L^q-L^r estimates of solution to the parabolic Maxwell equations and their application to the magnetohydrodynamic equations*

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1. Introduction and main results

Let Ω be a simply connected and unbounded domain in the three dimensional Euclidean space $\mathbf{R}^3$ whose boundary $\partial\Omega$ is a compact and sufficiently smooth hypersurface. Suppose that there is some $R_0 > 0$ such that $\partial\Omega \subset B_{R_0}(0) = \{x \in \mathbf{R}^3 \,|\, |x| < R_0\}$. In this paper we are concerned with the initial boundary value problem of the magnetohydrodynamic equations in $\Omega\times(0,\infty)$ concerning the velocity vector field $v = (v_1(x,t), v_2(x,t), v_3(x,t))$, the magnetic vector potential $H = (H_1(x,t), H_2(x,t), H_3(x,t))$ and the scalar pressure $p = p(x,t)$:

$$
\begin{cases}
v_t - \Delta v + (v\cdot\nabla)v + \nabla p + H\times \operatorname{curl} H = 0 & \text{in} \quad \Omega\times(0,\infty),\\
H_t + \operatorname{curl}\operatorname{curl} H + (v\cdot\nabla)H - (H\cdot\nabla)v = 0 & \text{in} \quad \Omega\times(0,\infty),\\
\operatorname{div} v = 0, \quad \operatorname{div} H = 0 & \text{in} \quad \Omega\times(0,\infty),\\
v = 0, \quad \operatorname{curl} H\times\nu = 0, \quad \nu\cdot H = 0 & \text{on} \quad \partial\Omega\times(0,\infty),\\
v(x,0) = a, \quad H(x,0) = b & \text{in} \quad \Omega.
\end{cases} \tag{MHD}
$$

Here $a = (a_1(x), a_2(x), a_3(x))$ and $b = (b_1(x), b_2(x), b_3(x))$ are prescribed initial data and ν is the unit outward normal on $\partial\Omega$. We impose the perfectly conducting wall condition on the magnetic vector potential on the boundary. The perfectly conducting wall condition means that the obstacle, $\mathbf{R}^3 \setminus \overline{\Omega}$, is the perfect conductor body. The magnetohydrodynamic equations were proposed by Cowling [3] or Landau and Lifshitz

*This is joint work with Prof. Y. Shibata of Waseda University.

[10]. They are known to be one of the mathematical model describing the motion of viscous incompressible resistive fluid.

The main purpose of this paper is to show the global solvability of (MHD). The initial boundary value problem of the magnetohydrodynamic equations was treated in a bounded domain by the Galerkin method. However, in general the Galerkin method does not work well in the unbounded domain case. Thus we shall take another approach. On the other hand, there are some works in exterior domain. Zhao [17] considered (MHD) with nonperfect conductor body, that is the boundary condition of the magnetic vector potential is replaced by the homogeneous Dirichlet condition. However from a physical viewpoint, the case of the perfectly conducting wall is also important. The author knows only the result by Kozono [9] concerning the perfectly conducting wall case, where the weak solution was dealt with. There has been no work on the global strong solvability to the (MHD) in the exterior domain.

Our approach is based on the argument of T. Kato [8]. Kato proved the global solvability of the Cauchy problem of the Navier-Stokes equations in $\mathbf{R}^N$ ($N \geq 2$) with small initial velocity with respect to L^N-norm. The argument of Kato is based on the estimates of various L^q-norm of the Stokes semigroup. In particular L^q-L^r type estimates play crucial role in it. The argument of Kato was extended to the case of exterior domain by Iwashita [7]. Our aim of this talk is to show the global solvability of (MHD), by use of the argument of Kato and Iwashita which is known to work well in exterior domain. In order to do this, one of the main points is to study the linearized problems corresponding to (MHD). They are consisted of two systems of equations. First is the system of well known nonstationary Stokes equations and second is the system of the Maxwell equations of parabolic type with perfectly conducting wall condition:

$$\begin{cases} u_t + \operatorname{curl}\operatorname{curl} u = 0 & \text{in} \quad \Omega \times (0,\infty), \\ \operatorname{div} u = 0 & \text{in} \quad \Omega \times (0,\infty), \\ \operatorname{curl} u \times \nu = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ \nu \cdot u = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ u(x,0) = b & \text{in} \quad \Omega. \end{cases} \tag{PM}$$

