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TAUBERIAN THEOREMS FOR FOURIER COSINE TRANSFORMS

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We prove Tauberian theorems for Fourier cosine transforms, which can be considered as analogues of the theorem of Soni and Soni for the boundary cases. They involve both Π-variation and improper integrals.

1. Introduction. We denote by R_0 the whole class of slowly varying functions at infinity, that is, R_0 is the class of positive measurable f, defined on some neighborhood of infinity, satisfying

$$\forall \lambda > 0, \qquad \lim_{x \to \infty} f(\lambda x)/f(x) = 1.$$

For $l \in R_0$, the class Π_l is the class of measurable f satisfying

$$\forall \lambda \geq 1, \qquad \lim_{x \to \infty} \{f(\lambda x) - f(x)\}/l(x) = c \log \lambda$$

for some constant c called the *l-index* of f. We write $\int_0^{\infty -}$ to denote an improper integral obtained from $\int_0^M$ by letting $M \uparrow \infty$. If $f : [0, \infty) \to \mathbb{R}$ is locally integrable and eventually non-increasing on $[0, \infty)$, $\lim_{\to \infty} f(t) = 0$, then we define its *Fourier cosine transform* g by

$$g(\xi) = \int_0^{\infty -} f(t) \cos t\xi dt \qquad (\xi > 0). \tag{1.1}$$

Since the improper integral on the right converges uniformly on each (a, ∞) with $a > 0$, g is a continuous function on $(0, \infty)$. See Theorem 6 of Titchmarsh (1948).

The aim of this paper is to prove the following theorems of Tauberian type:

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THEOREM 1.1. *Let $l \in R_0$. Let $f : [0,\infty) \to \mathbb{R}$ be locally integrable and eventually non-increasing on $[0,\infty)$, $\lim_{t\to\infty} f(t) = 0$. Let g be the Fourier cosine transform of f; define g by (1.1). Then*

$$g(1/\cdot) \in \Pi_l \text{ with index } 1 \tag{1.2}$$

implies

$$f(t) \sim t^{-1}l(t) \qquad (t \to \infty). \tag{1.3}$$

THEOREM 1.2. *Let $l \in R_0$. Let f and g be as in Theorem 1.1. Then*

$$g(\xi) \sim \xi^{-1}l(1/\xi)\pi 2^{-1} \qquad (\xi \to 0+) \tag{1.4}$$

implies

$$f \in \Pi_l \text{ with index } -1. \tag{1.5}$$

In Theorem 1.1, the Abelian implication (1.3) $\Rightarrow$ (1.2) also holds by Theorem 7(iii) of Pitman (1968), whence (1.2) and (1.3) are equivalent. We can consider Theorem 1.1 (Theorem 1.2 resp.) as an analogue of the following theorem for the boundary case $\alpha = 1$ ($\alpha = 0$ resp.).

THEOREM 1.3 [Soni and Soni (1975)]. *Let $0 < \alpha < 1$, and $l \in R_0$. Let f and g be as in Theorem 1.1. Then*

$$g(\xi) \sim \xi^{-(1-\alpha)}l(1/\xi)\Gamma(1-\alpha)\sin(\pi\alpha/2) \qquad (\xi \to 0+) \tag{1.6}$$

implies

$$f(t) \sim t^{-\alpha}l(t) \qquad (t \to \infty). \tag{1.7}$$

We note that, by Theorem 7(i) of Pitman (1968), the Abelian implication (1.7) $\Rightarrow$ (1.6) also holds in Theorem 1.3. For the previous works which are related to Theorems 1.1 and 1.2, we refer to Bingham, Goldie and Teugels [(1987), chapter 4]. In particular, de Haan (1976), and Bingham and Teugels (1980) include Abel-Tauber theorems involving Π-variation. Our results are different from the previous works in that they involve both Π-variation and improper integrals. The crucial idea of the proofs of Theorems 1.1 and 1.2 is to reduce the problem to the completely monotone case. The theorems have a direct application to stationary processes. We also apply Theorem 1.1 to Fourier cosine series to solve a problem in Boas (1967).

