Asymptotic behavior of spherically symmetric solutions to the compressible Navier–Stokes equation with external forces

Shinya Nishibata*

Department of Mathematical and Computing Sciences
Tokyo Institute of Technology
Tokyo 152-8552, Japan

We study the large time behavior of an isentropic and spherically symmetric motion of compressible viscous gas in a field of external force over an unbounded exterior domain in $\mathbb{R}^n$ ($n \geq 2$). The typical example of this problem appears in analysis of the behavior of atmosphere around the earth. First, we show that there exists a stationary solution satisfying an adhesion boundary condition and a positive spatial asymptotic condition. Then, it is shown that this stationary solution is a time asymptotic state to the initial boundary value problem with the same boundary and spatial asymptotic conditions. Here, the initial data can be chosen arbitrarily large if it belongs to the suitable Sobolev space. Moreover, if the external force is attractive, it also can be arbitrarily large. This condition includes the most typical external force, i.e., the gravitational force. In the proof of the stability theorem, it is the essential step to obtain the uniform positive lower bound for the density. It is derived through the energy method with the aid of a representation formula for the density.

The Navier-Stokes equation with external force for the isentropic motion of compressible viscous gas in the Eulerian coordinate is the system of equations given by

$$\rho_t + \nabla \cdot (\rho u) = 0, \tag{1a}$$

$$\rho\{u_t + (u \cdot \nabla)u\} = \mu_1 \Delta u + (\mu_1 + \mu_2)\nabla(\nabla \cdot u) - \nabla p(\rho) + \rho f. \tag{1b}$$

We study the asymptotic behavior of a solution (ρ, u) to (1) in an unbounded exterior domain $\Omega := \{\xi \in \mathbb{R}^n \; ; \; |\xi| > 1\}$, where n is a space dimension larger than or equal to 2. Here $\rho > 0$

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is the mass density; $u = (u_1, \dots, u_n)$ is the velocity of gas; $p(\rho) = K\rho^\gamma$ $(K > 0, \gamma \geq 1)$ is the pressure; f is the external force; μ_1 and μ_2 are constant viscosity-coefficients satisfying $\mu_1 > 0$ and $2\mu_1 + n\mu_2 > 0$.

It is assumed that the external force f is a spherically symmetric potential force and the initial data is also spherically symmetric. Namely, for $r := |\xi|$

$$\text{(A1)} \;\; f := -\nabla U = \frac{\xi}{r} U_r(r), \quad U_r \in C^1[1, \infty), \quad {}^\exists U_+ = \lim_{r\to\infty} U(r) = \lim_{r\to\infty} \int_1^r U_r(\eta) d\eta,$$

$$\text{(A2)} \;\; \rho_0(x) = \hat{\rho}_0(r), \quad u_0(\xi) = \frac{\xi}{r} \hat{u}_0(r).$$

Under the assumptions (A1) and (A2), it is shown in [4] that the solution (ρ, u) is spherically symmetric. Here, the spherically symmetric solution is a solution to (1) in the form of

$$\rho(\xi, t) = \hat{\rho}(r, t), \quad u(\xi, t) = \frac{\xi}{r} \hat{u}(r, t). \tag{2}$$

Substituting (2) in (1), we reduce the system (1) to the equations for $(\hat{\rho}, \hat{u})(r, t)$. Hereafter, we omit the hat to express a spherically symmetric function without confusion. Then, the spherically symmetric solution $(\rho, u)(r, t)$ satisfies the system of equations

$$\rho_t + \frac{(r^{n-1}\rho u)_r}{r^{n-1}} = 0, \tag{3a}$$

$$\rho(u_t + uu_r) = \mu \left\{ \frac{(r^{n-1}u)_r}{r^{n-1}} \right\}_r - p(\rho)_r - \rho U_r, \tag{3b}$$

where $\mu := 2\mu_1 + \mu_2$ is supposed to be positive. The initial data to (3) is prescribed to be asymptotically constant in space:

$$\rho(r, 0) = \rho_0(r) > 0, \quad u(r, 0) = u_0(r), \quad \lim_{r\to\infty} (\rho(r, t), u(r, t)) = (\rho_+, u_+), \;\; \rho_+ > 0. \tag{4}$$

As we interested in the behavior of gas around a solid sphere, an adhesion boundary condition is adopted:

$$u(1, t) = 0. \tag{5}$$

In addition, it is assumed that the initial data (4) is compatible with the boundary data (5). Since the characteristic speed of (3a) is zero on the boundary due to (5), one boundary condition is necessary and sufficient for the wellposedness of the initial boundary value problem (3), (4) and (5).