Here $u = (u_1(x,t), u_2(x,t), u_3(x,t)))$ is unknown vector valued function and b is a prescribed initial function. In order to derive good linear estimates, we have to study the parabolic Maxwell equations (PM). The parabolic Maxwell equations are important not only for (MHD) but also for another equations involving the Maxwell equations. For example, the time dependent Ginzburg-Landau-Maxwell superconductivity model and the magneto-micropolar fluid equations.

Before stating our main results we shall introduce some notations. Let $1 < q < \infty$.

It is well known that the Banach space $L^q(\Omega)^3$ admits the Helmholtz decomposition:

$$L^q(\Omega)^3 = L^q_\sigma(\Omega) \oplus G^q(\Omega), \quad \oplus : \text{direct sum}.$$

Here

$$\begin{aligned} L^q_\sigma(\Omega) &= \overline{\{f \in C_0^\infty(\Omega)^3 \mid \operatorname{div} f = 0 \text{ in } \Omega\}}^{\|\cdot\|_{L^q(\Omega)^3}}, \\ G^q(\Omega) &= \{f \in L^q(\Omega)^3 \mid f = \nabla p \text{ for some } p \in L^q_{\mathrm{loc}}(\overline{\Omega})\}. \end{aligned}$$

By the assumption that $\partial\Omega$ is sufficiently smooth, the space $L^q_\sigma(\Omega)$ is characterized as (see e.g., Galdi [6, Chapter 3])

$$L^q_\sigma(\Omega) = \{f \in L^q(\Omega)^3 \mid \operatorname{div} f = 0 \text{ in } \Omega,\ \nu \cdot f = 0 \text{ on } \partial\Omega\}.$$

Let $P = P_{q,\Omega}$ be a continuous projection from $L^q(\Omega)^3$ onto $L^q_\sigma(\Omega)$ and let us define the operators A and B as follows:

$$\begin{aligned} \mathcal{D}(A) &= L^q_\sigma(\Omega) \cap W^{2,q}(\Omega)^3 \cap W^{1,q}_0(\Omega)^3, \\ Av &= -P\Delta v \quad \text{for } v \in \mathcal{D}(A), \\ \mathcal{D}(B) &= L^q_\sigma(\Omega) \cap \{H \in W^{2,q}(\Omega)^3 \mid \operatorname{curl} H \times \nu = 0 \text{ on } \partial\Omega\}, \\ BH &= P(\operatorname{curl}\operatorname{curl} H) = \operatorname{curl}\operatorname{curl} H \quad \text{for } H \in \mathcal{D}(B). \end{aligned}$$

From Akiyama, Kasai, Shibata and M. Tsutsumi [1], Borchers and Sohr [2] and Miyakawa [12, 13] both $-A$ and $-B$ generate bounded analytic semigroups $\{e^{-tA}\}$ and $\{e^{-tB}\}$ in $L^q_\sigma(\Omega)$, respectively. By use of operators A and B, (MHD) is converted into the following system of integral equations:

$$\begin{aligned} v(t) &= e^{-tA}a - \int_0^t e^{-(t-\tau)A} P\big[(v(\tau)\cdot\nabla)v(\tau) + H(\tau) \times \operatorname{curl} H(\tau)\big]\, d\tau, \\ H(t) &= e^{-tB}b - \int_0^t e^{-(t-\tau)B} P\big[(v(\tau)\cdot\nabla)H(\tau) - (H(\tau)\cdot\nabla)v(\tau)\big]\, d\tau. \end{aligned} \tag{INT}$$

In analyzing (INT) we require L^q-L^r estimates for the semigroup $\{e^{-tA}\}$ and $\{e^{-tB}\}$. The first was already proved by Iwashita (see also Maremonti and Solonnikov [11] and Enomoto and Shibata [5]). Therefore we have to do is to derive L^q-L^r estimates for the semigroup $\{e^{-tB}\}$.