2. The completely monotone case.

In this section, as the first step to prove Theorems 1.1 and 1.2, we study a special case, that is, the completely monotone case.

Let σ be a finite Borel measure on $(0,\infty)$; in particular, σ has no mass at 0. We set

$$R(t) = \int_0^\infty e^{-|t|\lambda}\sigma(d\lambda) \qquad (t \in \mathbb{R}).$$

Since R is positive-definite, it is a correlation function of a real, weakly stationary process X. If we set

$$\Delta(\xi) = (1/\pi) \int_0^{\infty -} R(t) \cos t\xi dt \qquad (\xi \in \mathbb{R}), \tag{2.1}$$

then we have

$$\Delta(\xi) = (1/\pi) \int_0^{\infty} \frac{\lambda}{\lambda^2 + \xi^2} \sigma(d\lambda) \qquad (\xi \in \mathbb{R}). \tag{2.2}$$

It is easy to see that both $\Delta(\xi)$ and $(1+\xi^2)^{-1} \log \Delta(\xi)$ are integrable over $\mathbb{R}$. The function Δ corresponds to the spectral density of X [see section 4]. We define an outer function h for the Hardy class H^2 by

$$h(\zeta) = \exp\left\{ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1+\xi\zeta}{\xi - \zeta} \cdot \frac{\log \Delta(\xi)}{1+\xi^2} d\xi \right\} \qquad (\mathrm{Im}\, \zeta > 0).$$

Since h belongs to H^2, there exists $\mathrm{l.i.m.}_{\eta \downarrow 0} h(\cdot + i\eta)$ in $L^2(\mathbb{R})$, which we also denote by h. We define $E \in L^2(\mathbb{R})$ by $E = \hat{h}$, where $\hat{h}$ denotes the Fourier transform of h:

$$E(\cdot) = \operatorname*{l.i.m.}_{M \to \infty} \int_{-M}^{M} e^{-i\xi \cdot} h(\xi) d\xi.$$

Since the function E corresponds to the canonical kernel of X, we have the following relation between E and R:

$$R(t) = \frac{1}{2\pi} \int_0^{\infty} E(|t| + s) E(s) ds \qquad (t \in \mathbb{R}).$$

By Theorems 2.5 and 2.6 of Inoue (1993) which extend the results of Okabe (1986), there exists a unique Borel measure ν on $(0, \infty)$ such that

$$\int_0^{\infty} \int_0^{\infty} \frac{1}{\lambda + \lambda'} \nu(d\lambda) \nu(d\lambda') < \infty,$$
$$E(t) = \chi_{(0,\infty)}(t) \int_0^{\infty} e^{-t\lambda} \nu(d\lambda) \qquad (\text{a.e. } t \in \mathbb{R}).$$

For the details so far, we refer to Inoue (1993).

For any $l \in R_0$ such that $\int^{\infty} l(s) ds/s = \infty$, we set

$$\tilde{l}(t) = \int_M^t l(s) ds/s \qquad (t \geq M),$$

where we choose M so large that $l \in L^1_{\mathrm{loc}}[M, \infty)$. Then $\tilde{l}$ is also slowly varying [see Proposition 1.5.9a of Bingham, Goldie and Teugels (1987)].

In Inoue (1994), we have proved the following theorems:

THEOREM 2.1 [Inoue (1994), Theorem 5.1]. *Let $l \in R_0$. Then the following are equivalent:*

(1) $R \in \Pi_l$ *with index* -1,
(2) $\Delta(\xi) \sim \xi^{-1} l(1/\xi) 2^{-1}$ *as* $\xi \to 0+$,
(3) $h(iy) \sim \{y^{-1} l(1/y) 2^{-1}\}^{1/2}$ *as* $y \to 0+$,
(4) $E(t) \sim \{t^{-1} l(t) 2\pi\}^{1/2}$ *as* $t \to \infty$.

