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its applications

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Copula fields and its applications

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ABSTRACT.
We define the concept of the copula field, and give the applications to the stochastic quantization and to the stochastic control.

1. Introduction.

Let $\psi(t,x)$ $(t \geq 0,\ x \in R^d)$ be the solution of the following Schrödinger equation;

$$(-1)\partial\psi(t,x)/\partial t = -\triangle_x\psi(t,x)/2 + V(x)\psi(t,x) \quad (t > 0, x \in R^d), \\ \int_{R^d} |\psi(0,x)|^2 dx = 1, \tag{1.1}$$

for some function $V(\cdot) : R^d \mapsto R$. Here we put $\triangle_x \equiv \sum_{i=1}^d \partial^2/\partial x_i^2$.

One of basic problems in stochastic quantizations is to construct a stochastic process $\{X(t)\}_{t\geq 0}$ such that

$$P(X(t) \in dx) = |\psi(t,x)|^2 dx \tag{1.2}$$

for all $t \geq 0$(see [15], [16]).

From (1.1), for any infinitely differentiable function $f : R^d \mapsto R$ with a compact support,

$$d[\int_{R^d} f(x)|\psi(t,x)|^2 dx]/dt \\ = \int_{R^d} [\triangle_x f(x)/2 + < b(t,x;\psi), \nabla_x f(x) >]|\psi(t,x)|^2 dx, \tag{1.3}$$

where we put

$$b(t,x;\psi) \\ \equiv \begin{cases} Re(\nabla_x\psi(t,x)/\psi(t,x)) + Im(\nabla_x\psi(t,x)/\psi(t,x)) & \text{if } \psi(t,x) \neq 0, \\ 0 & \text{if } \psi(t,x) = 0. \end{cases} \tag{1.4}$$

Here we put $\nabla_x \equiv (\partial/\partial x_i)_{i=1}^d$, and $< \cdot, \cdot >$ denotes the inner product in R^d.

It is known that one can construct a Markov process $\{X(t)\}_{t\geq 0}$ such that (1.2) holds under the following condition; for all $t > 0$,

$$\int_0^t ds \int_{R^d} |b(s,x;\psi)|^2 |\psi(s,x)|^2 dx < \infty \qquad (1.5).$$

(see [5]-[7], [13], [14], [20]).

In section 2 we show the following; for any flow of Borel probability measures $\{\rho(t,dx)\}_{t\geq 0}$ on R, there exists a real valued stochastic process $\{X(t)\}_{t\geq 0}$ such that

$$P(X(t) \in dx) = \rho(t,dx) \qquad (1.6).$$

for all $t \geq 0$. Stochastic processes $\{X(t)\}_{t\geq 0}$ can be taken to have any kinds of dependence in time, to be a Markov process, or to be a one dimensional local Markov field, etc.(see Theorem 2.3 and Proposition 2.4). Main idea is that of copula in the multivariate analysis(see [8], [18], [19]). We give the definition of a **copula field**, extending the idea to the path space directly.

In section 3 we consider the application to the one dimensional stochastic quantization and show that Nelson's example in [17] does not deny, in the stochastic quantization, the approach to construct stochastic processes from given flow of marginals (probability measures). We also give the application to the stochastic control theory.

2. Copula fields.

In this section we show how to construct a real valued stochastic process from a flow of Borel probability measures on R, extending directly the idea of copula, to the path space. We also give the definition of the **copula field**, and its characterization. In this section we denote by I the parameter space.

Let us first introduce the definition of copulas.