We are now in a position to state our main results. The first result is concerned with the local energy decay property for the semigroup $\{e^{-tB}\}$.

Theorem 1.1 (Local energy decay)**.** *Let $1 < q < \infty$. For any $R > R_0$ and any integer $m \geq 0$, there exists $C = C(q, R, m) > 0$ such that*

$$\|\partial_t^m e^{-tB} f\|_{W^{2,q}(\Omega_R)} \leq C t^{-(3/2+m)} \|f\|_{L^q(\Omega)}, \quad t \geq 1,$$

for any $f \in L^q_\sigma(\Omega) \cap L^q_R(\Omega)$, where $\Omega_R = \Omega \cap B_R$ and $L^q_R(\Omega) = \{u \in L^q(\Omega)^3 \mid u = 0 \text{ for } |x| \geq R\}$.

By use of Theorem 1.1, one can obtain the following L^q-L^r estimates for $\{e^{-tB}\}$.

Theorem 1.2 (L^q-L^r estimates)**.**
(i) *Let* $1 \le q \le r \le \infty$ *and* $(q,r) \ne (1,1), (\infty,\infty)$. *Then there exists* $C = C(q,r) > 0$ *such that*

$$\|e^{-tB} f\|_{L^r(\Omega)} \le C t^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{r}\right)} \|f\|_{L^q(\Omega)}, \quad t > 0$$

for any $f \in L^q_\sigma(\Omega)$.
(ii) *Let* $1 < q \le r \le 3$. *Then there exists* $C = C(q,r) > 0$ *such that*

$$\|\nabla e^{-tB} f\|_{L^r(\Omega)} \le C t^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{r}\right)-\frac{1}{2}} \|f\|_{L^q(\Omega)}, \quad t > 0$$

for any $f \in L^q_\sigma(\Omega)$.

The basic idea to prove Theorem 1.1 and Theorem 1.2 is similar to that of Iwashita [7] which deals with the nonstationary Stokes equations. However the boundary condition of (PM), perfectly conducting wall condition, is quite different from the boundary condition of the Stokes equations that is homogeneous Dirichlet condition, nonslip boundary condition. Therefore in constructing the parametrix of the resolvent problem corresponding to (PM) (see (2.1)), we have to introduce a new idea which is based on a theorem due to von Wahl [16, Theorem 3.2].

The following result by Iwashita is concerning the L^q-L^r estimates for the Stokes semigroup, which is refined by Maremonti and Solonnikov and Enomoto and Shibata.

Theorem 1.3 ([7, 11, 5])**.**
(i) *Let* $1 \le q \le r \le \infty$ *and* $(q,r) \ne (1,1), (\infty,\infty)$. *Then there exists* $C = C(q,r) > 0$ *such that*

$$\|e^{-tA} f\|_{L^r(\Omega)} \le C t^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{r}\right)} \|f\|_{L^q(\Omega)}, \quad t > 0$$

for any $f \in L^q_\sigma(\Omega)$.
(ii) *Let* $1 < q \le r \le 3$. *Then there exists* $C = C(q,r) > 0$ *such that*

$$\|\nabla e^{-tA} f\|_{L^r(\Omega)} \le C t^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{r}\right)-\frac{1}{2}} \|f\|_{L^q(\Omega)}, \quad t > 0$$

for any $f \in L^q_\sigma(\Omega)$.

Combining Theorem 1.2 and Theorem 1.3 we obtain the global solvability of (MHD) with small initial data.

