UNIQUENESS IN INVERSE SCATTERING PROBLEMS WITH A SINGLE INCIDENT WAVE

J. CHENG AND M. YAMAMOTO

1. Introduction

Let $D \subset \mathbb{R}^2$ be a bounded domain and $k \in \mathbb{R}$. For $x \in \mathbb{R}^2$, we set $r = |x|$. We consider a scattering problem with sound-soft obstacle:

$$\Delta u + k^2 u = 0 \qquad in \quad \mathbb{R}^2 \setminus cl(D) \tag{1.1}$$

$$u = 0 \qquad on \quad \partial D \tag{1.2}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial}{\partial r}u^S(x) - iku^S(x)\right) = 0. \tag{1.3}$$

Henceforth $cl(D)$ denotes the closure of a domain D, and we set $i = \sqrt{-1}$, $d \in S^1 \equiv \{x \in \mathbb{R}^2; \ |x| = 1\}$ and

$$u^S(x) = u(x) - e^{ikx\cdot d}.$$

Under suitable conditions on D, for $k \in \mathbb{R}$ and $d \in S^1$, there exists a unique H^1-solution $u(x) = u(D)(x)$ to (1.1) - (1.3), and we can define the far field pattern $u_\infty(D)\left(\frac{x}{r}\right)$:

$$u^S(D)(x) = \frac{e^{ikr}}{\sqrt{r}}\left\{u_\infty(D)\left(\frac{x}{r}\right) + O\left(\frac{1}{r}\right)\right\} \quad \text{as } r \longrightarrow \infty. \tag{1.4}$$

Inverse scattering problem: Determine D from the far field pattern $u_\infty(D)$ for given k and d (possibly by changing them).

This inverse problem is also physically significant and has been studied by many authors. We refer for example to Colton and Kress [1].

The first basic topic for this inverse problem is the uniqueness: Does

$$u_\infty(D_1)(x) = u_\infty(D_2)(x), \qquad |x| = 1 \tag{1.5}$$

(for possible several d and k) imply $D_1 = D_2$?

There is a classical uniqueness result within smooth D_1, D_2 if (1.5) holds for an infinite number of $d \in S^1$, which is proved based on Schiffer's idea (see Theorem 5.1 in [1]). For the uniqueness by means of a finite number of $d \in S^1$, see Colton and Sleeman [2], Theorem 5.2 in [1]. Moreover the uniqueness is known with *a single* d, provided that D_1, D_2 are contained in a ball of radius ρ such that $k\rho < \pi$. See Corollary 5.3 in [1], [2].

An important open problem is the uniqueness in the inverse scattering problem with *a single* (d, k). This problem is interesting from the theoretical point of view, because the far field patterns with many d are overdetermining data for determination of D and we can expect the uniqueness with a single far field pattern. Moreover the formulation with a single (d, k) is helpful for justification of numerical reconstruction of D, because one can usually use far field patterns observed by taking a single or a finite number of d.

2. Main result

Let $k \in \mathbb{R}$ and $d \in S^1$ be arbitrarily fixed. Henceforth, for $P, Q \in \mathbb{R}^2$, we understand that $\overline{PQ}$ is an open segment (not including the end points P and Q). Moreover for a polygonal domain D and $P \in \partial D$, $Q \notin cl(D)$ such that $\overline{PQ} \in \mathbb{R}^2 \setminus cl(D)$, by $\angle(\overline{PQ}, \partial D)$ we denote the least angle among the two angles in $\mathbb{R}^2 \setminus cl(D)$ formed by $\overline{PQ}$ and ∂D. By a polygonal domain D, we mean that ∂D is composed of a finite number of segments.

Definition 2.1. Let $D \subset \mathbb{R}^2$ be a bounded polygonal domain. Let ℓ-points $P_1, ..., P_\ell$, $\ell \geq 2$, satisfy the following conditions (i) - (iv) :

(i) $P_1, ..., P_\ell \in \partial D$.

For $1 \leq j \leq \ell$, we set

$$\theta_j = \begin{cases} \text{the exterior angle of } D \text{ at } P_j, & \text{if } P_j \text{ is a vertex of a polygon } D, \\ \pi, & \text{otherwise.} \end{cases}$$

(ii) $\overline{P_j P_{j+1}} \subset \mathbb{R}^2 \setminus cl(D)$ for $1 \leq j \leq \ell$.

(iii) $\angle(\overline{P_{j-1}P_j}, \partial D) = \angle(\overline{P_j P_{j+1}}, \partial D)$, $1 \leq j \leq \ell$, if $\overline{P_{j-1}P_j}$ does not bisect θ_j at P_j.

(iv) For $1 \leq j \leq \ell$, we have $\frac{\theta_j}{\angle(\overline{P_{j-1}P_j}, \partial D)} \in \mathbb{Q}$.

