R}^n$. Our main attention is concentrated on the smoothing effects of the unitary propagator e^{-itH} in L^2. Looking through an extensive literature on the smoothing effects for Schrödinger evolution equations[1-4, 7, 9-13, 15, 16, 20-27, 29-31, 33-36, 38-42], we find that the methods are classified into three categories. The first method is related to the Strichartz type estimate, which is formulated in the L^p setting and depends essentially on the mapping properties of the Fourier transform with respect to the space variables. It implies in particular that for any data in L^2 the corresponding solution belongs for almost all times to $L^2 \cap L^p$ for all p with $2 < p < 2n/(n-2)$. The second method originates in the Kato smoothness estimate, which is formulated in the L^2 setting and depends essentially on the mapping properties of the Fourier transform in the time variable. It implies in particular that for any data in L^2 the corresponding solution belongs for almost all times to the local Sobolev space $H^{1/2}_{loc}$. Roughly speaking, both results above ensure that for almost all times propagators for Schrödinger evolution equations improve the spatial local regularity of the initial data. As for the time variable, we remark here that the restriction of almost everywhere convergence cannot be reduced to the pointwise convergence. In contrast, the third method holds pointwisely in time and depends exclusively on the commutation relations between the unperturbed operator $L = i\partial_t + (1/2)\Delta$

and the generators of transformations which leave the associated Lagrangean invariant. To be specific the only operator that has been used for that purpose is the generator $J = x + it\nabla$ of Galilei transformations, which commutes with L and provides the spatial local regularity through the formula $(x + it\nabla)U(t) = U(t)x$, where $U(t) = \exp(i(t/2)\Delta)$. It implies in particular that for any data $\phi \in L^2$ with $|x|^m\phi \in L^2$, m being a positive integer, the corresponding solution belongs to the local Sobolev space H^m_{loc} away from the initial time. In other words the last method reveals the connection between the spatial decay at infinity of the data measured by the position operator x and the regularity of the corresponding solutions measured by the order of local Sobolev spaces. Despite the fact that smoothing effects of pointwise times have been recognized as implications of the spatial decay at infinity, it does not logically mean that the decay at infinity is the only source of the local regularity of solutions. In fact the purpose of this paper is to prove that this is the case. We prove in particular that for any data $\phi \in L^2$ with $(x \cdot \nabla)^m\phi \in L^2$, the corresponding solution belongs to H^{2m}_{loc} away from the initial time. This reveals the connection between the regularity of the data with respect to the self-adjoint operator $A = (1/2i)(x \cdot \nabla + \nabla \cdot x)$ and that of the corresponding solutions measured by the order of local Sobolev spaces. As a corollary, it turns out that the propagator e^{-itH} maps C^∞-vectors for A into smooth functions for any $t \in \mathbf{R}\backslash\{0\}$. Our argument exploits the generator $P = x \cdot \nabla + 2t\partial_t$ of space-time dilations, which satisfies the commutation relation $[L, P] = 2L$. In this sense our method falls into the third category. We derive the local regularity from the time dilation part of P in the associated a priori estimates. To this end we prove that in the L^2 space with weight of negative index the effect of the space dilation part of P amounts to an infinitesmally small perturbation with respect to the time dilation part. As is expected by equation, the time derivative deserves the second derivatives in space, so that actually we prove that $\langle x\rangle^{-2}x \cdot \nabla$ is infinitesmally small with respect to $\langle x\rangle^{-2}\Delta$. The description of the resulting smoothing effect is therefore efficiently given by the use of the weighted Sobolev spaces, which traces back to Jensen[20]. The operator P has recently been used in the study of the nonlinear Schrödinger equation[5, 8, 9, 12]. To state our result precisely, we introduce several notations.