This problem is formulated to study the behavior of compressible viscous gas around the solid sphere in a field of external force. We show that the time asymptotic state of the solution to the initial boundary value problem (3), (4), (5) is the stationary solution, which is a solution to (3) independent of time t, satisfying the same conditions (4) and (5). Hence, the stationary solution $(\tilde{\rho}(r), \tilde{u}(r))$ satisfies the system of equations

$$\frac{1}{r^{n-1}}(r^{n-1}\tilde{\rho}\tilde{u})_r = 0, \tag{6a}$$

$$\tilde{\rho}\tilde{u}\tilde{u}_r = \mu \left\{ \frac{(r^{n-1}\tilde{u})_r}{r^{n-1}} \right\}_r - p(\tilde{\rho})_r - \tilde{\rho}U_r \tag{6b}$$

and the boundary and the spatial asymptotic conditions

$$\tilde{u}(1) = 0, \quad \lim_{r\to\infty} (\tilde{\rho}(r), \tilde{u}(r)) = (\rho_+, u_+). \tag{7}$$

Solving (6), we obtain an explicit formula of the stationary solution $(\tilde{\rho}(r), \tilde{u}(r))$:

$$\tilde{u}(r) \equiv 0, \tag{8a}$$

$$\tilde{\rho}(r) = \begin{cases} \left\{ \rho_+^{\gamma-1} + \dfrac{\gamma-1}{K\gamma}(U_+ - U(r)) \right\}^{\frac{1}{\gamma-1}} & \text{for} \quad \gamma > 1, \\ \rho_+ \exp\left\{ \dfrac{1}{K}(U_+ - U(r)) \right\} & \text{for} \quad \gamma = 1. \end{cases} \tag{8b}$$

Due to (8a), the spatial asymptotic data in (7) must satisfy $u_+ = 0$ for the existence of the stationary solution.The stability theorem on the stationary solution in (8) is summarized in the next theorem, which is the main result in the present research.

Theorem 1. *Suppose the initial data satisfies that for a certain $\sigma \in (0,1)$*

$$\rho_0 \in \mathcal{B}^{1+\sigma}_{\mathrm{loc}}[1,\infty), \quad u_0 \in \mathcal{B}^{2+\sigma}_{\mathrm{loc}}[1,\infty), \tag{9a}$$

$$r^{\frac{n-1}{2}}(\rho_0 - \tilde{\rho}),\ r^{\frac{n-1}{2}}u_0,\ r^{\frac{n-1}{2}}(\rho_0 - \tilde{\rho})_r,\ r^{\frac{n-1}{2}}u_{0r} \in L^2(1,\infty) \tag{9b}$$

and the compatibility condition holds. In addition, if there exists a positive constant δ, depending only on the initial data, such that $-\delta \le U_r(r)$, then the initial boundary value problem (3), (4) and (5) has a unique solution (ρ, u) globally in time and the solution converges to the corresponding stationary solution. Precisely, it holds that

$$\lim_{t\to\infty} \sup_{r\in[1,\infty)} |(\rho(r,t) - \tilde{\rho}(r), u(r,t))| = 0. \tag{10}$$

Notice that any smallness assumptions on the initial data is not necessary for the above stability theorem. Moreover, if $U_r \ge 0$, then U_r can be arbitrarily large. This is the case that the external force is attractive like the gravitational force. The Hölder continuity of the initial data (9a) is necessary to ensure the validity of the transformation between the Eulerian and the Lagrangian coordinates. (See (11) below.) Actually, we show the asymptotic stability of the stationary solution in the Lagrangian without the Hölder continuity. In translating this result to that in the Eulerian coordinate, we need the differentiability of solutions. This is the reason we assume (9a), which gives the Hölder continuity of the solution with the aid of the Schauder theory for parabolic equations. The remainder of the present paper is devoted to a brief outline of the proof of Theorem 1. The readers are referred to the paper [11] for the detailed discussions.

In the proof of Theorem 1, we show the uniform a priori estimate by employing the energy method. For this purpose, it is convenient to adopt the Lagrangian coordinate rather than the Eulerian coordinate. The transformation from the Eulerian coordinate (r, t) to the Lagrangian coordinate (x, t) is executed by the relation:

$$x = \int_1^r \eta^{n-1} \rho(\eta, t)\, d\eta, \quad r_t = u, \quad r_x = \frac{v}{r^{n-1}}, \tag{11}$$

where $v = 1/\rho$ is the specific volume. Using (11), we deduce the system (3) to

$$v_t = (r^{n-1}u)_x, \tag{12a}$$

$$u_t = \mu r^{n-1} \left(\frac{(r^{n-1}u)_x}{v} \right)_x - r^{n-1} p_x - U_r. \tag{12b}$$

The initial and the boundary conditions for (v, u) are derived from (4) and (5) as

$$v(x, 0) = v_0(x) := 1/\rho_0(x), \quad u(x, 0) = u_0(x), \quad \lim_{x \to \infty} v_0(x) = v_+ := 1/\rho_+, \tag{13}$$

$$u(0, t) = 0. \tag{14}$$

The spatial variable r in the Eulerian coordinate is regarded as a function of (x, t) in the Lagrangian coordinate. Thus, the density $\tilde{\rho}$ in the stationary solution also depends on (x, t), that is, $\tilde{\rho}(x, t) := \tilde{\rho}(r(x, t))$. Also, let $\tilde{\rho}_0(x) := \tilde{\rho}(r(x, 0))$.

