Large Deviations and Central Limit Theorems for Eyraud-Farlie-Gumbel-Morgenstern Processes

Toshio Mikami

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by

Toshio Mikami
Department of Mathematics
Hokkaido University
Sapporo 060, Japan

ABSTRACT

Let $\{X_n\}_{n=1}^{\infty}$ be a Eyraud-Farlie-Gumbel-Morgenstern process. Put $S_n \equiv \sum_{k=1}^{n} X_k$. In this paper we prove the large deviations theorem for S_n/n, and the central limit theorem for $S_n/n^{1/2}$, as $n \to \infty$.

1. Introduction.

Let $(\Omega, \mathbf{B}, P)$ be a probability space and $\{X_n\}_{n=1}^{\infty}$ be a sequence of real valued random variables on $(\Omega, \mathbf{B}, P)$. Put

$$S_n \equiv \sum_{n=1}^{\infty} X_n. \tag{1.1}$$

In this paper we prove the large deviations theorem for S_n/n, and the central limit theorem for $S_n/n^{1/2}$, as $n \to \infty$, when $\{X_n\}_{n=1}^{\infty}$ is a Eyraud-Farlie-Gumbel-Morgenstern (EFGM) process.

Let us give the definition of EFGM random process.

Definition 1.1(see [1], [2], [7]). A sequence $\{X_n\}_{n=1}^{\infty}$ of real valued random variables on a probability space $(\Omega, \mathbf{B}, P)$ is called a Eyraud-Farlie-Gumbel-Morgenstern (EFGM) process if there exists a sequence $\{\alpha_{kj}\}_{1\le k<j<\infty}$ such that for any $n \in N$ and any $x_1, \cdots, x_n \in R$,

$$\begin{aligned} &P(X_1 \le x_1, \cdots, X_n \le x_n) \\ &= \Pi_{i=1}^{n} F_i(x_i)(1 + \sum_{1\le k<j\le n} \alpha_{kj}(1-F_k(x_k))(1-F_j(x_j))), \end{aligned} \tag{1.2}$$

where we put $F_i(x_i) \equiv P(X_i \le x_i)$ $(1 \le i \le n)$, and where

$$1 + \sum_{1\le k<j\le n} \alpha_{kj}\epsilon_k\epsilon_j \ge 0 \tag{1.3}$$

for all

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$$\epsilon_i = \begin{cases} -\sup\{\{F_i(x); x \in R\} \backslash \{0,1\}\}, & \text{or} \\ 1 - \inf\{\{F_i(x); x \in R\} \backslash \{0,1\}\} & \end{cases} \qquad (1.4).$$

$(i = 1, 2, \cdots, n)$.

In this paper we restrict our attention to the following EFGM process.
(A.0). $\{X_n\}_{n=1}^{\infty}$ is a EFGM process for which (1.3) holds for all $\epsilon_i = 1$ or -1, and $E[X_n] = 0$ for all $n \in N$.

Remark 1.1. If $\{F_i(x)\}_{1 \le i \le n}$ are continuous, then $\epsilon_i = 1$ or -1 in (1.4). For any $m < n$, putting $x_i = \infty$ for $i \neq m, n$ in (1.2)-(1.4), we get

$$|\alpha_{mn}| < 1 \qquad (1.5).$$

under (A.0)(see [2], p. 208).

Let us consider the weak low of large numbers for S_n/n. By the Chebychef's inequality,

$$\begin{aligned} &P(|S_n/n| > \delta) \qquad (1.6). \\ &\le E[|S_n/n|^2]/\delta^2 \\ &= (\sum_{i=1}^{n} E[X_i^2] + 2 \sum_{1 \le k < j \le n} E[X_k X_j])/(n\delta)^2 \\ &= (\sum_{i=1}^{n} E[X_i^2] + 2 \sum_{1 \le k < j \le n} \alpha_{kj} E[X_k(1 - F_k(x_k) - F_k(x_k-))] \\ &\quad \times E[X_j(1 - F_j(x_j) - F_j(x_j-))])/(n\delta)^2. \end{aligned}$$

This is true, since from (1.2),

$$\begin{aligned} E[X_k X_j] = \int_{R^2} & x_k x_j (1 + \alpha_{kj}(1 - F_k(x_k) - F_k(x_k-)) \qquad (1.7). \\ & \times (1 - F_j(x_j) - F_j(x_j-))] \Pi_{i=1}^{n} dF_i(x_i), \end{aligned}$$

and since $E[X_k] = 0$ for $1 \le k$ from (A.0).

