Conical Shock Waves in Supersonic Flow

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Abstract

1 Physical background of shock waves

1. A steady shock front is produced as supersonic flow passes a solid projectile. The most important two typical configurations of the flying projectiles:
   - Long wing: two dimensional object;
   - Conical head: three dimensional object.

2. Depending upon the shape of the leading front of solid object, the shock front will be
   - detached from the projectile if the projectile has a blunt head;
   - attached to to the head of the projectile if the projectile has a narrow and sharp pointed head.

3. It is of practical importance because the sharp jump of pressure across shock front produces great resistance to the flying object and therefore is to be avoided.

4. We study the case of steady shock wave attached to sharp pointed front of projectile.

2 Mathematical models

1. Euler system of equations: quasi-linear, hyperbolic,

$$\begin{cases} \partial_t \rho + \sum_{j=1}^{3} \partial_{x_j}(\rho v_j) = 0, \\ \partial_t(\rho v_i) + \sum_{j=1}^{3} \partial_{x_j}(\rho v_i v_j + \delta_{ij} p) = 0, \quad i = 1, 2, 3 \\ \partial_t(\rho E) + \sum_{j=1}^{3} \partial_{x_j}(\rho v_j E + p v_j) = 0. \end{cases} \tag{1}$$

   ρ- density, $\mathbf{v}$- velocity, $E = e + \frac{1}{2}|\mathbf{v}|^2$- total energy, $p = p(\rho, E)$- pressure.

2. Various simplified models to (1) can be introduced.

   (a) Linearization: small perturbation.

   (b) Geometrical simplification:

   - One dimensional model: shock transition relations and Lax shock inequality.
   - Geometrically symmetric model: cylindrical and spherical model.

   (c) Steady flow model: time-independent flow.

   (d) Thermodynamical simplification:

   - Polytropic gas model: $p = A\rho^\gamma$;
   - Isentropic model: $p = p(\rho)$;
   - Irrotational model: $\nabla \times \mathbf{v} = 0$.

3 Conical shock wave for steady irrotational and isentropic flow

1. Irrotational flow: $\nabla \times \mathbf{v} = 0$ implies $\mathbf{v} = \nabla\phi$, ϕ- velocity potential.
2. Second order scalar equation for ϕ

$$\begin{aligned}\left(\frac{v_1^2}{a^2}-1\right)\phi_{x_1x_1}+\left(\frac{v_2^2}{a^2}-1\right)\phi_{x_2x_2}+\left(\frac{v_3^2}{a^2}-1\right)\phi_{x_3x_3}\\ +\frac{2v_1v_2}{a^2}\phi_{x_1x_2}+\frac{2v_1v_3}{a^2}\phi_{x_1x_3}+\frac{2v_2v_3}{a^2}\phi_{x_2x_3}=0.\end{aligned} \tag{2}$$

3. (2) is hyperbolic in the region of supersonic flow ($|\mathbf{v}| > a$), and is elliptic in the region of subsonic flow ($|\mathbf{v}| < a$).
4. Symmetric conical shock wave:
   - shock polar and apple curve;
   - weak and strong shock.
5. Theoremss: Linear stability of conical shock wave and existence.
6. Mathematical tools: conical coordinates, energy estimate for linearized problem, linear iteration
7. Generalized hodograph transformation: to transform the domain with free boundary into a fixed annular region.

4 Stability for oblique shock wave

My current work on the stability of oblique shock waves for isentropic system of Euler equations.

1. Boundary value problem for $m \times m$ hyperbolic system:

$$\begin{cases} \partial_t u + \sum_{j=1}^{n} A_j \partial_{x_j} u + Cu = f, \ \text{ in } \ x_1 > 0; \\ Pu = g \ \text{ on } \ x_1 = 0. \end{cases} \tag{3}$$

2. Well-posedness of (3): if there is energy estimate

$$\eta \|u\|_\eta^2 + |u|_\eta^2 \le C_0 \left(\frac{1}{\eta} \|f\|_\eta^2 + |g|_\eta^2 \right) \tag{4}$$

$$\|u\|_\eta = \left(\int_{R^1} \int_{R^{n-1}} \int_0^\infty e^{2\eta t} |u(t, x_1, x')|^2 dx_1 \, dx' \, dt \right)^{\frac{1}{2}},$$

$$|u|_\eta = \left(\int_{R^1} \int_{R^{n-1}} e^{2\eta t} |u(t, 0, x')|^2 dx' \, dt \right)^{\frac{1}{2}}.$$

3. Kreiss' condition for well-posedness:

   - $$M(s, i\omega) = -A_1^{-1} \left(sI + i \sum_{j=2}^{n} \omega_j A_j \right), \tag{5}$$

     with $s = \eta + i\tau$ and $\omega \in R^{n-1}$.
   - Eigenvectors and generalized eigenvectors with negative eigenvalues: u_j.
   - Kreiss' condition: Pu_j are uniformly linearly independent.

4. Analysis of stability of oblique shock front.

   - Oblique shock front is stable for weak shock.
   - Mathematical condition and its physical implication.