Definition 2.1.(see [18], [19]). $C : [0,1]^2 \mapsto [0,1]$ is called a 2-copula if the following holds; for all $u \in [0,1]$

$$C(u,0) = C(0,u) = 0, \qquad (2.1).$$

$$C(u,1) = C(1,u) = u, \qquad (2.2).$$

and for all $u_1, u_2, v_1, v_2 \in [0,1]$ for which $u_1 \leq v_1$, $u_2 \leq v_2$,

$$C(v_1, v_2) + C(u_1, u_2) - C(v_1, u_2) - C(u_1, v_2) \geq 0. \qquad (2.3).$$

For $n \geq 3$, $C : [0,1]^n \mapsto [0,1]$ is called a n-copula if the following holds; for all $i (1 \leq i \leq n)$ and all $u_1, \cdots, u_{i-1}, u_{i+1}, \cdots, u_n \in [0,1]$,

$$C(u_1, \cdots, u_{i-1}, 0, u_{i+1}, \cdots, u_n) = 0, \qquad (2.4).$$

and $C(u_1, \cdots, u_{i-1}, 1, u_{i+1}, \cdots, u_n)$ is a $(n-1)$-copula; for all $\{u_k, v_k; k = 1, \cdots, n\} \subset [0,1]$ for which $u_k \leq v_k$ $(k = 1, \cdots, n)$,

$$\sum_{\epsilon_k = u_k \text{ or } v_k (k=1,\cdots,n)} (-1)^{\#(\{k; \epsilon_k = u_k\})} C(\epsilon_1, \cdots, \epsilon_n) \geq 0. \qquad (2.5).$$

Remark 2.1. It is known that copulas are continuous(see [18], Lemma 6.2.3).

The following result is known and crucial in this paper.

Theorem 2.1. *([18], Theorems 6.2.4, 6.2.5).*

(I). For any $n \geq 1$, and any distribution function $F(x_1, \cdots, x_n)$ on R^n(see [2]), there exists a n-copula $C(u_1, \cdots, u_n)$ such that for all $x_1, \cdots, x_n \in R$,

$$F(x_1, \cdots, x_n) = C(F_1(x_1), \cdots, F_n(x_n)), \qquad (2.6).$$

where we put for k $(1 \leq k \leq n)$

$$F_k(x_k) = \lim_{i \neq k, x_i \to \infty} F(x_1, \cdots, x_n). \qquad (2.7).$$

(II). For any n-copula $C(u_1, \cdots, u_n)$ and any distribution functions $F_k(x_k)$ $(1 \leq k \leq n)$ on R, the function $F(x_1, \cdots, x_n) \equiv C(F_1(x_1), \cdots, F_n(x_n))$ is a distribution function on R^n which satisfies (2.7).

Let us give the definition of a copula for a real valued stochastic process which is well defined from Theorem 2.1.

Definition 2.2. For any real valued stochastic process $\{X(t)\}_{t\in I}$ on a probability space (Ω, B, P), the family $\{C_A^X(u_1, \cdots, u_{\#(A)})\}_{A\subset I, \#(A)<\infty}$ of copulas which satisfies the following is called a **copula for $\{\mathbf{X(t)}\}_{\mathbf{t}\in\mathbf{I}}$** ; for any $A = \{t_1^A, \cdots, t_{\#(A)}^A\} \subset I$ and any $x_1, \cdots, x_{\#(A)} \in R$

$$P(X(t_1^A) \leq x_1, \cdots, X(t_{\#(A)}^A) \leq x_{\#(A)}) = C_A^X(F_{t_1^A}^X(x_1), \cdots, F_{t_{\#(A)}^A}^X(x_{\#(A)})), \tag{2.8}$$

where we put $F_t^X(x) = P(X(t) \leq x)$.

Before we give the definition of a copulas field for a real valued stochastic process, let us give some notations. Denote by $DF(R)$ the set of all continuous distribution functions on R. For $F \in DF(R)$, we can define the functions $F^*(u)(0 \leq u \leq 1)$ by the following; put

$$\begin{aligned} F^*(0) &\equiv \begin{cases} \max\{x; F(x) = 0\} & \text{if } 0 \in Range(F), \\ -\infty & \text{if } 0 \notin Range(F), \end{cases} \\ F^*(u) &\equiv \min\{x; F(x) = u\} \quad \text{for } 0 < u < 1, \\ F^*(1) &\equiv \begin{cases} \min\{x; F(x) = 1\} & \text{if } 1 \in Range(F), \\ \infty & \text{if } 1 \notin Range(F) \end{cases} \end{aligned} \tag{2.9}$$

(see [18], p. 49). Put $DF(R)^* \equiv \{F^*; F \in DF(R)\}$; $DF(R)_I \equiv \{\{F_t\}_{t\in I};$ $F_t \in DF(R)(t \in I)\}$; $DF(R)_I^* \equiv \{\{F_t^*\}_{t\in I}; F_t \in DF(R)(t \in I)\}$.