Define the energy form $\mathcal{E}$ by

$$\mathcal{E} := \frac{1}{2}u^2 + \Psi(v, \tilde{v}),$$

$$\Psi(v, \tilde{v}) := p(\tilde{v})(v - \tilde{v}) - \varphi, \quad \varphi := \int_{\tilde{v}}^v p(\eta)\, d\eta, \quad p(v) := Kv^{-\gamma}, \quad \tilde{v} := 1/\tilde{\rho}.$$

If $c \le v(x, t) \le C$ for positive constants c and C, then $\Psi(v, \tilde{v})$ is equivalent to $|v - \tilde{v}|^2$. Namely, $c|v - \tilde{v}|^2 \le \Psi(v, \tilde{v}) \le C|v - \tilde{v}|^2$ for positive constants c and C. Then the energy form $\mathcal{E}$ is equivalent to $|u|^2 + |v - \tilde{v}|^2$.

We state several a priori estimates for the solution (v, u) without detailed proofs.

Proposition 2. (*Basic estimate*) *Suppose that* $v_0 - \tilde{v}_0,\ u_0 \in L^2(0,\infty)$. *Then the solution satisfies*

$$\int_0^\infty \mathcal{E}(x,t)\,dx + \mu \int_0^t \int_0^\infty (n-1)\frac{v}{r^2}u^2 + \frac{r^{2n-2}}{v}u_x^2\,dxd\tau \le \int_0^\infty \mathcal{E}(x,0)\,dx. \tag{15}$$

Applying the Sobolev inequality on (15), we have

Corollary 3.

$$\int_0^t \|(r^{n-2}u^2)(\tau)\|_\infty\,d\tau \le C, \tag{16}$$

where C *is a positive constant depending only on the initial data.*

In order to obtain the pointwise bound for the specific volume $v(x,t)$ uniformly in time, we employ a "cut-off-function" defined by

$$\eta(x) = \begin{cases} 1, & x \le k, \\ k+1-x, & k \le x \le k+1, \\ 0, & k+1 \le x. \end{cases} \quad \text{for} \quad k = 1, 2, \ldots$$

By using (12) with the cut-off-function $\eta(x)$, we have a representation formula of the density.

Lemma 4. $v(x,t)$ *is represented by*

$$v(x,t)^\gamma = \frac{v_0(x)^\gamma + \frac{K\gamma}{\mu}\int_0^t A(x,\tau)B(x,\tau)d\tau}{A(x,t)B(x,t)},$$

for $x \in [k-1,k)$ *and* $t \ge 0$, *where*

$$A(x,t) := \exp\left(\frac{K\gamma}{\mu}\int_0^t \int_k^{k+1} v^{-\gamma}dxd\tau + \frac{\gamma}{\mu}\int_0^t \int_x^\infty \frac{U_r}{r^{n-1}}\eta\,dxd\tau\right),$$

$$B(x,t) := \exp\left(\frac{\gamma}{\mu}\int_x^\infty \left(\frac{u}{r^{n-1}} - \frac{u_0}{r_0^{n-1}}\right)\eta\,dx + \frac{\gamma}{\mu}\int_0^t \int_x^\infty (n-1)\frac{u^2}{r^n}\eta\,dxd\tau - \gamma\int_k^{k+1} \log\frac{v}{v_0}dx\right).$$

Proposition 2 and Lemma 4 yield the upper and the lower bounds of $v(x,t)$.

Proposition 5. *There exist positive constants* c *and* C, *depending only on the initial data, such that*

$$c \le v(x,t) \le C \tag{17}$$

for $x \ge 0$ *and* $t \ge 0$.

