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UNIQUENESS IN INVERSE SCATTERING PROBLEMS WITH A
SINGLE INCIDENT WAVE

J. CHENG AND M. YAMAMOTO

1. INTRODUCTION

Let \( D \subset \mathbb{R}^2 \) be a bounded domain and \( k \in \mathbb{R} \). For \( x \in \mathbb{R}^2 \), we set \( r = |x| \). We consider a scattering problem with sound-soft obstacle:

\[
\Delta u + k^2 u = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \text{cl}(D)
\]

(1.1)

\[
u = 0 \quad \text{on} \quad \partial D
\]

(1.2)

\[
\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial}{\partial r} u^S(x) - iku^S(x) \right) = 0.
\]

(1.3)

Henceforth \( \text{cl}(D) \) denotes the closure of a domain \( D \), and we set \( i = \sqrt{-1}, \ d \in S^1 \equiv \{ x \in \mathbb{R}^2 ; |x| = 1 \} \) and

\[
u^S(x) = u(x) - e^{ikx \cdot d}.
\]

Under suitable conditions on \( D \), for \( k \in \mathbb{R} \) and \( d \in S^1 \), there exists a unique \( H^1 \)-solution \( u(x) = u(D)(x) \) to (1.1) - (1.3), and we can define the far field pattern \( u_{\infty}(D) \left( \frac{\cdot}{r} \right) \):

(1.4)

\[
u^S(D)(x) = \frac{e^{ikr}}{\sqrt{r}} \left\{ u_{\infty}(D) \left( \frac{x}{r} \right) + O \left( \frac{1}{r} \right) \right\} \quad \text{as} \ r \to \infty.
\]

**Inverse scattering problem:** Determine \( D \) from the far field pattern \( u_{\infty}(D) \) for given \( k \) and \( d \) (possibly by changing them).

This inverse problem is also physically significant and has been studied by many authors. We refer for example to Colton and Kress [1].

The first basic topic for this inverse problem is the uniqueness: Does

\[
u_{\infty}(D_1)(x) = u_{\infty}(D_2)(x), \quad |x| = 1
\]

(1.5)

(for possible several \( d \) and \( k \)) imply \( D_1 = D_2 \)?
There is a classical uniqueness result within smooth $D_1, D_2$ if (1.5) holds for an infinite number of $d \in S^1$, which is proved based on Schiffer's idea (see Theorem 5.1 in [1]). For the uniqueness by means of a finite number of $d \in S^1$, see Colton and Sleeman [2], Theorem 5.2 in [1]. Moreover the uniqueness is known with a single $d$, provided that $D_1, D_2$ are contained in a ball of radius $\rho$ such that $k\rho < \pi$. See Corollary 5.3 in [1], [2].

An important open problem is the uniqueness in the inverse scattering problem with a single $(d, k)$. This problem is interesting from the theoretical point of view, because the far field patterns with many $d$ are overdeteminig data for determination of $D$ and we can expect the uniqueness with a single far field pattern. Moreover the formulation with a single $(d, k)$ is helpful for justification of numerical reconstruction of $D$, because one can usually use far field patterns observed by taking a single or a finite number of $d$.

2. Main result

Let $k \in \mathbb{R}$ and $d \in S^1$ be arbitrarily fixed. Henceforth, for $P, Q \in \mathbb{R}^2$, we understand that $\overline{PQ}$ is an open segment (not including the end points $P$ and $Q$). Moreover for a polygonal domain $D$ and $P \in \partial D$, $Q \notin cl(D)$ such that $\overline{PQ} \in \mathbb{R}^2 \setminus cl(D)$, by $\angle(\overline{PQ}, \partial D)$ we denote the least angle among the two angles in $\mathbb{R}^2 \setminus cl(D)$ formed by $\overline{PQ}$ and $\partial D$. By a polygonal domain $D$, we mean that $\partial D$ is composed of a finite number of segments.

**Definition 2.1.** Let $D \subset \mathbb{R}^2$ be a bounded polygonal domain. Let $\ell$-points $P_1, \ldots, P_\ell$, $\ell \geq 2$, satisfy the following conditions (i) - (iv):

(i) $P_1, \ldots, P_\ell \in \partial D$.

For $1 \leq j \leq \ell$, we set

$$\theta_j = \begin{cases} 
\text{the exterior angle of } D \text{ at } P_j, & \text{if } P_j \text{ is a vertex of a polygon } D, \\
\pi, & \text{otherwise}. 
\end{cases}$$

(ii) $\overline{P_jP_{j+1}} \subset \mathbb{R}^2 \setminus cl(D)$ for $1 \leq j \leq \ell$.