If $\{X_n\}_{n=1}^{\infty}$ is identically distributed, then the last quantity in (1.6) converges to 0 if and only if

$$\lim_{n \to \infty} (\sum_{1 \le k < j \le n} \alpha_{kj}/n^2) = 0. \qquad (1.8).$$

From (1.6)-(1.8), it might seem that the dependence of $\{X_n\}_{n=1}^{\infty}$ controls the weak low of large numbers for S_n/n. But this is not true. This fact can be shown by proving the upper bound of the large deviations theorem for S_n/n.

In section 2, we prove the large deviations theorem for EFGM processes under the assumptions only on marginal distributions $\{P(X_n \in dx)\}_{n=1}^{\infty}$.

In section 3, we prove the central limit theorem for $S_n/n^{1/2}$.

2. Large deviations theorem for S_n/n.

In this section we prove the large deviations theorem for S_n/n. As an application of the upper bound on large deviations for S_n/n, we show the weak low of large numbers for S_n/n.

Let us give the assumptions on $\{P(X_n \in dx)\}_{n=1}^{\infty}$.

(A.1). For $z \in R$, put

$$H(z) \equiv \limsup_{n\to\infty}\{\sum_{i=1}^{n}\log(E[\exp(zX_i)])\}/n. \qquad (2.1).$$

Then $H(z)$ is finite in some neighborhood of $z = 0$.

Let us also give the notations;

$$L(u) \equiv \sup_{z\in R}(zu - H(z)), \qquad (2.2).$$

$$\Phi(s) \equiv \{u \in R; L(u) \le s\}. \qquad (2.3).$$

Remark 2.1. If $\{X_n\}_{n=1}^{\infty}$ are independent, then

$$\log(E[\exp(zS_n)]) = \sum_{i=1}^{n}\log(E[\exp(zX_i)]), \qquad (2.4).$$

which implies that (A.1) is a reasonable assumption(see [4] and [6], Chap. 5, section 1). $L(u)$ is lower semicontinuous from (2.2), and converges to $+\infty$ as $|u| \to \infty$ from (A.1)(see [8]).

The following result can be obtained almost in the same way as in [6](see also [4]).

Theorem 2.1.

Suppose that (A.0)-(A.1) hold. Then the following holds.

(O). For any $s > 0$, $\Phi(s)$ is compact in R.

(I). For any $\delta > 0$, and any $s > 0$,

$$\limsup_{n\to\infty}\{\log[P(dist(S_n/n, \Phi(s)) \ge \delta)]\}/n \le -s. \qquad (2.5).$$

(Proof). From the last part of Remark 2.1, (O) can be proved (see [6], Chap. 5, section 1).

From [6], pp. 138-139(see also [8]), we only have to show the following; for $z \in R$,

$$\limsup_{n\to\infty}\{\log(E[\exp(z\sum_{i=1}^{n}X_i)])\}/n \le H(z). \qquad (2.6).$$

Let us prove (2.6). From (1.2) and Remark 1.1,

$$
\begin{aligned}
&E[\exp(z\sum_{i=1}^{n}X_i)] \qquad (2.7).\\
&= \int_{R^n}[1+\sum_{1\le k<j\le n}\alpha_{kj}(1-F_k(x_k)-F_k(x_k-))\\
&\quad \times (1-F_j(x_j)-F_j(x_j-))]\exp(z\sum_{\ell=1}^{n}x_\ell)\Pi_{i=1}^{n}dF_i(x_i)\\
&\le \int_{R^n}[1+(n-1)n/2]\Pi_{i=1}^{n}\exp(zx_i)dF_i(x_i)\\
&= [1+(n-1)n/2]\Pi_{i=1}^{n}E[\exp(zX_i)].
\end{aligned}
$$

Q.E.D.