Theorem 1.4. *There exists a constant* $\epsilon > 0$ *such that if* $(a,b) \in L^3_\sigma(\Omega) \times L^3_\sigma(\Omega)$ *and* $\|(a,b)\|_3 < \epsilon$, *then a unique strong solution* (u,H) *to* (MHD) *exists and satisfies the following properties:*

$$t^{(1-3/q)/2}(u,H) \in BC([0,\infty); L^q_\sigma(\Omega) \times L^q_\sigma(\Omega)) \quad \text{for any } q,\ 3 \le q \le \infty, \tag{1.1}$$

$$t^{1/2}\nabla(u,H) \in BC([0,\infty); L^3(\Omega) \times L^3(\Omega)), \tag{1.2}$$

where $BC(\cdot)$ denotes the class of bounded continuous functions. All the values in (1.1)–(1.2) *vanish at $t=0$ except for $q=3$ in* (1.1), *and in case $q=3$, $(u(0), H(0)) = (a, b)$.*

2. Sketch of the proof of Theorem 1.1

As stated in the previous section, in view of argument of Iwashita we know that once obtaining the local energy decay, by cut-off technique we have L^q-L^r estimates. Therefore, in this section we will give a sketch of our proof of Theorem 1.1.

To prove Theorem 1.1 we have to study the resolvent problem corresponding to (PM). In view of Miyakawa [12], to do this it is suffices to study the following Laplace resolvent system with perfectly conducting wall condition:

$$\begin{cases} \lambda u - \Delta u = f & \text{in} \quad \Omega, \\ \operatorname{curl} u \times \nu = 0 & \text{on} \quad \partial\Omega, \\ \nu \cdot u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.1}$$

Here $\lambda \in \Sigma_\epsilon = \{\lambda \in \mathbf{C} \setminus \{0\} \mid |\arg\lambda| < \pi - \epsilon,\ 0 < \epsilon < \pi/2\}$ and $f = (f_1(x), f_2(x), f_3(x))$ is given function. Our aim of this section is to prove the following theorem.

Theorem 2.1. *Let $1 < q < \infty$ and m be a nonnegative integer. There exists solution operator $R(\lambda) \in B(W_R^{m,p}(\Omega), W^{m+2,p}(\Omega_{R+2}))$ such that $R(\lambda)$ depends on $\lambda \in \Sigma_\epsilon$ meromorphically and has the following properties*:

(i) *The set Λ of the poles is discrete.*
(ii) *$u = R(\lambda)f$ is a solution of* (2.1) *for $\lambda \in \Sigma_\epsilon \setminus \Lambda$ and $f \in W_R^{m,p}(\Omega)$.*
(iii) *$R(\lambda) \in B(W_R^{m,p}(\Omega), W^{m+2,p}(\Omega))$ for each $\lambda \in \Sigma_\epsilon \setminus \Lambda$.*
(iv) *Let $\Sigma_\epsilon(\delta) = \{\lambda \in \Sigma_\epsilon \mid |\lambda| < \delta\}$. There exists $\delta_0 > 0$ such that $\Sigma_\epsilon(\delta_0) \cap \Lambda = \emptyset$ and $R(\lambda)$ has the following expansion of $\lambda \in \Sigma_\epsilon(\delta_0)$ in $B(W_R^{m,p}(\Omega), W^{m+2,p}(\Omega_R))$:*

$$R(\lambda) = \lambda^{1/2} G_1 + G_2(\lambda) + \lambda^{1/2} G_3(\lambda). \tag{2.2}$$

Here $G_1 \in B(W_R^{m,p}(\Omega), W^{m+2,p}(\Omega_R))$, $G_2(\lambda)$ is $B(W_R^{m,p}(\Omega), W^{m+2,p}(\Omega_R))$-valued holomorphic function of $\lambda \in \Sigma_\epsilon(\delta_0)$ and $G_3(\lambda)$ is bounded.

Here $W_R^{m,p}(\Omega) = \{f \in W^{m,p}(\Omega) \mid f = 0$ for $|x| > R\}$.