THEOREM 2.2 [Inoue (1994), Theorem 5.2]. *Let $l \in R_0$ such that $\int^{\infty} l(s)ds/s = \infty$. Suppose $\int_0^{\infty} \lambda^{-1} \sigma(d\lambda) = \infty$. Then the following are equivalent:*

(1) $R(t) \sim t^{-1} l(t)$ *as* $t \to \infty$,
(2) $\Delta(1/\cdot) \in \Pi_l$ *with index* π^{-1},
(3) $h(i/\cdot) \in \Pi_{l_1}$ *with index* 1, *where* $l_1(t) = l(t) \widetilde{l}(t)^{-1/2} 2^{-1} \pi^{-1/2}$,
(4) $E(t) \sim t^{-1} l(t) \widetilde{l}(t)^{-1/2} \pi^{1/2}$ *as* $t \to \infty$.

In Theorem 5.1 of Inoue (1994), we have assumed $\int_0^{\infty} \lambda^{-1} \sigma(d\lambda) = \infty$ as well as $\int^{\infty} l(s)ds/s < \infty$. However we have not actually used both of them in the proof, whence we have Theorem 2.1. Similarly, in the implication (2) $\Rightarrow$ (1) of Theorem 2.2, we need not assume $\int^{\infty} l(s)ds/s = \infty$ beforehand. In fact, we have the following proposition.

PROPOSITION 2.3. *Let $l \in R_0$. Suppose $\int_0^{\infty} \lambda^{-1} \sigma(d\lambda) = \infty$. Then $\Delta(1/\cdot) \in \Pi_l$ with index π^{-1} implies $\int^{\infty} l(s)ds/s = \infty$.*

PROOF. By (2.2), we have

$$\Delta(1/x) = \Delta(1) + \int_1^x \{-\dot{\Delta}(1/u) u^{-2}\} du \qquad (x \geq 1),$$

$$-\dot{\Delta}(\xi) = \frac{2\xi}{\pi} \int_0^{\infty} \frac{\lambda}{(\lambda^2 + \xi^2)^2} \sigma(d\lambda) \qquad (\xi > 0).$$

Since $\log\{-\dot{\Delta}(1/x) x^{-2}\}$ is slowly decreasing on $(0, \infty)$, $\Delta(1/\cdot) \in \Pi_l$ with index π^{-1} implies $-\dot{\Delta}(1/x) x^{-2} \sim x^{-1} l(x) \pi^{-1}$ as $x \to \infty$ by Bingham, Goldie and Teugels [(1987), Theorem 3.6.10]. Now, in (2.2), the monotone convergence theorem yields $\Delta(1/x) \to (1/\pi) \int_0^{\infty} \lambda^{-1} \sigma(d\lambda) = \infty$ as $x \uparrow \infty$, whence $\int^{\infty} l(s)ds/s = \infty$. $\square$

The analogue of Theorem 2.2 for the case $\int_0^{\infty} \lambda^{-1} \sigma(d\lambda) < \infty$ is

THEOREM 2.4. *Let $l \in R_0$. Suppose $c := \int_0^{\infty} \lambda^{-1} d\sigma(\lambda)$ is finite. Then the following are equivalent:*

(1) $R(t) \sim t^{-1} l(t)$ *as* $t \to \infty$,
(2) $\Delta(1/\cdot) \in \Pi_l$ *with index* π^{-1},
(3) $h(i/\cdot) \in \Pi_l$ *with index* $(2\pi^{1/2} c^{1/2})^{-1}$,
(4) $E(t) \sim t^{-1} l(t) \pi^{1/2} c^{-1/2}$ *as* $t \to \infty$.