Before we prove the lower bound on large deviations for S_n/n, let us show the weak low of large numbers for S_n/n.

From (A.0), by Jensen's inequality, it is easy to see that the following holds(see [6], Chap. 5, section 1 and [8]);

$$
H(z)\begin{cases}\ge 0, & \text{if } z\in R,\\ = 0, & \text{if } z=0.\end{cases} \qquad (2.8).
$$

From (2.2) and (2.8),

$$
L(u)\begin{cases}\ge 0, & \text{if } u\in R,\\ = 0, & \text{if } u=0.\end{cases} \qquad (2.9).
$$

To prove the weak low of large numbers for S_n/n, we need the following assumption.

(A.2). The convex function $L(u)=0$ if and only if $u=0$.

The following result can be proved from Theorem 2.1 in the routine manner, and hence we only state the outline of proof(see [6], Chap. 5, section 1).

Theorem 2.2.

Suppose that (A.0)-(A.2) hold. Then for any $\delta>0$,

$$
\lim_{n\to\infty}P(|S_n/n|\ge\delta)=0. \qquad (2.10).
$$

(Proof). We only have to show that (2.10) is true for sufficiently small $\delta>0$, and that, from Theorem 2.1, for any $\delta>0$

$$
s_\delta \equiv \inf_{|u|\ge\delta}L(u)>0, \qquad (2.11).
$$

which is true from the last part of Remark 2.1, and from (A.2)(see [8]).

In fact

$$P(|S_n/n| \geq \delta) \tag{2.12}$$
$$\leq P(dist(S_n/n, \Phi(s_\delta/2)) \geq dist(\Phi(3s_\delta/4)^c, \Phi(s_\delta/2))).$$

From Theorem 2.1 and from the following which can be shown by (2.11), we get (2.10); for sufficiently small $\delta > 0$,

$$dist(\Phi(3s_\delta/4)^c, \Phi(s_\delta/2)) > 0. \tag{2.13}$$

Q.E.D.

Before we finally prove the lower bound on large deviations for S_n/n, let us give the following strong assumption.
(A.3). For any $z \in R$, the following limit exists including infinity;

$$\lim_{n\to\infty} \{\sum_{i=1}^{n} \log(E[\exp(zX_i)])\}/n \equiv H(z), \tag{2.14}$$

and $H(z)$ is finite in some neighborhood of $z = 0$. $L(u)$ is strictly convex in a dense subset of the set $\{u \in R; L(u) < \infty\}$.
(A.4). For any z for which $H(z) < \infty$,

$$\{\log(\min_{i=1}^{n} E[\exp(zX_i)(F_i(X_i) + F_i(X_i-))] \tag{2.15}$$
$$\times E[\exp(zX_i)(2 - F_i(X_i) - F_i(X_i-))]E[\exp(zX_i)]^{-2})\}/n$$
$$\to 0 \qquad \text{as } n \to \infty,$$

and for any z for which $H(z) = \infty$,

$$\lim_{n\to\infty} \{\sum_{i=1}^{n} \log(E[\exp(zX_i)]) \tag{2.16}$$
$$+ \log(\min_{i=1}^{n} E[\exp(zX_i)(F_i(X_i) + F_i(X_i-))]$$
$$\times E[\exp(zX_i)(2 - F_i(X_i) - F_i(X_i-))]E[\exp(zX_i)]^{-2})\}/n$$
$$= \infty.$$

Remark 2.2. The right hand side of (2.15) is nonpositive for each $n \geq 1$ (see the last part of (2.19) below).

The following result can be obtained in the same way as in [6], Chap. 5, section 1(see also [4]).

Theorem 2.3.