Definition 2.3. For any real valued stochastic process $\{X(t)\}_{t\in I}$ on a probability space (Ω, B, P), the **copula field** $\{\mathbf{C}^X(t, \mathbf{F}^*)\}_{t\in I, \mathbf{F}^*\in DF(R)_I^*}$ **for $\{\mathbf{X(t)}\}_{\mathbf{t}\in\mathbf{I}}$** is defined as follows; for all $t \in I$ and $\mathbf{F}^* = \{F_t^*\}_{t\in I} \in DF(R)_I^*$,

$$\mathbf{C}^X(t, \mathbf{F}^*) = F_t^*(F_t^X(X(t))). \tag{2.10}$$

Remark 2.2. The copula for a real valued stochastic process $\{X(t)\}_{t\in I}$ is uniquely determined if and only if $F_t^X(x)$ is continuous in $x \in R$ for all $t \in I$. Copula field for a real valued stochastic process is unique. F^* is a quasi-inverse of F(see [18], p. 49), and our choice in (2.9) is convenient as we show in the next proposition.

Proposition 2.2.

For any $F \in DF(R)$, F^* is strictly increasing, left continuous and has a right hand side limits, and the following holds; for any $u \in (0, 1)$, and any $y \in R$,

$$F^*(u) \leq y \quad \text{if and only if } u \leq F(y). \tag{2.11}$$

Proof. Since the first half of the proposition is trivial, we only show that (2.11) holds.

Suppose that $F^*(u) \leq y$. Then

$$u = F(F^*(u)) \leq F(y) \tag{2.12}$$

(see [2]). This is true. In fact, since F is continuous, $(0,1) \subset Range(F)$. Therefore for any $u \in (0,1)$,

$$F(F^*(u)) = u \qquad (2.13).$$

(see [18], (4.4.4)).

Suppose that $u \leq F(y)$. Then

$$F^*(u) \leq F^*(F(y)) \leq y, \qquad (2.14).$$

from (2.9), which completes the proof.

Q.E.D.

The next theorem shows that a copula field for a real valued stochastic process is a path space version of the idea of copula.

Theorem 2.3.

For any $\{F_t\}_{t\in I} \in DF(R)_I$, and any real valued stochastic process $\{X(t)\}_{t\in I}$ on a probability space (Ω, B, P) for which $\{F_t^X\}_{t\in I} \in DF(R)_I$, the stochastic process $\{Y(t) \equiv \mathbf{C}^X(t, \{F_t^*\}_{t\in I})\}_{t\in I}$ satisfies the following; for any $n \geq 1$, $t_1, \cdots, t_n \in I (t_i \neq t_j$ if $i \neq j)$, and $y_1, \cdots, y_n \in R$,

$$P(Y(t_1) \leq y_1, \cdots, Y(t_n) \leq y_n) = C^X_{\{t_1,\cdots,t_n\}}(F_{t_1}(y_1), \cdots, F_{t_n}(y_n)). \quad (2.15).$$

In particular, for any $y \in R$,

$$P(Y(t) \leq y) = F_t(y) \quad \text{for all } t \in I. \qquad (2.16).$$

Proof. Since (2.16) is a special case of (2.15)(see Definition 2.1), we only prove (2.16).