PROOF. Since the proof almost parallels that of Theorem 5.2 of Inoue (1994), we only outline it. The implication (1) $\Rightarrow$ (2) follows immediately from Theorem 7(iii) of Pitman (1968). The proof of the claim that (2) implies (3) requires the most tasks but, noting $\Delta(\xi) \to c$ as $\xi \to 0+$, we can carry it out almost in the same as that of Theorem 5.2 of Inoue (1994). The implication (3) $\Rightarrow$ (4) also follows in the same way. Now since

$$\int_0^\infty R(t)dt = \frac{1}{2\pi}\iint_{0\le u\le v<\infty} E(u)E(v)dudv = \frac{1}{4\pi}\left(\int_0^\infty E(t)dt\right)^2,$$

we have $\int_0^\infty E(t)dt = 2\pi^{1/2}c^{1/2}$. Therefore (4) implies (1) by Lemma 3.8 of Inoue (1991). Thus the theorem follows. □

By Theorems 2.1, 2.2, and 2.4, and Proposition 2.3, we obtain Theorems 1.1 and 1.2 in the completely monotone case:

COROLLARY 2.5. *Let σ be a finite Borel measure on $(0,\infty)$. If f is in the form $f(t) = \int_0^\infty e^{-t\lambda}\sigma(d\lambda)$ for $t \ge 0$, then Theorems 1.1 and 1.2 hold.*

By using this corollary, we will prove the general case in the next section.

3. Proofs of Theorems 1.1 and 1.2.

We complete the proofs of Theorems 1.1 and 1.2 in this section.

PROOF OF THEOREM 1.1. Choose $M > 0$ so large that f is non-increasing on $[M,\infty)$. Set $f_1(t) = f(M)$ on $[0,M)$, and $= f(t)$ on $[M,\infty)$. Also set

$$g_1(\xi) = \int_0^{\infty-} f_1(t)\cos t\xi dt \qquad (\xi > 0).$$

Then $g(1/\cdot) \in \Pi_l$ with index 1 if and only if $g_1(1/\cdot) \in \Pi_l$ with index 1 because, for any $\lambda > 1$ and $x > 0$,

$$\begin{aligned}&|g_1(1/\lambda x) - g_1(1/x) - g(1/\lambda x) + g(1/x)|\\ &= \left|\int_0^M \{f(M) - f(t)\}\{\cos(t/\lambda x) - \cos(t/x)\}dt\right| \le \text{const.}x^{-1}(1-\lambda^{-1}).\end{aligned}$$

Thus we may assume that f is finite and non-increasing on $[0,\infty)$.

We assume (1.2). By the second mean-value theorem for integrals, $\xi g(\xi)$ is bounded on $(0,\infty)$ (see Bingham, Goldie and Teugels [(1987), page 241]). By Theorem 3.7.4 of Bingham, Goldie and Teugels (1987), $|g(1/\cdot)| \in R_0$ and so $g(\cdot) \in L^1_{\text{loc}}[0,\infty)$. Then Theorem 38 of Titchmarsh (1948) gives

$$t^{-1/2}\int_0^t f(u)du = (2/\pi)\int_0^\infty \frac{g(\xi)}{\xi^{1/2}}\frac{\sin t\xi}{(t\xi)^{1/2}}d\xi \quad (t > 0). \tag{3.1}$$

See also Bingham, Goldie and Teugels [(1987), page 240]. Since $t^{-1/2}\sin t$ is bounded on $(0,\infty)$ and $\xi^{-1/2}g(\xi)$ is integrable over $(0,\infty)$, we have dominated convergence, as $t \to \infty$, in (3.1), and so

$$\lim_{t\to\infty} t^{-1/2}\int_0^t f(u)du = 0.$$

Therefore, integrating by parts,

$$\int_1^\infty f(t)dt/t = \int_1^\infty \left(t^{-1/2}\int_1^t f(u)du\right)t^{-3/2}dt < \infty.$$