In order to prove Theorem 2.1, first of all we construct the parametrix to (2.1). Choose a positive number $R > 0$ such that $R > R_0 + 3$. Here R_0 is introduced in the previous section. Let Φ be a mapping of $f \in L^q(\Omega_{R+3})$ to unique solution $u \in W^{2,p}(\Omega_{R+3})$ of the following problem:

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega_{R+3}, \\ \operatorname{curl} u \times \nu = 0 & \text{on} \quad \partial\Omega_{R+3}, \\ \nu \cdot u = 0 & \text{on} \quad \partial\Omega_{R+3}. \end{cases}$$

Then $\Phi \in B(L^q(\Omega_{R+3}), W^{2,q}(\Omega_{R+3}))$. Put $\varphi \in C_0^\infty(\mathbf{R}^3)$ such that $\varphi = 1$ for $|x| \leq R+1$ and $= 0$ for $|x| \geq R+2$. For $f \in L^q(\Omega)$, let f_{R+3} be the restriction of f to Ω_{R+3} and let f_0 be the zero extension of f to $\mathbf{R}^3$, that is, $f_0 = f$ in Ω and $f_0 = 0$ in $\mathbf{R}^3 \setminus \Omega$. Let us define an operator $A(\lambda)$ by

$$A(\lambda)f = (1-\varphi)R_0(\lambda)f_0 + \varphi\Phi f_{R+3}. \tag{2.3}$$

Here $R_0(\lambda)$ denotes the solution operator to $\lambda u - \Delta u = f$ in $\mathbf{R}^3$ (The properties of $R_0(\lambda)$ is stated in Murata [14]). For $A(\lambda)f$ we have

$$\begin{cases} (\lambda - \Delta)A(\lambda)f = f + S(\lambda)f & \text{in} \quad \Omega, \\ \operatorname{curl}(A(\lambda)f) \times \nu = 0 & \text{on} \quad \partial\Omega, \\ \nu \cdot (A(\lambda)f) = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where

$$S(\lambda)f = 2\nabla\varphi \cdot \nabla R_0(\lambda)f_0 + (\Delta\varphi)R_0(\lambda)f_0 + \lambda\varphi\Phi f_{R+3} - 2\nabla\varphi \cdot \nabla\Phi f_{R+3} - (\Delta\varphi)\Phi f_{R+3}$$

From the definition of the cut-off function φ, $\operatorname{supp} S(\lambda)f \subset \Omega_{R+3}$. If $f \in L^p_{R+2}(\Omega)$, then by the Fourier multiplier theorem and the property of $R_0(\lambda)$, we obtain

$$\|R_0(\lambda)f_0\|_{W^{2,p}(B_{R+3})} \leq C\|f\|_{L^p(\Omega)}.$$

Lemma 2.2. *The inverse $(I+S(\lambda))^{-1}$ of $I+S(\lambda)$ exists as a $B(L^p_{R+2}(\Omega), L^p_{R+2}(\Omega))$-valued meromorphic function of $\lambda \in \Sigma_\epsilon$. The set Λ of poles is discrete and has no intersection with $\Sigma_\epsilon(\delta_0)$ for some $\delta_0 > 0$. Furthermore, $(I+S(\lambda))^{-1}$ has the same type of expansion as* (2.2).

This lemma will follow from the following lemma.

Lemma 2.3. *$I + S(0)$ has the bounded inverse $(I+S(0))^{-1}$.*

Before stating the proof of Lemma 2.3, we introduce the following uniqueness result which will be required in the proof of Lemma 2.3.

Proposition 2.4. *Let $1 < q < \infty$. Suppose that $u \in W^{2,q}_{\mathrm{loc}}(\Omega)$ satisfies*

$$\begin{cases} -\Delta u = 0 & \text{in} \quad \Omega, \\ \operatorname{curl} u \times \nu = 0 & \text{on} \quad \partial\Omega, \\ \nu \cdot u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.4}$$

and $u(x) = O(|x|^{-1})$, $\nabla u(x) = O(|x|^{-2})$. Then $u = 0$ in Ω.

