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DILATION METHOD AND SMOOTHING EFFECT OF SOLUTIONS TO THE BENJAMIN-ONO EQUATION

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Abstract. In this paper we study smoothing effects of solutions to the Benjamin-Ono equation

$$\text{(BO)} \qquad \begin{cases} \partial_t u + u\partial_x u + H\partial_x^2 u = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(0) = \phi, & x \in \mathbb{R}, \end{cases}$$

where H is the Hilbert transform defined by

$$(Hf)(x) = p.v.\frac{1}{\pi}\int \frac{f(y)}{x-y}dy.$$

We prove that if $\phi \in H^4$ and $(x\partial_x)^4\phi$, then the solution u of (BO) belongs to $L^\infty_{loc}(\mathbb{R}\backslash\{0\}; H^{8,-4})$, where

$$H^{m,s} = \{f \in L^2; \|(1+x^2)^{\frac{s}{2}}(1-\partial_x^2)^{\frac{m}{2}} f\|_{L^2} < \infty\}.$$

§1 **Introduction.** The purpose of this paper is to study smoothing properties of solutions to the Benjamin-Ono equation

$$\text{(BO)} \qquad \begin{cases} \partial_t u + u\partial u + H\partial^2 u = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}, \\ u(0,x) = \phi(x), & x \in \mathbb{R}, \end{cases}$$

where $\partial_t = \partial/\partial t$, $\partial = \partial_x = \partial/\partial x$, $u = u(t,x)$ is real valued and H denotes the Hilbert transform

$$(Hf)(x) = p.v.\frac{1}{\pi}\int \frac{f(y)}{x-y}dy = \lim_{\epsilon\to 0}\frac{1}{\pi}\int_{|x-y|\geq\epsilon}\frac{f(y)}{x-y}dy.$$

H can be written as follows :

$$H = \mathcal{F}^{-1}(-isgn(\xi))\mathcal{F} = -(-\partial^2)^{\frac{1}{2}}\partial,$$

where

$$(\mathcal{F}f)(\xi) = \frac{1}{\sqrt{2\pi}}\int f(x)e^{-ix\xi}dx, \quad sgn(\xi) = \begin{cases} 1 & \xi > 0 \\ -1 & \xi \leq 0. \end{cases}$$

Hence the initial value problem (BO) is equivalent to

$$\begin{cases} \partial_t u + u\partial u - (-\partial^2)^{\frac{1}{2}}\partial u = 0, \\ u(0,x) = \phi(x). \end{cases}$$

The integro-differential equation in (BO) arises in the study of long internal gravity waves in deep stratified fluids [2],[19]. The existence problem of solutions of (BO) in Sobolev spaces H^s is studied by [1,4,7,13,21-23]. Roughly speaking, the previous results concerning smoothing properties say that if $\phi \in H^s$, then there exists a solution u of (BO) such that

$$u \in L^2_{loc}([-T,T]; H^{s+\frac{1}{2}}_{loc})$$

for some time T, for details see [18, Theorem 1.1] in which their results concerning (BO) are gathered.

In [18] the generalized Benjamin-Ono equation

$$\partial_t u + u^k\partial u + H\partial^2 u = 0, \quad k \geq 2$$

was studied in lower order Sobolev spaces. They established the local well-posedness result [18, Theorem 1.2] by making use of the smoothing effect of solutions to the associated linear problem

$$\partial_t u + H\partial^2 u = f$$

[18, Theorem 2.1]. Their results also say that the solution u gains the 1/2 spatial derivative, namely if $\phi \in H^s$, then

$$\|(1-\partial^2)^{\frac{1}{4}+\frac{s}{2}}u\|_{L_x^\infty L_T^2} < \infty$$

for some s which is a decreasing function of k. These results mentioned above were similar to those known for the (generalized) KdV equation in Sobolev spaces H^s (see [17] and references quoted therein).

However there is a remarkable difference concerning the existence result of the BO equation and the KdV equation which was found by Iorio [13] and [14]. He proved a global existence of solutions to (BO) in the weighted Sobolev space $\mathcal{F}_2 = H^2 \cap L_2^2$ where L_r^2 is defined by

$$L_r^2 = \{f \in L^2; \|f\|_{L_r^2}^2 = \int (1+x^2)^r |f(x)|^2 dx < \infty\}.$$

More generally for $r > 0$ let $\mathcal{F}_r = H^r \cap L_r^2$. Then he showed that there exists a solution of (BO) in $\mathcal{F}_3$ if and only if $\hat{\phi}(0) = 0$, and in this case it is global in time and of course unique. Furthermore he also showed that if $u \in C([0,T];\mathcal{F}_4)$, $T > 0$, is a solution of (BO), then it must be identically zero by using a combination of the non-smoothness of the symbol of the Hilbert transform and the nonlinearity.