Since

$$P(0 < F_{t_1}^X(X(t_1)), \cdots, F_{t_n}^X(X(t_n)) < 1) = 1, \qquad (2.17).$$

we have

$$\begin{aligned}
&P(Y(t_1) \leq y_1, \cdots, Y(t_n) \leq y_n) \qquad (2.18).\\
&= P(F_{t_1}^*(F_{t_1}^X(X(t_1))) \leq y_1, \cdots, F_{t_n}^*(F_{t_n}^X(X(t_n))) \leq y_n)\\
&= P(F_{t_1}^X(X(t_1)) \leq F_{t_1}(y_1), \cdots, F_{t_n}^X(X(t_n)) \leq F_{t_n}(y_n))\\
&\qquad \text{(from Proposition 2.2.)}\\
&= C^X_{\{t_1,\cdots,t_n\}}(F_{t_1}^X(z_1), \cdots, F_{t_n}^X(z_n))\\
&= C^X_{\{t_1,\cdots,t_n\}}(F_{t_1}(y_1), \cdots, F_{t_n}(y_n)).
\end{aligned}$$

Here we put for $1 \leq i \leq n$,

$$z_i = \sup\{x; F_{t_i}^X(x) \leq F_{t_i}(y_i)\}. \qquad (2.19).$$

Q.E.D.

We get the following proposition easily.

Proposition 2.4.

For any $\mathbf{F}^* \in DF(R)_I^*$, and any real valued stochastic process $\{X(t)\}_{t\in I}$ on a probability space $(\Omega, \mathbf{B}, P)$, the following holds.
(1). If $\{X(t)\}_{t\in I}$ is a Markov process, then so is $\{\mathbf{C}^X(t, \mathbf{F}^*)\}_{t\in I}$.
(2). If $\{X(t)\}_{t\in I}$ is a one dimensional local Markov field, then so is $\{\mathbf{C}^X(t, \mathbf{F}^*)\}_{t\in I}$.

In the following, we give the definition of a copula field.

Definition 2.4. A random field $\{\mathbf{C}(t, \mathbf{F}^*)\}_{t\in I, \mathbf{F}^* \in DF(R)_I^*}$ on a probability space $(\Omega, \mathbf{B}, P)$ is called a **copula field** if the following holds; there exists a real valued stochastic process $\{c(t)\}_{t\in I}$ on $(\Omega, \mathbf{B}, P)$ such that for all $t \in I$, all $u \in [0,1]$, and all $\mathbf{F}^* = \{F_t^*\}_{t\in I} \in DF(R)_I^*$,

$$P(c(t) \le u) = u, \qquad (2.20).$$

$$\mathbf{C}(t, \mathbf{F}^*) = F_t^*(c(t)). \qquad (2.21).$$

The following theorem can be obtained in the same way as in Theorem 2.3 and the proof is omitted.

Theorem 2.5.

For any copula field $\{\mathbf{C}(t, \mathbf{F}^*)\}_{t\in I, \mathbf{F}^* \in DF(R)_I^*}$ on a probability space $(\Omega, \mathbf{B}, P)$, the following holds; for any $\mathbf{F}^* = \{F_t^*\}_{t\in I} \in DF(R)_I^*$, $n \ge 1$, $t_1, \cdots, t_n \in I (t_i \ne t_j$ if $i \ne j)$, and $y_1, \cdots, y_n \in R$,

$$P(\mathbf{C}(t_1, \mathbf{F}^*) \le y_1, \cdots, \mathbf{C}(t_n, \mathbf{F}^*) \le y_n) = C^c_{\{t_1,\cdots,t_n\}}(F_{t_1}(y_1), \cdots, F_{t_n}(y_n)). \qquad (2.22).$$

In particular, for any $y \in R$,

$$P(\mathbf{C}(t, \mathbf{F}^*) \le y) = F_t(y) \quad \text{for all } t \in I. \qquad (2.23).$$

The following theorem together with Theorem 2.5 characterizes a copula field.

Theorem 2.6.