We define a measure σ on $(0,\infty)$ by

$$\sigma(d\lambda) = \chi_{(0,1)}(\lambda)f(1/\lambda)d\lambda/\lambda.$$

Then σ is finite because

$$\sigma(0,\infty) = \int_0^1 f(1/\lambda)d\lambda/\lambda = \int_1^\infty f(t)dt/t < \infty.$$

We set

$$F(t) = \int_0^\infty e^{-t\lambda}\sigma(d\lambda) \qquad (t \geq 0), \tag{3.2}$$

$$G(\xi) = \int_0^{\infty-} F(t)\cos t\xi dt \qquad (\xi > 0). \tag{3.3}$$

Then

$$G(\xi) = \int_0^\infty \frac{\lambda}{\lambda^2+\xi^2}\sigma(d\lambda) = \int_0^\infty \frac{f(t)}{1+t^2\xi^2}dt - \int_0^1 \frac{f(t)}{1+t^2\xi^2}dt \quad (\xi > 0).$$

Since

$$\frac{1}{1+t^2\xi^2} = \int_0^\infty \xi^{-1}e^{-u/\xi}\cos utdu \quad (t > 0,\ \xi > 0), \tag{3.4}$$

Theorem 36 of Titchmarsh (1948) yields

$$\int_0^\infty \frac{f(t)}{1+t^2\xi^2}dt = \int_0^\infty \xi^{-1}e^{-u/\xi}g(u)du \quad (\xi > 0).$$

Hence, for any $\lambda > 1$ and $x > 0$,

$$\begin{aligned}\frac{G(1/\lambda x) - G(1/x)}{l(x)} &= \int_0^\infty \frac{g(u/\lambda x) - g(u/x)}{l(x)}e^{-u}du \\ &\quad - \frac{(1-\lambda^{-2})}{x^2 l(x)}\int_0^1 \frac{t^2 f(t)}{\{1+(t/x)^2\}\{1+(t/\lambda x)^2\}}dt.\end{aligned}$$

The second term on the right clearly tends to zero as $x \to \infty$. Now since $g(\xi) \to 0$ as $\xi \to \infty$, $g(1/\cdot)$ can be extended to a continuous function on $[0,\infty)$. Therefore by Theorem 3.8.6 of Bingham, Goldie and Teugels (1987) we have dominated convergence, as $x \to \infty$, in the first term on the right, so it converges to $\log\lambda$. Thus $G(1/\cdot) \in \Pi_l$ with index 1, whence $F(t) \sim l(t)t^{-1}$ as $t \to \infty$ by Corollary 2.5. Since $\log x f(x)$ is slowly increasing, Karamata's Tauberian Theorem gives

$$\lambda^{-1}f(1/\lambda) \sim l(1/\lambda) \qquad (\lambda \to 0+),$$

whence (1.3). This completes the proof. □

To prove Theorem 1.2, we need the following version of the Abel-Tauber theorem of de Haan (1976).

THEOREM 3.1 [de Haan's Abel-Tauber Theorem; a version]. *Let $l \in R_0$ and $c \geq 0$. Let U be a non-decreasing, right-continuous function on $[0, \infty)$. Assume its Laplace-Stieltjes transform $\hat{U}(s) := \int_{[0,\infty)} e^{-s\lambda} dU(\lambda)$ is finite for any $s > 0$. Then the following are equivalent:*

(1) *$U(1/\cdot) \in \Pi_l$ with index $-c$,*
(2) *$\hat{U} \in \Pi_l$ with index $-c$.*

The proof of Theorem 3.1 is almost the same as that of Theorem 3.9.1 of Bingham, Goldie and Teugels (1987) except for that we use Theorem 3.7.1(iii) of Bingham, Goldie and Teugels (1987) instead of (ii).