On the other hand, the global existence of solutions to the KdV equation in $\mathcal{F}_n$ for any $n \geq 2$ was shown in [16],[24]. A similar result was also proved in [12],[24] for the nonlinear Schrödinger equations

$$i\partial_t u + \frac{1}{2}\Delta u = h(u), \quad u(0,x) = \phi(x) \tag{1.1}$$

under appropriate hypotheses on $h(\cdot)$.

G.Ponce [20] considered the Cauchy problem for the nonlinear dispersive equations of the form

$$\partial_t u + iP(D)u = F(u), \quad u(0,x) = \phi(x), \tag{1.2}$$

where $D = (1/i)(\partial/\partial x_1, \cdots, \partial/\partial x_n)$, $P(D)$ is defined by its real symbol $p(\xi)$, i.e.,

$$P(D)v(t,x) = \mathcal{F}^{-1}(p(\cdot)\mathcal{F}v(t,\cdot))(x)$$

and F is a nonlinear function of u and its derivatives. The initial value problem (1.2) includes the (generalized) KdV, nonlinear Schrödinger, BO equations as examples. He studied the regularity of solutions of (1.2) under decay assumptions on the derivatives of the data and obtained conditions which guarantee that this solution has the same regurarity as that of the associated linear problem. His method is based on a classical energy estimate and the fact that the linear operator $\partial_t + iP(D)$ commutes with the operators

$$\Gamma_j = \Gamma_j(t,x) = x_j + t\frac{\partial}{\partial x_j}P(D), \quad j = 1,2,\cdots,n. \tag{1.3}$$

When $P(D) = \Delta$ this commutation property was first used in [5],[6] to deduce the so-called psuedo-conformal identity for the nonlinear Schrödinger equation (1.1) with $h(u) = g(|u|^2)u$. For the same equation this idea was further developed in [10],[11] to obtain the smoothing property of solutions.

For the linear equation associated with the Benjamin-Ono equation the operator Γ given by (1.3) is written as

$$\Gamma = \Gamma(t,x) = x - 2tH\partial = x - 2t(-\partial^2)^{1/2},$$

and the symbol is given by

$$p(\xi) = \xi|\xi| \in W^{2,\infty}_{loc}.$$

Since $\Gamma(0,x) = x$, the use of the operator Γ suggests the gain of one derivative of solution to (BO) for $t \neq 0$. In view of the proofs of [13, Theorem 5.1] , [14, Theorem 1.1] and [20, Theorem 3.2] we could use the operators Γ and Γ^2 to study the smoothing effect of solutions to (BO). However the operator Γ^3 is useless since $p(\xi)$ is not smooth at the origin. This fact implies that the gain of three derivatives of solutions is not realized through the use of the operator Γ in general.

We now state our main result in this paper.

THEOREM 1. *Let m be an integer with $2 \leq m \leq 4$ and let $\phi \in H^m$, $(x\partial)^m\phi \in L^2$. Then there exists a unique global solution u of (BO) such that*

$$u \in C(\mathbb{R}; H^m) \cap L^\infty_{loc}(\mathbb{R}\backslash\{0\}; H^{2m,-2[m/2]}),$$

where $[m/2]$ is the integer less than or equal to $m/2$ and

$$H^{r,s} = \{f \in \mathcal{S}'; \|(1+x^2)^{\frac{s}{2}}(1-\partial^2)^{\frac{r}{2}}f\|_{L^2} < \infty\}.$$

Remark 1. Theorem 1 says that the solutions gain the spatial derivatives up to the order 4. We note here that our result can be extended to the generalized Benjamin-Ono equation easily.

Remark 2. For any integer $m \geq 1$ and $\phi \in H^m$, the condition $(x\partial)^m\phi \in L^2$ is equivalent to $(x\partial)^j\phi \in L^2$ for all j with $1 \leq j \leq m$ and to $x^m\partial^m\phi \in L^2$. This fact is verified by integration by parts.

Instead of Γ, we use the dilation operator of the linear equation associated with the Benjamin-Ono equation to obtain Theorem 1. The dilation operator P is given by

$$P = P(t,x) = x\partial + 2t\partial_t$$

which is the same as that of the Schrödinger equation $i\partial_t + \frac{1}{2}\Delta$ and has the commutation property that

$$[\partial_t + H\partial^2, P] = 2(\partial_t + H\partial^2).$$

The dilation operators were used to prove analyticity in time of solutions [8], Gevrey smoothing effect in space of solutions [3] to nonlinear Schrödinger equations (1.1) and the (generalized) KdV equation

$$\left\{ \begin{array}{l} \partial_t u + \partial^3 u = u^k \partial u, \\ u(0,x) = \phi(x), \end{array} \right. \tag{1.4}$$

where $k \in \mathbb{N}$. In the same way as in the proof of [3], we easily get the following results for (1.1) and (1.4).