For any random field $\{\mathbf{C}(t, \mathbf{F}^*)\}_{t\in I, \mathbf{F}^* \in DF(R)_I^*}$ on a probability space $(\Omega, \mathbf{B}, P)$ which satisfies (2.22) for a real valued stochastic process $\{c(t)\}_{t\in I}$ on $(\Omega, \mathbf{B}, P)$, the probability law of $\{\tilde{c}(t; \mathbf{F}^*) \equiv F_t(\mathbf{C}(t, \mathbf{F}^*))\}_{t\in I}$ is independent of $\mathbf{F}^* \in DF(R)_I^*$, and the marginal distribitions of $\{\tilde{c}(t; \mathbf{F}^*)\}_{t\in I}$ is the same as that of $\{c(t)\}_{t\in I}$, and $\{\tilde{\mathbf{C}}(t, \mathbf{F}^*) \equiv F_t^*(\tilde{c}(t; \mathbf{F}^*))\}_{t\in I}$ satisfies (2.22). In particular, for any $\mathbf{F}^* \in DF(R)_I^*$ for which $F_t(x)$ is strictly increasing in $x \in R$ for all $t \in I$, $\tilde{\mathbf{C}}(t, \mathbf{F}^*) = \mathbf{C}(t, \mathbf{F}^*)$.

Proof. For $\{\mathbf{C}(t, \mathbf{F}^*)\}_{t\in I, \mathbf{F}^* \in DF(R)_I^*}$ which satisfies (2.22) for a real valued stochastic process $\{c(t)\}_{t\in I}$, and any $\mathbf{F}^* \in DF(R)_I^*$, $\{\tilde{c}(t; \mathbf{F}^*)\}_{t\in I}$ satisfies the following; for any $n \ge 1$, $t_1, \cdots, t_n \in I (t_i \ne t_j$ if $i \ne j)$, and $u_1, \cdots, u_n \in [0,1]$,

$$P(\tilde{c}(t_1;\mathbf{F}^*) \le u_1, \cdots, \tilde{c}(t_n;\mathbf{F}^*) \le u_n) = C^c_{\{t_1,\cdots,t_n\}}(u_1,\cdots,u_n), \quad (2.24).$$

since $\mathbf{F} \in DF(R)_I$. In fact for any $y_1, \cdots, y_n \in R$,

$$\begin{aligned} &P(\tilde{c}(t_1;\mathbf{F}^*) \le F_{t_1}(y_1), \cdots, \tilde{c}(t_n;\mathbf{F}^*) \le F_{t_n}(y_n)) \\ &= C^c_{\{t_1,\cdots,t_n\}}(F_{t_1}(y_1), \cdots, F_{t_n}(y_n)), \end{aligned}$$

from (2.22). Continuity of $F_t(x)$ in $x \in R$ implies (2.24).

(2.24) means that the probability law of $\{\tilde{c}(t;\mathbf{F}^*)\}_{t\in I}$ is independent of $\mathbf{F}^* \in DF(R)^*_I$, and that the marginal distribitions of $\{\tilde{c}(t;\mathbf{F}^*)\}_{t\in I}$ is the same as that of $\{c(t)\}_{t\in I}$. It is easy to see that $\{\tilde{\mathrm{C}}(t,\mathbf{F}^*)\}_{t\in I}$ satisfies (2.22), from Proposition 2.2.

The last part of the theorem can be easily proved, since for any $\mathbf{F}^* \in DF(R)^*_I$ for which $F_t(x)$ is strictly increasing in $x \in R$ for all $t \in I$, $F^*_t = F_t^{-1}$ for all $t \in I$.

Q.E.D.

The following proposition shows the relation between the copulas fields for stochastic processes and copula fields. We omit the proof, since it is trivial.

Proposition 2.7.

The set of all copula fields is equal to the set of all copula fields for real valued stochastic processes.

As an application of Theorem 2.3, let us construct stochastic processes with special time dependence.

Theorem 2.8.