PROOF OF THEOREM 1.2. Without loss of generality, we may assume that f is finite, positive, non-increasing, and left-continuous on $[0, \infty)$. We set $U(\lambda) = f(1/\lambda)$ for $\lambda > 0$, and $= 0$ for $\lambda = 0$. Then U is finite, non-decreasing and right-continuous on $[0, \infty)$. We define a finite measure σ on $(0, \infty)$ by $\sigma(d\lambda) = dU(\lambda)$. We also define F and G by (3.2) and (3.3), respectively. Then integration by parts yields, for any $\xi > 0$,

$$G(\xi) = \int_0^\infty \frac{\lambda}{\lambda^2 + \xi^2} \sigma(d\lambda) = \int_0^\infty \frac{\lambda^2 - \xi^2}{(\lambda^2 + \xi^2)^2} f(1/\lambda) d\lambda = \int_0^\infty \frac{1 - t^2\xi^2}{(1 + t^2\xi^2)^2} f(t) dt.$$

By (3.4),

$$\frac{t}{1 + t^2\xi^2} = \int_0^\infty \xi^{-1} e^{-v/t\xi} \cos v dv \qquad (t > 0,\ \xi > 0),$$

and so

$$\frac{1 - t^2\xi^2}{(1 + t^2\xi^2)^2} = \frac{\partial}{\partial t} \left(\frac{t}{1 + t^2\xi^2} \right) = \int_0^\infty \xi^{-2} u e^{-u/\xi} \cos ut du \quad (t > 0,\ \xi > 0),$$

whence Theorem 36 of Titchmarsh (1948) gives

$$G(\xi) = \xi^{-2} \int_0^\infty g(u) u e^{-u/\xi} du \qquad (\xi > 0).$$

Since $ug(u)$ is bounded on $(0, \infty)$, $g(\xi) \sim \xi^{-1} l(1/\xi) \pi 2^{-1}$ as $\xi \to 0+$ implies $G(\xi) \sim \xi^{-1} l(1/\xi) \pi 2^{-1}$ as $\xi \to 0+$. Therefore, by Corollary 2.5, $F \in \Pi_l$ with index -1, which by Theorem 3.1 is equivalent to $f \in \Pi_l$ with index -1. This completes the proof. □

4. Application to stationary processes.

In this section, we apply Theorems 1.1 and 1.2 to stationary processes.

Let $X = (X(t) : t \in \mathbb{R})$ be a real, weakly stationary process with zero expectation, and let R be the correlation function of X: $R(t) = E(X(t)X(0))$ for $t \in \mathbb{R}$. Let μ_X be the spectral measure of X: $R(t) = \int_{-\infty}^{\infty} e^{-it\xi} \mu_X(d\xi)$ for $t \in \mathbb{R}$. If μ_X is absolutely continuous with respect to the Lebesgue measure $d\xi$, then we call the density Δ the spectral density of X: $\mu_X(\xi) = \Delta(\xi) d\xi$. The spectral density Δ is a non-negative, even and integrable function on $\mathbb{R}$.

PROPOSITION 4.1. *Assume that the correlation function R is eventually monotone on $[0,\infty)$, $\lim_{t\to\infty} R(t) = 0$. Then the spectral measure of X is absolutely continuous with respect to the Lebesgue measure, and the spectral density Δ is given by (2.1).*

Proof. By the assumption, R is even, continuous, and either eventually positive and non-increasing, or eventually negative and non-decreasing. By Lévy's inversion formula, if a and b are continuity points of μ_X such that $0 < a < b$, then

$$\mu_X(a,b) = \lim_{M\to\infty} \frac{1}{2\pi} \int_{-M}^{M} \frac{e^{itb} - e^{ita}}{it} R(t)dt = F(b) - F(a),$$

where

$$F(\xi) := \frac{1}{\pi} \int_0^{\infty-} \frac{R(t)}{t} \sin \xi t dt \qquad (\xi > 0).$$

Since the improper integral $\int_0^{\infty-} R(t) \cos \xi t dt$ converges uniform in $\xi > \epsilon$ for any $\epsilon > 0$, the function F is of C^1-class in $(0,\infty)$ and satisfies

$$F'(\xi) = \frac{1}{\pi} \int_0^{\infty-} R(t) \cos \xi t dt \qquad (\xi > 0).$$