Notation. $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$, $\nabla = (\partial_1, \ldots, \partial_n)$, $x = (x_1, \ldots, x_n)$; $\partial^\alpha = \Pi_{j=1}^n \partial^{\alpha_j}$, $x^\alpha = \Pi_{j=1}^n x_j^{\alpha_j}$; L^p denotes the usual Lebesgue space on $\mathbf{R}^n$ with norm $\|\cdot\|_p$; $\|\cdot\|$ and $(\cdot,\cdot)$ denote the norm and scalar product in L^2, respectively; $H^{m,s}$ denotes the weighted Sobolev space defined by $H^{m,s} = \{\psi \in \mathcal{S}'; \|\psi\|_{m,s} \equiv \|\langle x\rangle^s(1-\Delta)^{m/2}\psi\| < \infty\}$ for m, $s \in \mathbf{R}$, where $\mathcal{S}'$ denotes the space of temperate distributions and $\langle x\rangle = (1 + |x|^2)^{1/2}$; $D_m = D(A^m) = \{\psi \in L^2; A^j\psi \in L^2$ for all j with $1 \le j \le m\}$ denotes the domain of the m-th power of the self-adjoint dilation operator $A = (1/2i)(x \cdot \nabla + \nabla \cdot x)$, equipped with the graph norm, which is equivalent to $|\!|\!|\psi|\!|\!|_m \equiv \sum_{j=0}^m \|(x \cdot \nabla)^j\psi\|$ and to $\|\psi\| + \|(x \cdot \nabla)^m\psi\|$.

With these notations we state our main results. In this paper we prove

Theorem 1. *Let $m \geq 1$ be an integer. Suppose that $\partial^\alpha V \in L^\infty$ for all α with $|\alpha| \leq 2m-2$ and that $(x \cdot \nabla)^j V \in L^\infty$ for all j with $1 \leq j \leq m$. Then e^{-itH} is bounded from D_m to $H^{2m,-2m}$ for any $t \in \mathbf{R}\backslash\{0\}$ with the associated operator norm bounded by $C(1+|t|^{-2m})$.*

Combining Theorem 1 with the results in [27] (see also [20, 26, 29]), we obtain

Theorem 2. *Let $m \geq 1$ be an integer. In addition to the assumptions in Theorem 1, if $x \cdot \nabla\partial^\alpha V \in L^\infty$ for all α with $|\alpha| \leq 2m-2$ then e^{-itH} is bounded from $D_m + H^{0,2m}$ to $H^{2m,-2m}$ for any $t \in \mathbf{R}\backslash\{0\}$ with the associated operator norm bounded by $C(1+|t|^{-2m})$.*

Remark 1. *The condition that $|\partial^\alpha V(x)| \leq C_\alpha \langle x\rangle^{-|\alpha|}$ for all α may be the simplest one that ensures all the assumptions above. The following potentials illustrate other examples.*

(1) $V(x) = x^\alpha/|x|^{|\alpha|}$ *with $\alpha \neq 0$ satisfies the assumptions with $m = 1$, while the first derivatives are not bounded at the origin. A general example in this direction is a smooth function on $\mathbf{R}^n\backslash\{0\}$ with homogeneity of degree zero. The asymptotic behavior of propagators for Schrödinger operators with such a class of potentials is studied by Herbst[17].*
(2) $V(x) = \sin\log(x^\alpha)$ *with $\alpha \neq 0$ satisfies the assumptions with $m = 1$, while the first derivatives need not stay bounded along the axes.*
(3) $V(x) = \cos(|x|/(1+|x|))$ *satisfies the assumptions with $m = 1, 2$.*
(4) $V(x) = \exp(-|x|^\sigma)$ *with $\sigma > 0$ satisfies the assumptions with $m \leq \sigma/2 + 1$. If σ is even, V satisfies all the assumptions.*

Remark 2. *Let V satisfy all the assumptions in Theorems 1 and 2, respectively. Then for any $t \neq 0$ the propagator e^{-itH} maps D_∞ and $D_\infty + H^{0,\infty}$ into C^∞, respectively, where $D_\infty = \bigcap_{m\geq 0} D_m$ is the space of C^∞-vectors for A and $H^{0,\infty} = \bigcap_{m\geq 0} H^{0,m}$.*

Remark 3. $\phi(x) = |x|^{-n/2+\epsilon}(1+|x|)^{-2\epsilon}$ *with $\epsilon > 0$ satisfies $\phi \in D_\infty$ while $\phi \notin H^{0,\epsilon}$. By the preceding remark, $e^{-itH}\phi \in C^\infty$ for any $t \neq 0$, provided that V satisfies all the assumptions in Theorem 1. This example falls beyond the scope of the previous literature even when restricted to the case $V = 0$.*

2. Proof of Theorem 1

The proof of Theorem 1 requires the following lemma.