THEOREM 0.1. *(1) We assume that $\phi \in H^m$, $(x\partial)^m\phi \in L^2$, $m \geq 3$. Then there exists a unique solution u of (1.4) and a positive constant T such that*

$$u \in L^\infty([-T,T]\backslash\{0\}; H^{3m}_{loc}(\mathbb{R})).$$

(2) We assume that $n = 1$, $\phi \in H^m$, $(x\partial)^m\phi \in L^2$, $m \geq 1$ and the nonlinear term $h(u)$ is a polynomial of u and $\bar{u}$. Then there exists a unique solution u of (1.1) and a positive constant T such that

$$u \in L^\infty([-T,T]\backslash\{0\}; H^{2m}_{loc}(\mathbb{R})).$$

For the linear Schrödinger equation with real potential V

$$\left\{ \begin{array}{ll} i\partial_t u + \frac{1}{2}\Delta u = Vu, & (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0,x) = \phi(x), & x \in \mathbb{R}^n \end{array} \right.$$

we have obtained the following result [9] through the dilation operator.

THEOREM 0.2. *We assume that $m \geq 1$ is an integer, $(x\cdot\nabla)^j\phi \in L^2(\mathbb{R}^n)$ for all j with $0 \leq j \leq m$, $\partial^\alpha V \in L^\infty(\mathbb{R}^n)$ for all multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n)$ with $|\alpha| \leq 2m-2$ and $(x\cdot\nabla)^j V \in L^\infty(\mathbb{R}^n)$ for all j with $0 \leq j \leq m$. Then the solution u satisfies the following estimate*

$$\|u(t)\|_{H^{2m,-2m}(\mathbb{R}^n)} \leq C(1+|t|^{-2m}) \sum_{j=0}^{m} \|(x\cdot\nabla)^j\phi\|_{L^2(\mathbb{R}^n)},$$

where
$$H^{r,s}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n); \|(1+|x|^2)^{\frac{s}{2}}(1-\Delta)^{\frac{r}{2}} f\|_{L^2(\mathbb{R}^n)} < \infty\}.$$

However it seems unlikely that the similar result as above holds in the case of the Benjamin-Ono equation since the Hilbert transform is a nonlocal operator.

We now explain our strategy of the proof of Theorem 1. We first obtain the estimates of $P^j u$ in L^2 space by using the commutation relation

$$[\partial_t + H\partial^2, P] = 2(\partial_t + H\partial^2)$$

and a classical energy method. Secondly by using the estimates of $\|P^j u\|$ we derive the estimates of $t^j\|\partial_t^j u\|$. Finally we obtain the desired estimates by combining the estimates of $t^j\|\partial_t^j u\|$ and the commutation relation concerning the Hilbert transform H and $< x >^{-2m} = (1+x^2)^{-m}$ given in Lemma 2.1. We prove the theorem only for the case $m = 4$, since the proof of the other cases is similar and simpler.

We note here that the final stage of the proof prevents us to obtain the theorem for $m \geq 5$. The difficulty arises from the commutator estimates for the Hilbert transform and thus, we meet a sharp contrast between the KdV and BO equations in the smoothing properties.

§2 Preliminaries. In this section we use the notation $\|\cdot\| = \|\cdot\|_{L^2}$. We first prove the commutator estimate.

LEMMA 2.1. *Let $m \geq 1$ be an integer. Then there exists a constant C depending only on m such that*

$$\|[H, < x >^{-2m}]\partial^2 f\| \leq C(\| < x >^{-2m} \partial f\| + \| < x >^{-2} f\|).$$

Proof. By integration by parts we have

$$\pi[H, < x >^{-2m}]\partial^2 f = p.v. \int (x-y)^{-1}(< y >^{-2m} - < x >^{-2m})\partial_y^2 f(y)dy \tag{2.1}$$

$$= p.v. \int \partial_y((x-y)^{-1}(< x >^{-2m} - < y >^{-2m}))\partial_y f(y)dy.$$

We compute

$$\partial_y((x-y)^{-1}(< x >^{-2m} - < y >^{-2m})) \tag{2.2}$$

$$= (x-y)^{-2}(< x >^{-2m} - < y >^{-2m}) + 2m(x-y)^{-1} < y >^{-2m-2} y,$$

$$< x >^{-2m} - < y >^{-2m} = ((1+y^2)^m - (1+x^2)^m) < x >^{-2m} < y >^{-2m} \tag{2.3}$$

$$= \sum_{k=1}^{m} \binom{m}{k} (y^{2k} - x^{2k}) < x >^{-2m} < y >^{-2m}$$

$$= \sum_{k=1}^{m} \sum_{j=0}^{k-1} \binom{m}{k} x^{2j} y^{2k-2j-2}(y^2 - x^2) < x >^{-2m} < y >^{-2m},$$

and therefore
(2.4)

$$[H, < x >^{-2m}]\partial^2 f = -\sum_{k=1}^{m} \sum_{j=0}^{k-1} \binom{m}{k} < x >^{-2m} x^{2j} H < y >^{-2m} y^{2k-2j-1}\partial_y f$$

$$-\sum_{k=1}^{m} \sum_{j=0}^{k-1} \binom{m}{k} < x >^{-2m} x^{2j+1} H < y >^{-2m} y^{2k-2j-2}\partial_y f + 2mH < y >^{-2m-2} y\partial_y f,$$

where the multiplication operators acting before and after the Hilbert transform are denoted with the arguments y and x, respectively. We now prove by induction on l that