Remark 2.5. The assumption that Ω is simply connected is essentially required to prove Proposition 2.4 only.

Proof of Proposition 2.4. By virtue of the local theory for the elliptic equations, one can take $u \in W^{2,r}_{\mathrm{loc}}(\Omega)$ for any $r \in (1, \infty)$. In particular, now we take $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$. We consider a function $\psi \in C_0^\infty(\mathbf{R}^3)$ with the properties $0 \leq \psi(x) \leq 1$, $\psi(x) = 1$ for $|x| \leq 1/2$ and $= 0$ for $|x| \geq 1$ and define $\psi_R(x) := \psi(x/R)$. According to the well known formula $\Delta u = \nabla \operatorname{div} u - \operatorname{curl}\operatorname{curl} u$, the divergence theorem and the assumption we get

$$\begin{aligned} 0 = \int_\Omega -\Delta u \cdot \psi_R u \, dx &= \int_{\Omega_R} \operatorname{curl} u \cdot (\nabla\psi_R \times u) \, dx + \int_{\Omega_R} (\operatorname{div} u)(\nabla\psi_R \cdot u) \, dx \\ &\quad + \int_{\Omega_R} \psi_R[(\operatorname{div} u)^2 + \operatorname{curl} u \cdot \operatorname{curl} u] \, dx. \end{aligned}$$

Since $\operatorname{supp} \psi_R \subset \{x \in \mathbf{R}^3 \mid R/2 < |x| < R\}$, we have

$$\left| \int_{\Omega_R} \operatorname{curl} u \cdot (\nabla\psi_R \times u) \, dx + \int_{\Omega_R} (\operatorname{div} u)(\nabla\psi_R \cdot u) \, dx \right| \leq \frac{C}{R}.$$

Therefore letting $R \to \infty$, we have $\| \operatorname{curl} u\|^2_{L^2(\Omega)} + \| \operatorname{div} u\|^2_{L^2(\Omega)} = 0$. This implies that $\operatorname{curl} u = 0$ and $\operatorname{div} u = 0$ in Ω and moreover by virtue of theorem due to von Wahl [16], we obtain $\nabla u = 0$. Hence $u = \text{const}$ in Ω. From the assumption that u satisfies $\nu \cdot u = 0$ on $\partial\Omega$, we have $u = 0$ in Ω in Ω. This completes the proof. ∎

Now we shall show Lemma 2.3.

Proof of Lemma 2.3. Since the operator $S(0)$ is compact, by the Fredholm alternative theorem it suffices to show injectivity of $I + S(0)$. Let us pick up $f \in L^p_{R+2}(\Omega)$ so that $(I + S(0))f = 0$. Then it follows from (2.3), $A(0)f$ satisfies (2.4) and moreover $A(0)f$ has the properties that $A(0)f = O(|x|^{-1})$ and $\nabla(A(0)f) = O(|x|^{-2})$. Therefore from Proposition 2.4, $A(0)f = 0$. Namely we have

$$(1 - \varphi)R_0(0)f_0 + \varphi\Phi f_{R+3} = 0 \quad \text{in } \Omega.$$

By the definition of the cut-off function φ we have $\Phi f_{R+3} = 0$ for $|x| \leq R+1$ and $R_0(0)f_0 = 0$ for $|x| \geq R+2$. Put $w = \Phi f_{R+3}$ for $x \in \Omega_{R+3}$ and $= 0$ for $x \notin \Omega$. Then w satisfies

$$\begin{cases} -\Delta w = f_0 & \text{in} \quad B_{R+3}, \\ \operatorname{curl} w \times \nu = 0 & \text{on} \quad \partial B_{R+3}, \\ \nu \cdot w = 0 & \text{on} \quad \partial B_{R+3}. \end{cases}$$