Lemma. *Let $m \geq 1$ be an integer. Then there exists a constant C such that for* *all* $\psi \in H^{2m,-2m}$

(1)

$$\|\psi\|_{2m-1,-2m+1} \leq C\left(\sum_{j=0}^{m-1}\|\Delta^j\psi\|_{0,-2j}\right)^{1/2}\|\Delta^m\psi\|_{0,-2m}^{1/2} + C\sum_{j=0}^{m-1}\|\Delta^j\psi\|_{0,-2j},$$

(2)
$$\sum_{j=0}^{2m}\|\psi\|_{j,-j} \leq C\sum_{k=0}^{m}\|\Delta^k\psi\|_{0,-2k},$$

(3)
$$\|\psi\|_{m,-m} \leq C\|\psi\|^{1/2}\left(\sum_{j=0}^{m}\|\Delta^j\psi\|_{0,-2j}\right)^{1/2}.$$

Proof. By density it suffices to prove the estimates for $\psi \in \mathcal{S}$. Let α satisfy $|\alpha| = 2m-2$. By the relation $|\partial_k\partial^\alpha\psi|^2 = (1/2)\partial_k^2|\partial^\alpha\psi|^2 - \mathrm{Re}(\overline{\partial^\alpha\psi}\partial_k^2\partial^\alpha\psi)$, integration by parts, and the Schwarz inequality, we obtain

$$\begin{aligned}\|\partial_k\partial^\alpha\psi\|_{0,-2m+1}^2 &= \frac{1}{2}\left(\langle x\rangle^{-4m+2}, \partial_k^2|\partial^\alpha\psi|^2\right) - \mathrm{Re}\left(\langle x\rangle^{-4m+2}\partial^\alpha\psi, \partial_k^2\partial^\alpha\psi\right),\\ &\leq C\|\partial^\alpha\psi\|_{0,-2m+2}^2 + \|\partial^\alpha\psi\|_{0,-2m+2}\|\partial_k^2\partial^\alpha\psi\|_{0,-2m}.\end{aligned}$$

This implies (1) since

(4)
$$\sum_{|\beta|=2l}\|\partial^\beta\psi\|_{0,-2l} \leq C\|\psi\|_{2l,-2l} \leq C\sum_{j=0}^{l}\|\Delta^j\psi\|_{0,-2l} \leq C\sum_{j=0}^{l}\|\Delta^j\psi\|_{0,-2j}$$

and the norm defined by $\|\psi\|_{k,s}^* = \|\psi\|_{0,s} + \sum_{|\alpha|=k}\|\partial^\alpha\psi\|_{0,s}$ is an equivalent norm in $H^{k,s}$ with $k \in \mathrm{N}$, $s \in \mathbf{R}$[37]. Similarly, (2) follows from (1) and (4). The proof of (3) proceeds similarly since for $|\alpha| = m$

$$\begin{aligned}\|\partial^\alpha\psi\|_{0,-m}^2 &= (\partial^\alpha\psi, \langle x\rangle^{-2m}\partial^\alpha\psi)\\ &= (-1)^{|\alpha|}(\psi, \partial^\alpha(\langle x\rangle^{-2m}\partial^\alpha\psi))\\ &\leq \|\psi\|\|\partial^\alpha(\langle x\rangle^{-2m}\partial^\alpha\psi)\| \leq C\|\psi\|\,\|\psi\|_{2m,-2m}. \qquad \square\end{aligned}$$

Proof of Theorem 1. We first regularize V by a mollifier. Let $\rho \in C_0^\infty(\mathbf{R}^n)$ satisfy $\rho \geq 0$ and $\|\rho\|_1 = 1$ and let $\rho_\epsilon(x) = \epsilon^{-n}\rho(\epsilon^{-1}x)$ for $\epsilon > 0$. Let $V_\epsilon = V * \rho_\epsilon$ and let $u_\epsilon(t) = \exp(-it(H_0 + V_\epsilon))\phi$ for $\phi \in \mathcal{S}$. Then $u_\epsilon \in C^\infty(\mathbf{R}; \mathcal{S})$ since $\partial^\alpha V_\epsilon \in L^\infty$ for all α[18, 19, 28, 29, 32]. We prove that

(5.m) $\quad \|\Delta^j u_\epsilon(t)\|_{0,-2j} \leq C(1 + |t|^{-2j})|\!|\!|\phi|\!|\!|_j,\ 0 \leq j \leq m,\ t \in \mathbf{R}\backslash\{0\},$