(2.5)

$$Hy^l < y >^{-2m} \partial f = x^l H < y >^{-2m} \partial f + \sum_{k=0}^{l-1} x^k (xH - Hy)\{\partial(y^{l-1-k} < y >^{-2m})\} f.$$

For $l = 1$, we write

$$y < y >^{-2m} \partial f = y\partial(< y >^{-2m} f) - y(\partial < y >^{-2m}) f$$

and use the commutation relation

$$Hy\partial = xH\partial$$

to obtain (2.5). A similar argument with induction assumption yields (2.5) for all integer $l \geq 1$. Substituting (2.5) into (2.4), we obtain

$$(2.6) \qquad [H, < x >^{-2m}]\partial^2 f$$

$$= -\sum_{k=1}^{m}\sum_{j=0}^{k-1} \binom{m}{k} (< x >^{-2m} x^{2k-1} H < y >^{-2m} \partial f$$

$$+ \sum_{l=0}^{2k-2j-2} < x >^{-2m} x^{2j+l}(xH - Hy)\{\partial(y^{2k-2j-2-l} < y >^{-2m})\} f)$$

$$-\sum_{k=1}^{m}\sum_{j=0}^{k-1} \binom{m}{k} (< x >^{-2m} x^{2k-1} H < y >^{-2m} \partial f$$

$$+ \sum_{l=0}^{2k-2j-3} < x >^{-2m} x^{2j+l+1}(xH - Hy)\{\partial(y^{2k-2j-3-l} < y >^{-2m})\} f)$$

$$+2mH < y >^{-2m-2} y\partial f$$

$$= -2\sum_{k=1}^{m} \binom{m}{k} kx^{2k-1} < x >^{-2m} H < y >^{-2m} \partial f + 2mH < y >^{-2m-2} y\partial f$$

$$-\sum_{k=1}^{m}\sum_{j=0}^{k-1}\sum_{l=0}^{2k-2j-2} \binom{m}{k} < x >^{-2m} x^{2j+l}(xH - Hy)\{\partial(y^{2k-2j-2-l} < y >^{-2m})\} f$$

$$-\sum_{k=1}^{m}\sum_{j=0}^{k-1}\sum_{l=0}^{2k-2j-3}\binom{m}{k} < x >^{-2m} x^{2j+l+1}(xH - Hy)\{\partial(y^{2k-2j-3-l} < y >^{-2m})\}f,$$

where we have followed the convention that $\sum_{l=0}^{-1}(\cdots) = 0$. The first term of the right hand side of the last equality of (2.6) is rewritten as

$$-2m < x >^{-2} xH < y >^{-2m} \partial f$$

and estimated in the L^2 norm as $C\| < x >^{-2m} \partial f\|$ by the isometry of H. Similarly, the second term is estimated as $C\| < x >^{-2m-1} \partial f\|$. The third term has the factor x^{2j+l} with $2j + l \leq 2j + (2k - 2j - 2) = 2k - 2 \leq 2m - 2$ and the weight $\partial(y^{2k-2j-2-l} < y >^{-2m})$ with $2k - 2j - 2 - l \leq 2k - 2 \leq 2m - 2$, which is estimated as $|\partial(y^{2k-2j-l} < y >^{-2m})| \leq C < y >^{-3}$. Hence the third term is estimated in the L^2 norm as $C\| < x >^{-2} f\|$. Similarly, the fourth term is estimated in the L^2 norm as $C\| < x >^{-3} f\|$. Collecting these estimates, we arrive at the required inequality.

LEMMA 2.2. *Let m and k be nonnegative real numbers. Then*

$$\| < x >^{-m} \partial f\| \leq \| < x >^{-m-k} \partial^2 f\|^{1/2}\| < x >^{-m+k} f\|^{1/2} + C\| < x >^{-m-1} f\|.$$

Proof. Integration by parts with the identity $\partial^2(f^2) = 2f\partial^2 f + 2(\partial f)^2$ gives

$$\| < x >^{-m} \partial f\|^2 = (< x >^{-2m} \partial f, \partial f) = \frac{1}{2}(\partial^2(< x >^{-2m}), f^2) - \frac{1}{2}(< x >^{-2m} \partial^2 f, f)$$

$$\leq C\| < x >^{-m-1} f\|^2 + \| < x >^{-m-k} \partial^2 f\|\| < x >^{-m+k} f\|$$
$$\leq (\| < x >^{-m-k} \partial^2 f\|^{1/2}\| < x >^{-m+k} f\|^{1/2} + C\| < x >^{-m-1} f\|)^2.$$

This completes the proof of the lemma.