On the other hand, from $R(0)f_0 = 0$ for $|x| \geq R+2$, we also have

$$\begin{cases} -\Delta R_0(0)f_0 = f_0 & \text{in} \quad B_{R+3}, \\ \operatorname{curl}(R_0(0)f_0) \times \nu = 0 & \text{on} \quad \partial B_{R+3}, \\ \nu \cdot (R_0(0)f_0) = 0 & \text{on} \quad \partial B_{R+3}. \end{cases}$$

Hence we obtain $w = R_0(0)f_0$ in Ω_{R+3}. Therefore

$$0 = A(0)f = R_0(0)f_0 + \varphi(\Phi f_{R+3} - R_0(0)f_0) = R_0(0)f_0.$$

This implies $f_0 = 0$ in Ω. ∎

Proof of Lemma 2.2. Let $M = \|(I + S(\lambda))^{-1}\|$, where $\|\cdot\|$ denotes the operator norm. From the fact that $S(\lambda)$ is continuous in $\Sigma_\epsilon \cup \{0\}$, there is some $\delta_0 > 0$ such that $\|S(\lambda) - S(0)\| < 1/2M$ for any $\lambda \in \Sigma_\epsilon(\delta_0)$. Hence, for $\lambda \in \Sigma_\epsilon(\delta_0)$,

$$(I + S(\lambda))^{-1} = \sum_{j=1}^{\infty} [(I + S(0))^{-1}(S(0) - S(\lambda))]^j (I + S(0))^{-1}. \tag{2.5}$$

Since $S(\lambda)$ is holomorphic in $\lambda \in \Sigma_\epsilon$, by analytic Fredholm's alternative theorem (see e.g., Dunford and Schwartz [4, p. 592, Lemma 13]) we obtain $(I + S(\lambda))^{-1}$ for any $\lambda \in \Sigma_\epsilon$ as a meromorphic function and we see that the set Λ of poles is discrete in Σ_ϵ. The expansion of $(I + S(\lambda))^{-1}$ follows from that of the expansion of $R_0(\lambda)$, Lemma 2.3 and (2.5). ∎

With help of above results, we prove Theorem 2.1.

Proof of Theorem 2.1. Define $R(\lambda)$ by

$$R(\lambda) = A(\lambda)(I + S(\lambda))^{-1}. \tag{2.6}$$

Then the assertions are immediately derived from the expansion of $R_0(\lambda)$, Lemma 2.2 and (2.6). ∎

In the rest of this paper, we shall state the strategy of the proof of Theorem 1.1. Let $0 < \epsilon < \epsilon_1 < \pi/2$ and let γ be a contour as follows: $\gamma = \gamma_1 \cup \gamma_2$, where

$$\gamma_1 = \{\lambda \in \mathbf{C} \,|\, 0 < |\lambda| \le \delta_0/2, |\arg\lambda| = \pi - \epsilon_1\},$$
$$\gamma_2 = \{\lambda \in \mathbf{C} \,|\, |\lambda| > \delta_0/2, |\arg\lambda| = \pi - \epsilon_1\}.$$

According to Theorem 2.1, the semigroup e^{-tB} is represented as

$$e^{-tB} = \frac{1}{2\pi i}\int_{\gamma_1} e^{\lambda t} R(\lambda)\,d\lambda + \frac{1}{2\pi i}\int_{\gamma_2} e^{\lambda t}(\lambda + B)^{-1}\,d\lambda,$$

in $L^q_\sigma(\Omega) \cap L^q_{R+2}(\Omega)$. To prove Theorem 1.1, we introduce the following well known lemma concerning the gamma function $\Gamma(\sigma)$.

Lemma 2.6. *For $\sigma > 0$ and $t > 0$, it holds that*

$$\left| \frac{1}{2\pi i}\int_{\gamma_1} e^{\lambda t}\lambda^{\sigma-1}\,d\lambda - \frac{\sin\sigma\pi}{\pi}\Gamma(\sigma)t^{-\sigma} \right| \le Ce^{-ct}.$$

Finally combining Theorem 2.1 and Lemma 2.6, we have the local energy decay property of the semigroup e^{-tB}, Theorem 1.1.

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