The estimate (17) immediately gives the pointwise bounds for the density, $0 < c \leq \rho \leq C$. To obtain the a priori estimate for the derivatives of the solution, it is convenient to use the function

$$\varphi(x,t) := \int_{\tilde{v}}^{v} p(\eta)\, d\eta.$$

Proposition 6. *Suppose that $v_0 - \tilde{v}_0$, u_0, $r_0^{n-2}(v_0 - \tilde{v}_0)_x$, $r_0^{n-2} u_{0x} \in L^2(0,\infty)$. Then we have*

$$\int_0^\infty \left(r^{2n-4}\varphi_x^2\right)(x,t)\, dx + c\int_0^t \int_0^\infty \left(r^{2n-4}\varphi_x^2\right)(x,\tau)\, dx d\tau \leq C, \tag{18}$$

$$\int_0^\infty \left(r^{2n-4}u_x^2\right)(x,t)\, dx + c\int_0^t \int_0^\infty \left(r^{4n-6}u_{xx}^2\right)(x,\tau)\, dx d\tau \leq C, \tag{19}$$

where c and C are positive constants depending only on the initial data.

The estimate for $(v - \tilde{v})_x$ follows from (18);

$$\int_0^\infty \left(r^{2n-4}(v-\tilde{v})_x^2\right)(x,t)\, dx \leq C. \tag{20}$$

Using the estimates (15), (18), (19) and (20), we show the global existence of the solution (v,u) in the Lagrangian coordinate. Moreover, these estimates give the asymptotic stability of the solution in (10). In these discussions, the Hölder continuity of the solution is not necessary. It is used to ensure the validity of the translation of these results to those in Eulerian coordinate. Actually, we show that, by applying the Schauder theory for the parabolic equations (see [2]), the solution (ρ, u) is also belonging to the Hölder space if the initial data satisfies (9a). It immediately gives the corresponding stability theorem in the Eulerian coordinate.

Related results. The Navier–Stokes equation has been attracting interests of a lot of researchers in the fields of not only physics but also mathematics for these decays. Thus, we have so many preceding researches and have to restrict ourselves to a certain problem. Here we mainly state several results on the spherically symmetric motion of compressible viscous fluid in an exterior domain.

The first of all, we need to mention the book [1] written by S. N. Antontsev, A. V. Kazhikhov, and V. N. Monakhov, which gives comprehensive introduction to the mathematical theory of the compressible and viscous fluid. The first notable research in the equations on the exterior domain is given by A. Matsumura and T. Nishida in [9], where the stability of the stationary solution is first proved under the smallness assumptions. Note that this research covers more general solutions on more general domain than the present research, studying the spherically symmetric solution on the exterior domain.

Another pioneering work is given by N. Itaya [4], which establishes the existence of the spherically symmetric solution globally in time on a bounded annulus domain without smallness assumptions on the initial data. This paper has drawn attention of the researchers to the spherically symmetric solution. Then, A. Matsumura in [8] shows that the spherically symmetric solution to the isothermal model with external force on the annulus domain exists globally in time and it converges to the corresponding stationary solution as time tends to infinity. Moreover, it shows that the convergence is exponentially fast. The research by K. Higuchi in [3] extends this result to the isentropic model. In addition, it considers the equations of heat-conductive ideal gas on the annulus domain. The present research aims to extend the results in [8] and [3] to those on an unbounded exterior domain.

The study of the spherically symmetric solution over an unbounded exterior domain is started by S. Jiang in [5], where the global existence of the solution is established for the model of heat-conductive ideal gas. Moreover, the partial result on the asymptotic state is obtained. Precisely, it shows that, for the space dimension $n = 3$, $\|u(t)\|_{2j} \to 0$ as $t \to \infty$, where j (≥ 2) is an arbitrarily fixed integer.

In the case of one dimensional space $n = 1$, the problem on the unbounded exterior domain is coincide with the half-space problem. A. Matsumura and K. Nishihara in [10] start to investigate this problem for the compressible Navier–Stokes equation. In [10], several kinds of boundary conditions are proposed. Namely, inflow, outflow and no flow boundary conditions. Then, it classifies the asymptotic behaviors of the solution into the several cases subject to the relation between the boundary data and the spatial asymptotic data. Moreover, it proves the stability theorem for some cases by using the Lagrangian coordinate. The research [6] by S. Kawashima, S. Nishibata and P. Zhu also studies the same one dimensional half space problem. It obtains the a priori estimates directly in the Eulerian coordinate and proves the stability of the stationary solution. The Hölder continuity of the solution is also discussed in [6].

Notation. For a region Ω, an integer l and $0 < \sigma < 1$, $\mathcal{B}^{l+\sigma}(\Omega)$ denotes the space of Hölder continuous functions over Ω which have the l-th order derivatives of Hölder continuity with exponent σ. $\mathcal{B}^{l+\sigma}_{\mathrm{loc}}(\Omega)$ is the space of functions belonging to $\mathcal{B}^{l+\sigma}(\omega)$ for an arbitrarily compact set $\omega \subset \Omega$. For $1 \leq p \leq \infty$, $L^p(\Omega)$ denotes the standard Lebesgue space over Ω equipped with the norm $\|\cdot\|_p$. c and C denote several generic positive constants.

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