§3 **Proof of Theorem 1.** We prove the theorem only for $m = 4$ since the proof for $m \leq 3$ is similar and simpler. The restriction $m \leq 4$ in the theorem arises in connection with the second term on the right hand side of the inequality in Lemma 2.1. The argument irrelevant to Lemma 2.1 proceeds without the restriction $m \leq 4$. Accordingly, we do not restrict ourselves to the specific choice of m before it is needed, namely Lemma 3.5 below. In this section we use the following notation

$$X^m = \{f \in L^2; \|f\|_{X^m} = \sum_{j=0}^{m} \|(x\partial)^j f\| < \infty\}.$$

In what follows, we let $M(a,b,c)$ be a nonnegative, locally bounded, monotone increasing function with respect to a, b and c.

In the same way as in the proofs of the existence results obtained in [13], [15] we have the following existence theorem.

THEOREM 3.1. *We assume that $\phi \in H^{2m} \cap X^m$, $m \in \mathbb{N}$. Then there exists a unique global solution u of (BO) such that*

$$u \in \cap_{j=0}^{m} C^j(\mathbb{R}; H^{2(m-j)} \cap X^{m-j}).$$

In order to obtain Theorem 1 we prepare some a-priori estimates of solutions constructed in Theorem 3.1.

LEMMA 3.2. *Let u be the solution constructed in Theorem 3.1. Then*

$$\|P^m u\| \leq M(t, \|\phi\|_{H^m}, \|\phi\|_{X^m}), \quad \|\partial_t^l u\|_{H^{m-2l}} \leq M(t, \|\phi\|_{H^m}) \quad \text{for} \quad m \geq 2l.$$

Proof. The second inequality follows from the existence theorem in H^m spaces ([13]). To prove the first one, it is sufficient to prove that the solution u satisfies the following a-priori estimate

$$\|P^m u\| \leq C \exp(C \int_0^t \|\partial u\|_\infty d\tau) \tag{3.1}$$

$$\times (\sum_{l=0}^{m-1} t^{2^l - 1} \sum_{j=0}^{l} \sum_{k=0}^{m-l} \|\partial^j (x\partial)^k \phi\|^{2^l} + t^{2^{m-1}-1} \sum_{j=0}^{m} (\int_0^t \|\partial^j u\|^2 d\tau)^{2^{m-1}}),$$

where $\|\cdot\|_\infty = \|\cdot\|_{L^\infty}$. A-priori estimate (3.1) is obtained if we prove

$$\|P^m u\| \leq C \exp(C \int_0^t \|\partial u\|_\infty d\tau) \tag{3.2}$$

$$\times(\sum_{l=0}^{m'} t^{2^l-1}\sum_{j=0}^{l}\sum_{k=0}^{m-l}\|\partial^j(x\partial)^k\phi\|^{2^l}+t^{2^{m'}-1}\sum_{j=0}^{m'+1}\sum_{k=0}^{m-m'-1}(\int_0^t\|\partial^jP^ku\|^2d\tau)^{2^{m'}})$$

$$\text{for}\quad 0\le m'\le m-1.$$

Indeed we see that (3.2) is equal to (3.1) if we put $m'=m-1$ in (3.2).

We prove (3.2) by induction on m'. We first prove (3.2) for $m'=0$. Applying P^m to both sides of (BO), using the fact that $[\partial_t+H\partial^2,P]=2(\partial_t+H\partial^2)$ and Leibniz' formula, we obtain

$$2(\partial_t+H\partial^2)P^mu=-(P+2)^m\partial(u^2)=-\partial(P+1)^m(u^2) \tag{3.3}$$

$$=-\sum_{k=0}^{m}\binom{m}{k}\partial P^k(u^2)=-\sum_{k=0}^{m}\sum_{j=0}^{k}\binom{m}{k}\binom{k}{j}(\partial P^{k-j}u\cdot P^ju+P^{k-j}u\cdot\partial P^ju)$$

$$=-2u\partial P^mu+2\sum_{j=0}^{m-1}P^{m-j}u\cdot\partial P^ju+2\sum_{k=0}^{m-1}\sum_{j=0}^{k}\binom{m}{k}\binom{k}{j}P^{k-j}u\cdot\partial P^ju.$$

We multiply both sides of (3.3) by P^mu and use the integration by parts to get

$$\frac{d}{dt}\|P^mu\|^2\le C(\|\partial u\|_\infty\|P^mu\|^2$$

$$+\sum_{j=1}^{m-1}\|P^{m-j}u\|_\infty\|\partial P^ju\|\|P^mu\|$$

$$+\sum_{k=0}^{m-1}\sum_{j=0}^{k}\|P^{k-j}u\|_\infty\|\partial P^ju\|\|P^mu\|)$$

$$\le C(\|\partial u\|_\infty\|P^mu\|^2+\sum_{k=0}^{m-1}(\|P^ku\|^2+\|\partial P^ku\|^2)\|P^mu\|).$$

Hence Gronwall's inequality gives

$$\|P^mu\|\le C\exp(C\int_0^t\|\partial u\|_\infty d\tau)(\|(x\partial)^m\phi\|+\sum_{j=0}^{1}\sum_{k=0}^{m-1}\int_0^t\|\partial^jP^ku\|^2d\tau). \tag{3.4}$$