Suppose that (A.0), (A.3) and (A.4) hold. Then
(II). For any $\delta > 0$, and $u \in R$,

$$\liminf_{n\to\infty}\{\log[P(|S_n/n - u| < \delta)]\}/n \geq -L(u). \qquad (2.17).$$

(Proof). From [5], Chap. 5, Theorem 1.2, it is enough to show the following; for all $z \in R$,

$$\lim_{n\to\infty}\{\log(E[\exp(zS_n)])\}/n = H(z). \qquad (2.18).$$

Let us prove (2.18) from (A.4). In the same way as in (2.7),

$$\begin{aligned}
&E[\exp(z\sum_{i=1}^{n} X_i)] \qquad (2.19).\\
&= \int_{R^n}[1 + \sum_{1\leq k<j\leq n} \alpha_{kj}(1 - F_k(x_k) - F_k(x_k-))\\
&\quad \times (1 - F_j(x_j) - F_j(x_j-))]\Pi_{i=1}^{n}\exp(zx_i)dF_i(x_i)\\
&= \Pi_{i=1}^{n}E[\exp(zX_i)](1\\
&\quad - (\max_{i=1}^{n}\{E[\exp(zX_i)(1 - F_i(X_i) - F_i(X_i-))]/E[\exp(zX_i)]\}^2)\\
&\quad + (\max_{i=1}^{n}\{E[\exp(zX_i)(1 - F_i(X_i) - F_i(X_i-))]/E[\exp(zX_i)]\}^2)\\
&\quad \times (1 + \sum_{1\leq k<j\leq n}\alpha_{kj}\\
&\quad \times (E[\exp(zX_k)(1 - F_k(X_k) - F_k(X_k-))]/E[\exp(zX_k)])\\
&\quad \times (\max_{i=1}^{n}|E[\exp(zX_i)(1 - F_i(X_i) - F_i(X_i-))]/E[\exp(zX_i)]|)^{-1}\\
&\quad \times (E[\exp(zX_j)(1 - F_j(X_j) - F_j(X_j-))]/E[\exp(zX_j)])\\
&\quad \times (\max_{i=1}^{n}|E[\exp(zX_i)(1 - F_i(X_i) - F_i(X_i-))]/E[\exp(zX_i)]|)^{-1}\\
&\geq \Pi_{i=1}^{n}E[\exp(zX_i)]\{1 - (\max_{i=1}^{n}\{E[\exp(zX_i)(1 - F_i(X_i) - F_i(X_i-))]\\
&\quad /E[\exp(zX_i)]\}^2)\} \qquad \text{(from (1.3))}\\
&= \Pi_{i=1}^{n}E[\exp(zX_i)]\min_{i=1}^{n}\{E[\exp(zX_i)(F_i(X_i) + F_i(X_i-))]\\
&\quad \times E[\exp(zX_i)(2 - F_i(X_i) - F_i(X_i-))]E[\exp(zX_i)]^{-2}\}.
\end{aligned}$$

From (2.6), and (A.3)-(A.4), the proof is over.

Q. E. D.

Let us give the following assumption to state the corollary to Theorems 2.1-2.3.

(A.5). $\{X_n\}_{n=1}^{\infty}$ is identically distributed and for all $z \in R$,

$$E[\exp(zX_1)] < \infty, \tag{2.20.}$$
$$E[|X_1|^2] > 0. \tag{2.21.}$$

Remark 2.3. (A.3) is stronger than (A.1). (A.5) implies (A.2)-(A.4).

Corollary 2.4.

Suppose that (A.0) and (A.5) holds. Then (O), (I) in Theorem 2.1, and (II) in Theorem 2.3 holds. In particular (2.10) in Theorem 2.2 holds.