For any $T > 0$, any flow of distribution functions $\{F_t\}_{0\le t\le T}$ on R for which $F_t \in DF(R)$ for $0 < t < T$, and any 2-copula $C(u,v)$, there exists a real valued, one dimensional local Markov field $\{Y(t)\}_{0\le t\le T}$ on a probability space (Ω, B, P) such that for all $x, y \in R$

$$P(Y(t) \le x) = F_t(x) \quad \textit{for all}\ 0 \le t \le T, \qquad (2.25).$$

$$P(Y(0) \le x, Y(T) \le y) = C(F_0(x), F_T(y)). \qquad (2.26).$$

Proof. From Theorem 2.1 in [11], and the first part of section 3 in [11], there exists a real valued one dimensional local Markov field $\{X(t)\}_{0\le t\le T}$ on a probability space (Ω, B, P) such that for $x, y \in R$

$$P(X(0) \le x, X(T) \le y) = C(F_0(x), F_T(y)). \qquad (2.27).$$

In fact, put for $x, y \in R$,

$$F(x,y) = C(F_0(x), F_T(y)) \qquad (2.28).$$

which is a distribution function on R^2 from Theorem 2.1, and put for $0 \le s < t < u \le T$, $x, y, z \in R$,

$$q(s,x;t,y) = (2\pi(t-s))^{-1/2} \exp(-|y-x|^2/(2(t-s))) \quad (2.29).$$
$$p(s,x;t,y;u,z) = q(s,x;t,y)q(t,y;u,z)/q(s,x;u,z). \quad (2.30).$$

Then $p(s,x;t,y;u,z)$ is a reciprocal transition probability density function(see [11], section 3). For $F(x,y)$ and $p(s,x;t,y;u,z)$, there exists a real valued one dimensional local Markov field (or reciprocal process) $\{X(t)\}_{0\le t\le T}$ on a probability space (Ω, B, P) such that (2.27) holds(see [11], Theorem 2.1). In particular, from the construction,

$$C(u,v) = C^X_{\{0,T\}}(u,v). \quad (2.31).$$

Putting

$$Y(t) \equiv \begin{cases} \mathrm{C}^X(t, \{F_t^*\}_{0<t<T}) & \text{if } 0 < t < T, \\ X(t) & \text{if } t = 0 \text{ or } T, \end{cases} \quad (2.32).$$

the proof is over from Theorem 2.3.

Q. E. D.

As a corollary to Theorem 2.8, we get the following.

Corollary 2.9.

For any $T > 0$, and any flow of distribution functions $\{F_t\}_{0\le t\le T}$ on R for which $F_t \in DF(R)$ for $0 < t < T$, there exists a real valued, one dimensional local Markov field $\{Y(t)\}_{0\le t\le T}$ on a probability space (Ω, B, P) such that (2.25) holds, and that $Y(0)$ and $Y(T)$ are independent of each other. If $F_0 = F_T$, then there exists a real valued, one dimensional local Markov field $\{Y(t)\}_{0\le t\le T}$ on a probability space (Ω, B, P) such that (2.25) holds, and that $Y(0) = Y(T)$.

Proof. $Y(0)$ and $Y(T)$ are independent of each other if and only if for all x, $y \in R$,

$$P(Y(0) \le x, Y(T) \le y) = P(Y(0) \le x)P(Y(T) \le y), \quad (2.33).$$

in which case the 2-copula $C^Y_{\{0,T\}}(u,v)$ can be taken to be uv(see [18] or [19]).

$Y(0) = Y(T)$, if and only if for all x, $y \in R$,

$$\begin{aligned} F_0^Y(x) &= F_T^Y(x), \\ P(Y(0) \le x, Y(T) \le y) &= \min(P(Y(0) \le x), P(Y(T) \le y)), \end{aligned} \quad (2.34).$$

in which case the 2-copula $C^Y_{\{0,T\}}(u,v)$ can be taken to be $\min(u,v)$. In fact if $Y(0) = Y(T)$, then it is easy to see that (2.34) holds. Suppose that (2.34) holds. Then for any $r \in R$,

$$
\begin{aligned}
P(Y(0) \leq r < Y(T)) &= P(Y(0) \leq r) - P(Y(0), Y(T) \leq r) \quad (2.35). \\
&= F_0^Y(r) - \min(F_0^Y(r), F_T^Y(r)) \\
&= 0.
\end{aligned}
$$

In the same way,

$$P(Y(T) \leq r < Y(0)) = 0. \qquad (2.36).$$

From (2.35)-(2.36), we get $Y(0) = Y(T)$.