This implies (3.2) for $m'=0$. In the same way as in the proof of (3.4) we have

$$\|\partial^lP^mu\|\le C\exp(C\int_0^t\|\partial u\|_\infty d\tau)(\|\partial^l(x\partial)^m\phi\|+\sum_{j=0}^{l+1}\sum_{k=0}^{m-1}\int_0^t\|\partial^jP^ku\|^2d\tau). \tag{3.5}$$

We assume that (3.2) holds for m'. By (3.5) we have for $m' \leq m-2$

$$\sum_{j=0}^{m'+1}\sum_{k=0}^{m-m'-1}\left(\int_0^t \|\partial^j P^k u\|^2 d\tau\right)^{2^{m'}} \tag{3.6}$$

$$\leq C\sum_{j=0}^{m'+1}\sum_{k=0}^{m-m'-1}\left(\int_0^t \exp(C\int_0^\tau \|\partial u\|_\infty d\sigma)(\|\partial^j (x\partial)^k \phi\|^2\right.$$

$$\left.+(\sum_{j'=0}^{j+1}\sum_{k'=0}^{k-1}\int_0^\tau \|\partial^{j'} P^{k'} u\|^2 d\sigma)^2)d\tau\right)^{2^{m'}}$$

$$\leq C\sum_{j=0}^{m'+1}\sum_{k=0}^{m-m'-1}\exp(C\int_0^t \|\partial u\|_\infty d\tau)t^{2^{m'}}\|\partial^j (x\partial)^k \phi\|^{2^{m'+1}}$$

$$+C\sum_{j=0}^{m'+1}\sum_{k=0}^{m-m'-2}\exp(C\int_0^t \|\partial u\|_\infty d\tau)\left(\int_0^t \left(\int_0^\tau \|\partial^j P^k u\|^2 d\sigma\right)d\tau\right)^{2^{m'}}$$

$$\leq C\exp(C\int_0^t \|\partial u\|_\infty d\tau)t^{2m'}$$

$$\times\left(\sum_{j=0}^{m+1}\sum_{k=0}^{m-m'-1}\|\partial^j (x\partial)^k \phi\|^{2^{m'+1}}+\sum_{j=0}^{m'+1}\sum_{k=0}^{m-m'-2}\left(\int_0^t \|\partial^j P^k u\|^2 d\tau\right)^{2^{m'+1}}\right).$$

Therefore by (3.6)
(3.7)

$$\text{The R.H.S. of (3.2)} \quad \leq C\exp(C\int_0^t \|\partial u\|_\infty d\tau)\left(\sum_{l=0}^{m'} t^{2^l-1}\sum_{j=0}^{l}\sum_{k=0}^{m-l}\|\partial^j (x\partial)^k \phi\|^{2^l}\right.$$

$$\left.+t^{2^{m'+1}-1}\left(\sum_{j=0}^{m+1}\sum_{k=0}^{m-m'-1}\|\partial^j (x\partial)^k \phi\|^{2^{m'+1}}+\sum_{j=0}^{m'+1}\sum_{k=0}^{m-m'-2}\left(\int_0^t \|\partial^j P^k u\|^2 d\tau\right)^{2^{m'+1}}\right)\right)$$

$$\leq C\exp(C\int_0^t \|\partial u\|_\infty d\tau)$$

$$\times\left(\sum_{l=0}^{m'+1} t^{2^l-1}\sum_{j=0}^{l}\sum_{k=0}^{m-l}\|\partial^j (x\partial)^k \phi\|^{2^l}+t^{2^{m'+1}-1}\sum_{j=0}^{m'+1}\sum_{k=0}^{m-m'-2}\left(\int_0^t \|\partial^j P^k u\|^2 d\tau\right)^{2^{m'+1}}\right).$$

This implies that (3.2) holds for any $m' \leq m-1$.

In the same way as in the proof of Lemma 3.2 we have

LEMMA 3.3. *Let u be the solution of (BO) constructed in Theorem 3.1. Then*

$$\sum_{j=0}^{m} \|\partial^j P^{m-j} u\| \le M(t, \|\phi\|_{H^m}, \|\phi\|_{X^m}).$$

LEMMA 3.4. *Let u be the solution of (BO) constructed in Theorem 3.1. Then*

$$t^m \| < x >^{-m} \partial_t^m u\| \le M(t, \|\phi\|_{H^m}, \|\phi\|_{X^m}).$$

Proof. From the identity $2t\partial_t = P - x\partial$ and Leibniz' formula

$$\| < x >^{-m} (t\partial_t)^m u\| \le C \sum_{j=0}^{m} \| < x >^{-m} (x\partial)^j P^{m-j} u\|.$$

Then the lemma follows from Lemma 3.3.