(iii) $\angle(\overline{P_{j-1}P_j}, \partial D) = \angle(\overline{P_jP_{j+1}}, \partial D)$, $1 \leq j \leq \ell$, if $\overline{P_{j-1}P_j}$ does not bisect $\theta_j$ at $P_j$.

(iv) For $1 \leq j \leq \ell$, we have $\frac{\theta_j}{\angle(\overline{P_{j-1}P_j}, \partial D)} \in \mathbb{Q}$. 

Here we set $P_0 = P_\ell$ and $P_{\ell+1} = P_1$ and
\[
TR(D : P_1, ..., P_\ell) = \begin{cases} 
\text{a closed broken line } P_1 \to P_2 \to \cdots \to P_\ell \to P_1 & 
\text{if } P_1 P_\ell \text{ does not bisect } \theta_1 \text{ at } P_1, \\
\text{a non-closed broken line } P_1 \to P_2 \to \cdots \to P_\ell, & \text{otherwise.}
\end{cases}
\]
We call $TR(D : P_1, ..., P_\ell)$ a trapped ray of $D$ with rational angles.

By $TR(D)$, we denote the sum of all the trapped rays of $D$ with rational angles. If $TR(D) \neq \emptyset$, then we call $D$ trapping with rational angles.

In other words, if $TR(D) = \emptyset$, then there are no rays in $\mathbb{R}^2 \setminus d(D)$ which go out to $\infty$ after finite times reflecting on $\partial D$ subject to physical law (iii) with stricter constraint (iv) for angles of incidence.

We can state our main result:

**Theorem 2.2.** Let $k \in \mathbb{R}$ and $d \in S^1$ be arbitrarily fixed and let
\[
(2.1) \quad \partial D_1 \cap TR(D_2) = \emptyset \quad \text{and} \quad \partial D_2 \cap TR(D_1) = \emptyset.
\]
Then $u_\infty(D_1)(x) = u_\infty(D_2)(x), \ |x| = 1$, implies $D_1 = D_2$.

**Corollary 2.3.** Let $D_1$ and $D_2$ be star-shaped polygons. Then $u_\infty(D_1)(x) = u_\infty(D_2)(x), \ |x| = 1$, implies $D_1 = D_2$.

By the definition, the break of condition (2.1) happens rarely. However we do not know the uniqueness if (2.1) does not hold. In fact, we have the following trapping $D_1, D_2$ where our proof does not work.

**Example 1.** Let us form $D_1, D_2$ as follows.

1. We take a square $A_1 A_2 A_3 A_4$. For convenience, we set $A_1 = (0, 0), A_2 = (1, 0), A_3 = (1, 1), A_4 = (0, 1)$.

2. In the interior of the square $A_1 A_2 A_3 A_4$, we take a regular triangle $B_1 B_2 B_3$ (i.e., the lengths of the sides are equal). Here we choose vertices $B_1, B_2, B_3$ such that $B_1 \to B_2 \to B_3$ is counterclockwise and that $\overline{B_1 B_2} \parallel \overline{A_1 A_2}$.

3. Take the midpoints $P_1$ and $P_2$ of the sides $\overline{B_1 B_3}$ and $\overline{B_2 B_3}$ respectively.

4. Take a point $Q_1$ on the segment $\overline{B_2 P_2}$ arbitrarily.

5. Take two points $Q_2, Q_3$ on the side $A_2 A_3$ such that $\overline{B_3 Q_3} \parallel \overline{A_1 A_2}$ and $\overline{Q_1 Q_2} \parallel \overline{A_1 A_2}$. 
(6) By $D_1$ we denote the interior bounded by the closed broken line $A_1A_2Q_2Q_1B_2B_1B_3Q_3A_3A_4$ (which is a non-convex polygon with those vertices). By $D_2$ we denote the interior bounded by the closed broken line $A_1A_2Q_2Q_1P_2P_1B_3Q_3A_3A_4$ (Figure 1).

Then $D_1$ is trapping with rational angles. In fact, let $P_3$ be the midpoint of the side $B_1B_2$. For $D_1$, we can see that $P_1P_2P_3$ satisfies conditions (i) - (iv), and we have $TR(D_1) \cap \partial D_2 \supset P_1P_2 \neq \emptyset$, that is, condition (2.1) does not hold. In this example, we note that $TR(D_1 : P_1, P_2, P_3)$ is a closed broken line $P_1 \to P_2 \to P_3 \to P_1$. For these $D_1$ and $D_2$, our proof does not work.

![Figure 1](image)

REFERENCES


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