(6.m) $\quad \|P^j u_\epsilon(t)\| \leq C(1 + |t|^j)|\!|\!|\phi|\!|\!|_j,\ 0 \leq j \leq m,\ t \in \mathbf{R}\backslash\{0\},$

with constant C independent of ϵ, t and ϕ. We prove (5.m) and (6.m) by induction on m. The case $m = 0$ follows from the unitarity of the propagator $\exp(-it(H_0 + V_\epsilon))$. Let $m \geq 1$ and suppose that (5.$m-1$)and (6.$m-1$) hold. We proceed to the case m. By the commutation relation $[L, P] = 2L$, we have, omitting the time variable t,

$$
\begin{aligned}
LP^m u_\epsilon &= (P+2)^m L u_\epsilon = (P+2)^m (V_\epsilon u_\epsilon) \\
&= \sum_{j=0}^{m} \binom{m}{j} 2^{m-j} P^j (V_\epsilon u_\epsilon) \\
&= V_\epsilon P^m u_\epsilon + \sum_{k=0}^{m-1} \sum_{j=0}^{m-k} \binom{m}{k} \binom{m-k}{j} 2^{m-j-k} (P^j V_\epsilon) P^k u_\epsilon .
\end{aligned}
$$

Taking the imaginary part of the scalar product in L^2 between $LP^m u_\epsilon$ with the last form and $P^m u_\epsilon$ and integrating by parts, we have

$$
(7) \qquad \frac{d}{dt} \|P^m u_\epsilon\|^2 = \sum_{k=0}^{m-1} \sum_{j=0}^{m-k} \binom{m}{k} \binom{m-k}{j} 2^{m-j-k+1} \mathrm{Im}((P^j V_\epsilon) P^k u_\epsilon, P^m u_\epsilon).
$$

Noting that $(x \cdot \nabla) V * \rho_\epsilon = (x \cdot \nabla V) * \rho_\epsilon + V * \nabla \cdot (x \rho_\epsilon)$, we obtain

$$
(8) \qquad \sum_{j=0}^{m} \|P^j V_\epsilon\|_\infty \leq C \sum_{j=0}^{m} \|(x \cdot \nabla)^j V\|_\infty .
$$

with constant C independent of ϵ. By (7), (8), and (6.$m-1$),

$$
\begin{aligned}
(9) \qquad \left| \frac{d}{dt} (\|P^m u_\epsilon\|^2 + \delta) \right| &\leq C \sum_{k=0}^{m-1} \|P^k u_\epsilon\| \|P^m u_\epsilon\| \\
&\leq C(1 + |t|^{m-1}) |\!|\!| \phi |\!|\!|_{m-1} (\|P^m u_\epsilon\|^2 + \delta)^{1/2},
\end{aligned}
$$

for any $\delta > 0$. Dividing both sides of (9) by $(\|P^m u_\epsilon\| + \delta)^{1/2}$, integrating in t, and letting $\delta \downarrow 0$, we obtain (6.m). We next prove (5.m) by making use of (5.$m-1$) and (6.m) . By expanding the m-th power of P, $(2t)^m \partial_t^m$ is expressed as

$$
\begin{aligned}
(2t)^m \partial_t^m = & P^m - (x \cdot \nabla)^m - \sum_{k=0}^{m-1} a_{m,k} t^k \partial_t^k \\
& - \sum_{j=1}^{m-1} \binom{m}{j} (x \cdot \nabla)^{m-j} \left((2t)^j \partial_t^j + \sum_{k=1}^{j-1} a_{j,k} t^k \partial_t^k \right)
\end{aligned}
$$

with constants $a_{j,k}$, $1 \le j \le m$ and $1 \le k \le m-1$, where we follow the convention $\sum_{k=1}^{0}(\cdots) = 0$. Letting (10) act on u_ϵ, using the equation $i\partial_t u_\epsilon = (H_0 + V_\epsilon)u_\epsilon$, and estimating the result, we have

$$
\begin{aligned}
(11)\qquad & (2|t|)^m \|(H_0+V_\epsilon)^m u_\epsilon\|_{0,-2m} \\
\le & \|P^m u_\epsilon\| + \|(x\cdot\nabla)^m u_\epsilon\|_{0,-2m} + C\sum_{k=1}^{m-1} |t|^k \|(H_0+V_\epsilon)^k u_\epsilon\|_{0,-2m} \\
& + C\sum_{j=1}^{m-j}\sum_{k=1}^{j} |t|^k \|(x\cdot\nabla)^{m-j}(H_0+V_\epsilon)^k u_\epsilon\|_{0,-2m}.
\end{aligned}
$$