LEMMA 3.5. *Let u be the solution constructed in Theorem 3.1. Then*

$$t^2(\| < x >^{-2} \partial_t^3 u\| + \sum_{j=1}^{2} \| < x >^{-2} \partial_t^2 \partial^j u\|) \le M(t, \|\phi\|_{H^4}, \|\phi\|_{X^4}).$$

Proof. By Lemma 3.4 we have

$$\left\{ \begin{array}{l} t^2 \| < x >^{-2} \partial_t^3 u\| \le M(t, \|\partial_t u(0)\|_{H^2}, \|\partial_t u(0)\|_{X^2}), \\ t^2 \| < x >^{-2} \partial_t^2 \partial^j u\| \le M(t, \|\phi\|_{H^{2+j}}, \|\partial^j \phi\|_{X^2}). \end{array} \right. \tag{3.8}$$

From (BO) we see that

$$\partial_t u(0) = -H\partial^2\phi - \phi\partial\phi. \tag{3.9}$$

Sobolev's inequality, (3.8) and (3.9) yield the lemma.

LEMMA 3.6. *Let u be the solution constructed in Theorem 3.1. Then*

$$\sum_{j=0}^{3} t^2 \| < x >^{-2} \partial_t^j \partial^{2(3-j)} u\| \le M(t, \|\phi\|_{H^4}, \|\phi\|_{X^4}).$$

Proof. From Lemma 3.5 we have the lemma for $j = 2, 3$. We now prove the lemma for $j = 1$. We have by (BO)

$$H < x >^{-2} \partial_t \partial^4 u = [H, < x >^{-2}] \partial^4 \partial_t u + < x >^{-2} \partial^2 \partial_t (-\partial_t u - \frac{1}{2} \partial(u^2))$$

$$= [H, < x >^{-2}] \partial^4 \partial_t u - < x >^{-2} \partial^2 \partial_t^2 u$$

$$- < x >^{-2} (u \partial^3 \partial_t u + 3 \partial u \partial^2 \partial_t u + 3 \partial^2 u \partial \partial_t u + \partial^3 u \partial_t u).$$

By Lemmas 2.1, 3.2, 3.5 and Sobolev's inequality

$$t^2 \| < x >^{-2} \partial_t \partial^4 u \| \leq C t^2 (\| < x >^{-2} \partial^3 \partial_t u \| \tag{3.10}$$

$$+ \| < x >^{-2} \partial^2 \partial_t u \| + \|u\|_{H^1} \| < x >^{-2} \partial^3 \partial_t u \|) + M(t, \|\phi\|_{H^4}, \|\phi\|_{X^4}).$$

Lemma 2.2 gives
(3.11)

$$\| < x >^{-2} \partial^3 \partial_t u \| \leq C (\| < x >^{-2} \partial^4 \partial_t u \|^{1/2} \| < x >^{-2} \partial^2 \partial_t u \|^{1/2} + \| \partial^2 \partial_t u \|).$$

We use Schwarz' inequality on the right hand side of (3.11) and apply the resulting inequality to (3.10) to obtain

$$t^2 \| < x >^{-2} \partial_t \partial^4 u \| \leq M(t, \|\phi\|_{H^4}, \|\phi\|_{X^4})(1 + \| < x >^{-2} \partial^2 \partial_t u \|). \tag{3.12}$$

The lemma for $j = 1$ follows from (3.12) and Lemma 3.2. We finally consider the case $j = 0$. We have

$$H < x >^{-2} \partial^6 u = [H, < x >^{-2}] \partial^6 u + < x >^{-2} \partial^4 (-\partial_t u - \frac{1}{2} \partial(u^2))$$

$$= [H, < x >^{-2}] \partial^6 u - < x >^{-2} \partial^4 \partial_t u - < x >^{-2} (u \partial^5 u + 5 \partial u \partial^4 u + 10 \partial^2 u \partial^3 u)$$

Lemma 2.1, the lemma for $j = 1$, Sobolev's inequality yield

$$t^2 \| < x >^{-2} \partial^6 u \|$$

$$\leq C t^2 (\| < x >^{-2} \partial^5 u \| + \|u\|_{H^1} \| < x >^{-2} \partial^5 u \|) + M(t, \|\phi\|_{H^4}, \|\phi\|_{X^4}).$$

We again apply Lemma 2.1 and Schwarz' inequality to get the lemma for $j = 0$. This completes the proof of the lemma.