Theorem 2.8 together with (2.33)-(2.34) completes the proof.

Q. E. D.

For other typical 2-copulas, see [3], [4], [8], [9], [12], [18], [19].

3. Applications.

In this section, we give the application of the results in section 2 to the stochastic quantization and to the stochastic control.

The next theorem is a generalizations of [5], [7], [13], [14], [20] when $d = 1$, and can be obtained from Theorem 2.3.

Theorem 3.1.

Suppose that (1.1) has a solution $\psi(t, x)$ $(t \geq 0,\ x \in R)$. Then there exists a Markov process such that (1.2) holds.

Proof. Put

$$X(t) = X(0) + W(t), \qquad (3.1).$$

where $W(t)$ is a one dimensional Wiener process(see [10]), and where $X(0)$ is a real valued random variable which is independent of $\{W(t)\}_{0 \leq t}$ and for which $P(X(0) \leq x) = F_0^{\psi}(x)$. Here we put $F_t^{\psi}(x) = \int_{-\infty}^{x} |\phi(t, y)|^2 dy$ for all $t \geq 0$.

Put

$$Y(t) \equiv \begin{cases} \mathrm{C}^X(t, \{(F_t^{\psi})^*\}_{0<t}) & \text{if } 0 < t, \\ X(0) & \text{if } t = 0. \end{cases} \qquad (3.2).$$

Then the proof is over from Theorem 2.3.

Q. E. D.

Let $\{X(t)\}_{t \geq 0}$ be a solution of the following stochastic differential equation;

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dW(t), \qquad (3.3).$$

where $\sigma(t, x)$ and $b(t, x) : [0, \infty) \times R :\mapsto R$, are bounded, and are globally Lipschitz continuous. We assume that $F_t^X(x)$ is continuously differentiable in $t > 0$, and is strictly increasing and twice continuously differentiable in $x \in R$(see [10] for sufficient conditions).

The following proposition can be obtained in the same way as in Theorem 3.1.

Proposition 3.2.

For any flow of distribution functions $\{F_t\}_{t \geq 0}$ for which $F_t(x)$ is continuously differentiable in $t > 0$ and is strictly increasing and twice continuously differentiable in $x \in R$, $\{Y(t)\}_{t \geq 0}$ which is constructed from $\{X(t)\}_{t \geq 0}$ in (3.3) as in (3.2) satisfies the following stochastic differential equation;

$$dY(t) = \tilde{b}(t, Y(t))dt + \tilde{\sigma}(t, Y(t))dW(t), \qquad (3.4).$$

where we put

$$
\begin{aligned}
&\tilde{b}(t,y) && (3.5).\\
&= \{\partial F_t^X((F_t^X)^{-1}(F_t(y)))/\partial t - \partial F_t(y)/\partial t \\
&\quad + b(t,(F_t^X)^{-1}(F_t(y)))\partial F_t^X((F_t^X)^{-1}(F_t(y)))/\partial x \\
&\quad + \sigma(t,(F_t^X)^{-1}(F_t(y)))^2[\partial^2 F_t^X((F_t^X)^{-1}(F_t(y)))/\partial x^2 - \partial^2 F_t(y)/\partial x^2 \\
&\quad \times \{[\partial F_t^X((F_t^X)^{-1}(F_t(y)))/\partial x]/[\partial F_t(y)/\partial x]\}^2]/2\}/[\partial F_t(y)/\partial x],\\
&\tilde{\sigma}(t,y) && (3.6).\\
&= \sigma(t,(F_t^X)^{-1}(F_t(y)))[\partial F_t^X((F_t^X)^{-1}(F_t(y)))/\partial x]/[\partial F_t(y)/\partial x].
\end{aligned}
$$