3. Central Limit Theorem for $S_n/n^{1/2}$.

In this section we prove the central limit theorem for $S_n/n^{1/2}$. In this section we assume the following.
(A.6). $\{X_n\}_{n=1}^{\infty}$ is an identically distributed EFGM process such that $E[X_n] = 0$ for all $n \in N$, and such that $0 < s_n \equiv (nE[X_1^2])^{1/2} < \infty$ $(n \geq 1)$, and such that the following holds;

$$\lim_{n\to\infty} \Big(\sum_{1\leq k<j\leq n} \alpha_{kj} \Big)/n = 0. \tag{3.1.}$$

In the same way as in (2.7), for any $z \in R$,

$$\begin{aligned}
&E[\exp((-1)^{1/2} z S_n/s_n)] \\
&= \Pi_{i=1}^{n} E[\exp((-1)^{1/2} z X_i/s_n)](1 + \sum_{1\leq k<j\leq n} \alpha_{kj} \\
&\quad \times E[\exp((-1)^{1/2} z X_k/s_n)(1 - F_k(X_k) - F_k(X_k-))] \\
&\qquad \times E[\exp((-1)^{1/2} z X_k/s_n)]^{-1} \\
&\quad \times E[\exp((-1)^{1/2} z X_j/s_n)(1 - F_j(X_j) - F_j(X_j-))] \\
&\qquad \times E[\exp((-1)^{1/2} z X_j/s_n)]^{-1} \\
&\equiv (I)_n \times (II)_n.
\end{aligned} \tag{3.2.}$$

In (3.2), $(I)_n$ is a characteristic function of $\sum_{i=1}^{n} Y_i/s_n$, where $\{Y_i\}_{i=1}^{n}$ are independent, real valued random variables such that

$$P(Y_i \in dx) = P(X_i \in dx), \tag{3.3.}$$

for all $i = 1, \cdots, n$.

The following result is well known (see [5], p. 259, Theorem 1).

Theorem 3.1.

Suppose that $\{Y_n\}_{n=1}^{\infty}$ is identically distributed, and that $0 < E[Y_1^2] < \infty$. Then $\sum_{i=1}^{n} Y_i/s_n$ converges to the normal distribution $N(0,1)$ with zero expection and unit variance, as $n \to \infty$, in distribution, that is, for any $z \in R$,

$$\lim_{n\to\infty} E[\exp((-1)^{1/2}z\sum_{i=1}^{n} Y_i/s_n)] = \exp(-z^2/2). \tag{3.4}$$

By Theorem 3.1 above, we get the following result.

Theorem 3.2.

Suppose that (A.6) holds. Then S_n/s_n converges, as $n \to \infty$, to the normal distribution $N(0,1)$ with zero expection and unit variance in distribution. (Proof). Since $\{X_n\}_{n=1}^{\infty}$ is identically distributed, $(II)_n$ in (3.2) behaves, as $n \to \infty$, as follows;

$$\begin{aligned}(II)_n &= 1 + (\sum_{1\le k<j\le n} \alpha_{kj})(E[\exp((-1)^{1/2}zX_1/s_n) \\ &\quad \times (1 - F_1(X_1) - F_1(X_1-))]/E[\exp((-1)^{1/2}zX_1/s_n)])^2 \\ &\to 1 \qquad \text{as } n \to \infty.\end{aligned} \tag{3.5}$$

Here we used the following;

$$E[1 - F_1(X_1) - F_1(X_1-)] = 1 - \int_R dF_1^2(dy) = 0, \tag{3.6}$$

and

$$\begin{aligned}&|E[\exp((-1)^{1/2}zX_1/s_n)(1 - F_1(X_1) - F_1(X_1-))]| \\ &= |E[(\exp((-1)^{1/2}zX_1/s_n) - 1)(1 - F_1(X_1) - F_1(X_1-))]| \\ &\le |z|E[|X_1|]/s_n.\end{aligned} \tag{3.7}$$

From (3.5) and Theorem 3.1 above, the proof is over.

Q.E.D.

Remark 3.1. Suppose that (A.6) holds. Then the following is true;

$$\lim_{n\to\infty} (E[S_n^2]/s_n^2) = 1. \tag{3.8}$$

This is true, since

$$\begin{aligned}&E[S_n^2]/s_n^2 \\ &= (s_n^2 + 2(\sum_{1\le k<j\le n} \alpha_{kj})E[X_1(1 - F_1(X_1) - F_1(X_1-))]^2)/s_n^2 \\ &\to 1 \qquad \text{as } n \to \infty.\end{aligned} \tag{3.9}$$

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