By expanding the j-th power of $H_0 + V_\epsilon$,

$$
\begin{aligned}
(12)\qquad & (H_0+V_\epsilon)^j \\
= & \sum_{k=0}^{j} \binom{j}{k} V_\epsilon^{j-k} H_0^k \\
& + \sum_{k=1}^{j-1} \sum_{\substack{|\beta|\le 2(j-k)-1 \\ |\alpha_1+\cdots+\alpha_k+\beta| = 2(j-k)}} C(j,k,\{\alpha_l\},\beta)\left(\prod_{l=1}^{k} \partial^{\alpha_l} V_\epsilon\right)\partial^\beta
\end{aligned}
$$

and therefore by (2)

$$
\begin{aligned}
(13)\qquad & \|((H_0+V_\epsilon)^j - H_0^j)u_\epsilon\|_{0,-2j} \\
\le & C\sum_{k=0}^{j-1} \|V_\epsilon\|_\infty^{j-k} \|H_0^k u_\epsilon\|_{0,-2k} \\
& + C\sum_{k=1}^{j-1}\left(\sum_{|\alpha|\le 2(j-k)} \|\partial^\alpha V_\epsilon\|_\infty\right)^k \sum_{|\beta|\le 2(j-k)-1} \|\partial^\beta u_\epsilon\|_{0,-|\beta|} \\
\le & C\left(1 + \sum_{|\alpha|\le 2j-2} \|\partial^\alpha V\|_\infty^j\right)\sum_{k=0}^{j-1} \|\Delta^k u_\epsilon\|_{0,-2k}.
\end{aligned}
$$

By (13) and $(5.m-1)$,

$$
\begin{aligned}
(14)\qquad & \|(t\Delta)^m u_\epsilon\|_{0,-2m} \\
& \le (2|t|)^m \|(H_0+V_\epsilon)^m u_\epsilon\|_{0,-2m} + C(|t|^m + |t|^{-m+2})\|\phi\|_{m-1},
\end{aligned}
$$

$$
\begin{aligned}
(15)\qquad \sum_{k=1}^{m-1} |t|^k \|(H_0+V_\epsilon)^k u_\epsilon\|_{0,-2m} & \le C\sum_{k=1}^{m-1} |t|^k \sum_{j=0}^{k} \|\Delta^j u_\epsilon\|_{0,-2j} \\
& \le C(|t|^{m-1} + |t|^{-m+1})\|\phi\|_{m-1}.
\end{aligned}
$$

Combining (6.m), (11), (14) and (15), we have

$$
(16)\qquad
\begin{aligned}
&\|(t\Delta)^m u_\epsilon\|_{0,-2m}\\
\leq & C(|t|^m + |t|^{-m+1})\|\!|\phi|\!\|_m + \|(x\cdot\nabla)^m u_\epsilon\|_{0,-2m}\\
&+ C\sum_{j=1}^{m-1}\sum_{k=1}^{j}|t|^k\|(x\cdot\nabla)^{m-j}(H_0+V_\epsilon)^k u_\epsilon\|_{0,-2m}.
\end{aligned}
$$

By (2), (3), (5.$m-1$), the middle term of the RHS of (16) is estimated as

$$
(17)\qquad
\begin{aligned}
&C\sum_{|\alpha|\leq m}\|\partial^\alpha u_\epsilon\|_{|\alpha|-2m} \leq C\sum_{|\alpha|\leq m}\|\partial^\alpha u_\epsilon\|_{0,-|\alpha|}\\
\leq & C\sum_{j=0}^{m-1}\|u_\epsilon\|_{j,-j} + C\|u_\epsilon\|_{m,-m}\\
\leq & C\sum_{k=0}^{[m/2]}\|\Delta^k u_\epsilon\|_{0,-2k} + C(|t|^{-m}\|\phi\|)^{1/2}\left(\sum_{j=0}^{m}|t|^m\|\Delta^j u_\epsilon\|_{0,-2j}\right)^{1/2}\\
\leq & \frac{1}{4}\|(t\Delta)^m u_\epsilon\|_{0,-2m} + C(1+|t|^m)\sum_{j=0}^{m-1}\|\Delta^j u_\epsilon\|_{0,-2j} + C|t|^{-m}\|\phi\|\\
\leq & \frac{1}{4}\|(t\Delta)^m u_\epsilon\|_{0,-2m} + C(|t|^m + |t|^{-m})\|\!|\phi|\!\|_{m-1},
\end{aligned}
$$