Therefore μ_X is absolutely continuous in $(0,\infty)$, and the density there is equal to the derivative F'. If we put $\Delta(\xi) := F'(\xi)$ for $\xi > 0$, then we obtain

$$R(t) = \mu_X\{0\} + 2 \int_0^{\infty} \Delta(\xi) \cos \xi t d\xi \qquad (t \in \mathbb{R}).$$

By the Riemann-Lebesgue Lemma, the second term on the right converges to zero as $t \to \infty$, so that $\mu_X\{0\} = 0$. This completes the proof. □

By Proposition 4.1, Theorems 1.1 and 1.2, and Theorem 7(iii) of Pitman (1968), we immediately obtain the following theorem:

THEOREM 4.1. *Let $l \in R_0$. Assume that the correlation function R is eventually non-increasing on $[0,\infty)$, $\lim_{t\to\infty} R(t) = 0$. Then*

(1) *$R(t) \sim t^{-1} l(t)$ as $t \to \infty$ if and only if $\Delta(1/\cdot) \in \Pi_l$ with index π^{-1}.*
(2) *$R \in \Pi_l$ with index -1 if $\Delta(\xi) \sim \xi^{-1} l(1/\xi) 2^{-1}$ as $\xi \to 0+$.*

REMARK 4.2. In Theorem 4.1, we set $V(s) := \int_0^s \xi \Delta(\xi) d\xi$ for $s \geq 0$. Then the proof of Theorem 1.2 implies that $R \in \Pi_l$ with index -1 if and only if $V(s) \sim s l(1/s) 2^{-1}$ as $s \to 0+$. Therefore, if $\xi \Delta(\xi)$ satisfies a Tauberian condition near 0, e.g., $\Delta(1/\cdot)$ is eventually positive and $\log \Delta(1/\cdot)$ is slowly decreasing, then $R \in \Pi_l$ with index -1 is equivalent to $\Delta(\xi) \sim \xi^{-1} l(1/\xi) 2^{-1}$ as $\xi \to 0+$.

5. Application to Fourier cosine series.

In this section, we apply Theorem 1.1 to Fourier cosine series. Suppose $\{a_n\}$ is non-increasing, and tends to 0 as $n \to \infty$. We set

$$f(t) = \begin{cases} 2a_0 & (0 \leq t \leq 1/2), \\ a_n & (n - 1/2 < t \leq n + 1/2, \quad n = 1, 2, \cdots). \end{cases}$$

Then

$$g(\xi) = \frac{\sin(\xi/2)}{(\xi/2)} G(\xi) \qquad (0 < \xi < \pi), \tag{5.1}$$

where

$$g(\xi) = \int_0^{\infty-} f(t) \cos t\xi dt \qquad (\xi > 0), \tag{5.2}$$

$$G(\xi) = \sum_{n=0}^{\infty} a_n \cos n\xi \qquad (0 < \xi < \pi). \tag{5.3}$$

Let $l \in R_0$. Then, by (5.1) and Theorem 3.7.4 of Bingham, Goldie and Teugels (1987), we easily see that $g(1/\cdot) \in \Pi_l$ with index 1 if and only if $G(1/\cdot) \in \Pi_l$ with index 1. Therefore, by Theorem 1.1 and Theorem 7(iii) of Pitman (1968), we obtain the following theorem:

THEOREM 5.1. *Let $l \in R_0$. Assume that $\{a_n\}$ is non-increasing, and tends to zero as $n \to \infty$. Define G by (5.3). Then $a_n \sim n^{-1} l(n)$ as $n \to \infty$ if and only if $G(1/\cdot) \in \Pi_l$ with index* 1.

Let K be a positive constant. If we set $l \equiv K$ in Theorem 5.1, then we obtain an answer to a question in Boas [(1967), page 45].

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