Here we set $P_0 = P_\ell$ and $P_{\ell+1} = P_1$ and

$$TR(D : P_1, ..., P_\ell) = \begin{cases} \text{a closed broken line } P_1 \to P_2 \to \cdots \to P_\ell \to P_1 \\ \text{if } \overline{P_1 P_\ell} \text{ does not bisect } \theta_1 \text{ at } P_1, \\ \text{a non-closed broken line } P_1 \to P_2 \to \cdots \to P_\ell, \text{ otherwise.} \end{cases}$$

We call $TR(D : P_1, ..., P_\ell)$ a *trapped ray of D with rational angles.*

By $TR(D)$, we denote the sum of all the trapped rays of D with rational angles. If $TR(D) \neq \emptyset$, then we call D trapping with rational angles.

In other words, if $TR(D) = \emptyset$, then there are no rays in $\mathbb{R}^2 \setminus cl(D)$ which go out to ∞ after finite times reflecting on ∂D subject to physical law (iii) with stricter constraint (iv) for angles of incidence.

We can state our main result:

Theorem 2.2. *Let $k \in \mathbb{R}$ and $d \in S^1$ be arbitrarily fixed and let*

$$\partial D_1 \cap TR(D_2) = \emptyset \quad \text{and} \quad \partial D_2 \cap TR(D_1) = \emptyset. \tag{2.1}$$

Then $u_\infty(D_1)(x) = u_\infty(D_2)(x)$, $|x| = 1$, implies $D_1 = D_2$.

Corollary 2.3. *Let D_1 and D_2 be star-shaped polygons. Then $u_\infty(D_1)(x) = u_\infty(D_2)(x)$, $|x| = 1$, implies $D_1 = D_2$.*

By the definition, the break of condition (2.1) happens rarely. However we do not know the uniqueness if (2.1) does not hold. In fact, we have the following trapping D_1, D_2 where our proof does not work.

Example 1. *Let us form D_1, D_2 as follows.*

(1) We take a square $A_1A_2A_3A_4$. For convenience, we set $A_1 = (0,0)$, $A_2 = (1,0)$, $A_3 = (1,1)$, $A_4 = (0,1)$.

(2) In the interior of the square $A_1A_2A_3A_4$, we take a regular triangle $B_1B_2B_3$ (i.e., the lengths of the sides are equal). Here we choose vertices B_1, B_2, B_3 such that $B_1 \to B_2 \to B_3$ is counterclockwise and that $\overline{B_1B_2} \,||\, \overline{A_1A_2}$.

(3) Take the midpoints P_1 and P_2 of the sides $\overline{B_1B_3}$ and $\overline{B_2B_3}$ respectively.

(4) Take a point Q_1 on the segment $\overline{B_3P_2}$ arbitrarily.

(5) Take two points Q_2, Q_3 on the side $\overline{A_2A_3}$ such that $\overline{B_3Q_3} \,||\, \overline{A_1A_2}$ and $\overline{Q_1Q_2} \,||\, \overline{A_1A_2}$.

(6) By D_1 we denote the interior bounded by the closed broken line $A_1A_2Q_2Q_1B_2B_1B_3Q_3A_3A_4$ (which is a non-convex polygon with those vertices). By D_2 we denote the interior bounded by the closed broken line $A_1A_2Q_2Q_1P_2P_1B_3Q_3A_3A_4$ (Figure 1).

Then D_1 is trapping with rational angles. In fact, let P_3 be the midpoint of the side $\overline{B_1B_2}$. For D_1, we can see that $P_1P_2P_3$ satisfies conditions (i) - (iv), and we have $TR(D_1) \cap \partial D_2 \supset \overline{P_1P_2} \neq \emptyset$, that is, condition (2.1) does not hold. In this example, we note that $TR(D_1 : P_1, P_2, P_3)$ is a closed broken line $P_1 \to P_2 \to P_3 \to P_1$. For these D_1 and D_2, our proof does not work.

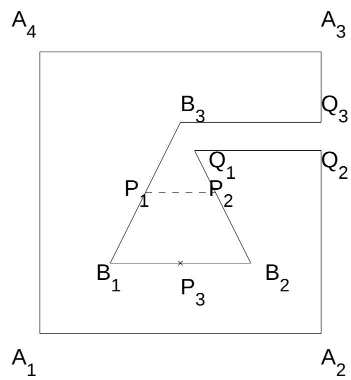

Figure 1

References

1. Colton, D. and Kress, R., Inverse Acoustic and Electromagnetic Scattering Theory. the second edition, Springer-Verlag, Berlin, 1998.
2. Colton, D. and Sleeman, B.D., Uniqueness theorems for the inverse problem of acoustic scattering. IMA J. Appl. Math. 31 (1983), 253–259.

Department of Mathematics, Fudan University, Shanghai 200433, China

E-mail address: jcheng@fudan.edu.cn

Department of Mathematical Sciences, University of Tokyo, Toyko 153-8914, Japan

E-mail address: myama@ms.u-tokyo.ac.jp