LEMMA 3.7. *Let u be the solution constructed in Theorem 3.1. Then*

$$\sum_{j=0}^{4} t^4 \| < x >^{-4} \partial_t^j \partial^{2(4-j)} u \| \leq M(t, \|\phi\|_{H^4}, \|\phi\|_{X^4}).$$

Proof. The lemma for $j = 4$ follows from Lemma 3.4. By (BO) and Leibniz' formula we have

$$(3.13)\qquad H < x >^{-4} \partial^2\partial_t^3 u = [H, < x >^{-4}]\partial^2\partial_t^3 u + < x >^{-4} \partial_t^3(-\partial_t u - \frac{1}{2}\partial(u^2))$$

$$= [H, < x >^{-4}]\partial^2\partial_t^3 u - < x >^{-4} \partial_t^4 u$$
$$- < x >^{-4} (\partial_t^3 u\partial u + 3\partial_t^2 u\partial_t\partial u + 3\partial_t u\partial_t^2\partial u + u\partial_t^3\partial u),$$

$$(3.14)\qquad H < x >^{-4} \partial^4\partial_t^2 u = [H, < x >^{-4}]\partial^4\partial_t^2 u + < x >^{-4} \partial^2\partial_t^2(-\partial_t u - \frac{1}{2}\partial(u^2))$$

$$= [H, < x >^{-4}]\partial^4\partial_t^2 u - < x >^{-4} \partial^2\partial_t^3 u$$
$$- < x >^{-4} (u\partial^3\partial_t^2 u + 3\partial u\partial^2\partial_t^2 u + 3\partial^2 u\partial\partial_t^2 u + \partial^3 u\partial_t^2 u + 6\partial\partial_t u\partial^2\partial_t u + 2\partial_t u\partial^3\partial_t u),$$

$$(3.15)\qquad H < x >^{-4} \partial^6\partial_t u = [H, < x >^{-4}]\partial^6\partial_t u + < x >^{-4} \partial^4\partial_t(-\partial_t u - \frac{1}{2}\partial(u^2))$$

$$= [H, < x >^{-4}]\partial^6\partial_t u - < x >^{-4} \partial^4\partial_t^2 u$$
$$- < x >^{-4} (\partial^5 u\partial_t u + 5\partial^4 u\partial\partial_t u + 10\partial^3 u\partial^2\partial_t u + 10\partial^2 u\partial^3\partial_t u$$
$$+5\partial u\partial^4\partial_t u + 5\partial u\partial^4\partial_t u + u\partial^5\partial_t u),$$

and

$$(3.16)\qquad H < x >^{-4} \partial^8 u = [H, < x >^{-4}]\partial^8 u + < x >^{-4} \partial^6(-\partial_t u - \frac{1}{2}\partial(u^2))$$

$$= [H, < x >^{-4}]\partial^8 u - < x >^{-4} \partial^6\partial_t u$$
$$- < x >^{-4} (u\partial^7 u + 7\partial u\partial^6 u + 21\partial^2 u\partial^5 u + 35\partial^3 u\partial^4 u).$$

We apply Lemmas 2.1, 3.5-3.7 and Sobolev's inequality to (3.13)-(3.16) to get

$$\| < x >^{-4} \partial^2\partial_t^3 u\| \le C(1 + \|u\|_{H^1})\| < x >^{-4} \partial\partial_t^3 u\| + t^{-4}M(t, \|\phi\|_{H^4}, \|\phi\|_{X^4}),$$

$$\| < x >^{-4} \partial^4\partial_t^2 u\|$$
$$\le C((1 + \|u\|_{H^1})\| < x >^{-4} \partial^3\partial_t^2 u\| + \| < x >^{-4} \partial^2\partial_t^3 u\|) + t^{-4}M(t, \|\phi\|_{H^4}, \|\phi\|_{X^4}),$$

$$\| < x >^{-4} \partial^6\partial_t u\|$$
$$\le C((1 + \|u\|_{H^1})\| < x >^{-4} \partial^5\partial_t^1 u\| + \| < x >^{-4} \partial^4\partial_t^2 u\|) + t^{-4}M(t, \|\phi\|_{H^4}, \|\phi\|_{X^4}),$$

$$\| < x >^{-4} \partial^8 u\|$$

$$\leq C((1+\|u\|_{H^1})\| < x >^{-4} \partial^7 u\| + \| < x >^{-4} \partial^6 \partial_t^1 u\|) + t^{-4} M(t, \|\phi\|_{H^4}, \|\phi\|_{X^4}).$$

Collecting everything, we have

$$\sum_{j=1}^{4} t^4 \| < x >^{-4} \partial^{2j} \partial_t^{4-j} u\|$$

$$\leq C(1+\|u\|_{H^1}) \sum_{j=1}^{4} t^4 \| < x >^{-4} \partial^{2j-1} \partial_t^{4-j} u\| + M(t, \|\phi\|_{H^4}, \|\phi\|_{X^4}).$$

We apply Lemma 2.2 and Schwarz' inequality to the above to obtain the lemma.

We are now in a position to prove Theorem 1.

Proof of Theorem 1. We let $\phi_n \in H^8 \cap X^4$ satisfy

$$\|\phi_n - \phi\|_{H^4} + \|\phi_n - \phi\|_{X^4} \to 0 \quad \text{as} \quad n \to \infty.$$

By Theorem 3.1 we see that for any n there exists a unique global solution u_n of (BO). From Lemmas 3.2, 3.6, 3.7 it follows that

$$t^4 \| < x >^{-4} (1-\partial^2)^4 u_n\| \leq M(t, \|\phi_n\|_{H^4}, \|\phi_n\|_{X^4}).$$

This gives the result.

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