Proof. Put for $t > 0$,

$$\Phi(t,x) = F_t^{-1}(F_t^X(x)). \tag{3.7}$$

Then $Y(t) = \Phi(t, X(t))$ and by the Ito formula(see [10]),

$$
\begin{aligned}
dY(t) = [&\partial\Phi(t,X(t))/\partial t + b(t,X(t))\partial\Phi(t,X(t))/\partial x \\
&+ [\sigma(t,X(t))^2\partial^2\Phi(t,X(t))/\partial x^2]/2]dt \\
&+ [\partial\Phi(t,X(t))/\partial x]\sigma(t,X(t))dW(t).
\end{aligned} \tag{3.8}
$$

Since

$$\Phi^{-1}(t,x) = (F_t^X)^{-1}(F_t(x)), \tag{3.9}$$

the proof is over from the following;

$$
\begin{aligned}
\partial\Phi(t,x)/\partial t &= [\partial F_t^X(x)/\partial t - \partial F_t(\Phi(t,x))/\partial t]/[\partial F_t(\Phi(t,x))/\partial x], && (3.10).\\
\partial\Phi(t,x)/\partial x &= [\partial F_t^X(x)/\partial x]/[\partial F_t(\Phi(t,x))/\partial x], && (3.11).\\
\partial^2\Phi(t,x)/\partial x^2 &= [\partial^2 F_t^X(x)/\partial x^2 - \partial^2 F_t(\Phi(t,x))/\partial x^2 && (3.12).\\
&\quad \times ([\partial F_t^X(x)/\partial x]/[\partial F_t(\Phi(t,x))/\partial x])^2]/[\partial F_t(\Phi(t,x))/\partial x].
\end{aligned}
$$

Q.E.D.

Remark 3.1. The above approach can be generalized to non Markov semimartingales.

The following theorem shows that Nelson's example in [17], pp. 441-442 does not deny, in the stochastic quantization, the approach to construct stochastic processes from given flow of marginals (probability measures), and can be obtained from Corollary 2.9.

Theorem 3.3.

Suppose that (1.1) has a solution $\psi(t,x)$ ($t \geq 0$, $x \in R$), and that there exists $T > 0$ for which $|\psi(0,x)| = |\psi(T,x)|$ ($x \in R$). Then there exists a real valued, one dimensional local Markov field $\{X(t)\}_{0 \leq t \leq T}$ on a probability space $(\Omega, \mathbf{B}, P)$ such that (1.2) holds, and that $X(0) = X(T)$.

Next we give the application to the stochastic control.
Fix $T > 0$ and $t_0 \in [0, T]$. Let

$$k(t, x) : [0, T] \times R \mapsto R, \qquad (3.13).$$

$$G(x) : R \mapsto R \qquad (3.14).$$

be bounded measurable, and put for a real valued stochastic process $\{X(t)\}_{0 \leq t \leq T}$,

$$J(X) \equiv E[\int_0^T k(t, X(t))dt + G(X(t_0))]. \qquad (3.15).$$

The following theorem can be obtained in the same way as in Theorem 3.1, from Proposition 2.4.

Theorem 3.4.

For any real valued stochastic process $\{X(t)\}_{0 \leq t \leq T}$ on a probability space $(\Omega, \mathbf{B}, P)$ for which $F_t^X \in DF(R)$ for all $t \in (0, T]$, there exists a Markov proces $\{X^M(t)\}_{0 \leq t \leq T}$ such that for all $x \in R$, and all $t (0 \leq t \leq T)$,

$$P(X(t) \leq x) = P(X^M(t) \leq x). \qquad (3.16).$$

In particular, for any $A \subset DF(R)_{(0,T]}$ and any subset B of the set of all distribution functions on R, the infimum of $J(\cdot)$ over all real valued stochastic processes $\{X(t)\}_{0 \leq t \leq T}$ for which $\{F_t^X\}_{t \in (0,T]} \in A$ and for which $F_0^X \in B$ is equal to that over all real valued Markov processes $\{X(t)\}_{0 \leq t \leq T}$ for which $\{F_t^X\}_{t \in (0,T]} \in A$ and for which $F_0^X \in B$.

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