where $[m/2]$ is the largest integer less than or equal to $m/2$. By (1), (2), (12) and (5.$m-1$), the last term of the RHS of (16) is estimated as

$$
(18)\qquad
\begin{aligned}
&C\sum_{j=1}^{m-1}\sum_{k=1}^{j}\sum_{|\alpha|\leq m-j}|t|^k\|\partial^\alpha(H_0+V_\epsilon)^k u_\epsilon\|_{0,|\alpha|-2m}\\
\leq & C\sum_{j=1}^{m-1}\sum_{k=1}^{j}\sum_{|\alpha|\leq m-j}\sum_{|\beta|\leq|\alpha|+2k}|t|^k\|\partial^\beta u_\epsilon\|_{0,|\alpha|-2m}\\
\leq & C\sum_{j=1}^{m-1}\sum_{k=1}^{j}\sum_{|\alpha|\leq m-j}\sum_{|\beta|\leq|\alpha|+2k}|t|^k\|\partial^\beta u_\epsilon\|_{0,-|\beta|}\\
\leq & C\sum_{j=1}^{m-1}\sum_{k=1}^{j}\sum_{l=0}^{m-j+2k}|t|^k\|u_\epsilon\|_{l,-l}\\
\leq & C\sum_{j=1}^{m-2}\sum_{l=0}^{m-1}(1+|t|^j)\|u_\epsilon\|_{l,-l} + C|t|^{m-1}\|u_\epsilon\|_{2m-1,-2m+1}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C(1+|t|^{m-1})\sum_{j=0}^{m-1}\|\Delta^j u_\epsilon\|_{0,-2j} \\
&\qquad + C\left(\sum_{j=0}^{m-1}|t|^{m-2}\|\Delta^j u_\epsilon\|_{0,-2j}\right)^{1/2}\|(t\Delta)^m u_\epsilon\|_{0,-2m}^{1/2} \\
&\leq \frac{1}{4}\|(t\Delta)^m u_\epsilon\|_{0,-2m} + C(|t|^m+|t|^{-m})|\!|\!|\phi|\!|\!|_{m-1}.
\end{aligned}
$$

By (16), (17) and (18), we have $\|(t\Delta)^m u_\epsilon\|_{0,-2m} \leq C(|t|^m+|t|^{-m})|\!|\!|\phi|\!|\!|_m$ and hence (5.m), as required. We note that (5.m) and (6.m) extends to $\phi \in D_m$ since $\mathcal{S}$ is dense in D_m. By (5.m) we have

$$
\|u_\epsilon(t)\|_{2m,-2m} \leq C(1+|t|^{-2m})|\!|\!|\phi|\!|\!|_m,\ \phi \in D_m,\ t \in \mathbf{R}\backslash\{0\}, \tag{19}
$$

where C is independent of ϵ, t and ϕ. Finally we consider (19) in the limit $\epsilon \downarrow 0$. For $\psi \in L^2$, $V_\epsilon\psi \to V\psi$ almost everwhere as $\epsilon \downarrow 0$ and $|(V_\epsilon - V)\psi| \leq 2\|V\|_\infty|\psi|$. This implies $\|(V_\epsilon - V)\psi\| \to 0$ as $\epsilon \downarrow 0$, which in turn implies

$$
\begin{aligned}
&\|((H_0+V_\epsilon+i)^{-1}-(H_0+V+i)^{-1})\psi\| \\
&= \|(H_0+V_\epsilon+i)^{-1}(V_\epsilon-V)(H_0+V+i)^{-1}\psi\| \\
&\leq \|(V_\epsilon-V)(H_0+V+i)^{-1}\psi\| \to 0 \quad \text{as} \quad \epsilon\downarrow 0.
\end{aligned}
$$

The Trotter-Kato theorem shows that $u_\epsilon(t) \to e^{-itH}\phi$ in L^2 locally uniformly on t-intervals as $\epsilon \downarrow 0$. Since the operator $\langle x\rangle^{-2m}(1-\Delta)^m$ is closed in L^2, we conclude from (19) that $e^{-itH}\phi \in H^{2m,-2m}$ for $t \in \mathbf{R}\backslash\{0\}$ and $\|e^{-itH}\phi\|_{2m,-2m} \leq C(1+|t|^{-2m})|\!|\!|\phi|\!|\!|_m$. □

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