HOMOLOGY OF HYPERELLIPTIC MAPPING CLASS GROUPS FOR SURFACES

Nariya Kawazumi

Series ♯262. August 1994

HOKKAIDO UNIVERSITY

PREPRINT SERIES IN MATHEMATICS

♯ 237 T. Ozawa, Characterization of Trudinger's inequality, 7 pages. 1994.

♯ 238 T. Hibi, Buchsbaum complexes with linear resolutions, 12 pages. 1994.

♯ 239 Y. Giga, N. Mizoguchi, On time periodic solutions of the Dirichlet problem for degenerate parabolic equations of nondivergence type, 23 pages. 1994.

♯ 240 C. Dohmen, Existence of Fast Decaying Solutions to a Haraux-Weissler Equation With a Prescribed Number of Zeroes, 12 pages. 1994.

♯ 241 K. Sugano, Note on H-separable Frobenius extensions, 8 pages. 1994.

♯ 242 J. Zhai, Some Estimates For The Blowing up Solutions of Semilinear Heat Equations, 11 pages. 1994.

♯ 243 N. Hayashi, K. Kato and T. Ozawa, Dilation Method and Smoothing Effect of the Schrödinger Evolution Group, 10 pages. 1994.

♯ 244 D. Lehmann, T. Suwa, Residues of holomorphic vector fields relative to singular invariant subvarieties, 26 pages. 1994.

♯ 245 H. Kubo, Slowly decaying solutions for semilinear wave equations in odd space dimensions, 30 pages. 1994.

♯ 246 T. Nakazi, M. Yamada, (A_2)-Conditions and Carleson Inequalities, 27 pages. 1994.

♯ 247 N. Hayashi, K. Kato and T. Ozawa, Dilation Method and smoothing Effect of Solutions to the Benjamin-ono Equation, 17 pages. 1994.

♯ 248 H. Kikuchi, Sheaf cohomology theory for measurable spaces, 12 pages. 1994.

♯ 249 A. Inoue, Tauberian theorems for Fourier cosine transforms, 9 pages. 1994.

♯ 250 S. Izumiya, G. T. Kossioris, Singularities for viscosity solutions of Hamilton-Jacobi equations, 23 pages. 1994.

♯ 251 H. Kubo, K. Kubota, Asymptotic behaviors of radially symmetric solutions of $\Box u = |u|^p$ for super critical values p in odd space dimensions, 51 pages. 1994.

♯ 252 T. Mikami, Large Deviations and Central Limit Theorems for Eyraud-Farlie-Gumbel-Morgenstern Processes, 9 pages. 1994.

♯ 253 T. Nishimori, Some remarks in a qualitative theory of similarity pseudogroups, 19 pages. 1994.

♯ 254 T. Suwa, Residues of complex analytic foliations relative to singular invariant subvarieties, 15 pages. 1994.

♯ 255 T. Tsujishita, On Triple Mutual Information, 7 pages. 1994.

♯ 256 T. Tsujishita, Construction of Universal Modal World based on Hyperset Theory, 15 pages. 1994.

♯ 257 A. Arai, Trace Formulas, a Golden-Thompson Inequality and Classical Limit in Boson Fock Space, 35 pages. 1994.

♯ 258 Y-G. Chen, Y. Giga, T. Hitaka and M. Honma, A Stable Difference Scheme for Computing Motion of Level Surfaces by the Mean Curvature, 18 pages. 1994.

♯ 259 K. Iwata, J. Schäfer, Markov property and cokernels of local operators, 7 pages. 1994.

♯ 260 T. Mikami, Copula fields and its applications, 14 pages. 1994.

♯ 261 A. Inoue, An Abel-Tauber theorem for Fourier sine transforms, 6 pages. 1994.

HOMOLOGY OF HYPERELLIPTIC MAPPING CLASS GROUPS FOR SURFACES.

NARIYA KAWAZUMI

Department of Mathematics, Faculty of Sciences,
Hokkaido University, Sapporo, 060 Japan

INTRODUCTION.

The simplest of compact Riemann surfaces are hyperelliptic curves. By definition these curves are the compactifications of the plane curves in the (z, w) plane

$$w^2 = \prod_{i=1}^{2g+2} (z - a_i), \quad (a_i \neq a_j, \quad (i \neq j)),$$

and admit the hyperelliptic involutions given by

$$(z, w) \mapsto (z, -w).$$

The hyperelliptic mapping class group Δ_g is the subgroup of the mapping class group Γ_g corresponding to the hyperelliptic curves. As is known, $\Delta_2 = \Gamma_2$, and $\Delta_g \neq \Gamma_g$ for $g \geq 3$. The cohomology of the group Δ_g has been studied by Benson-Cohen [2], Boedigheimer-Cohen-Peim [4], Cohen [5] [6] and so on. They utilize sophisticated techniques coming from algebraic topology. The purpose of the present paper is to show an alternative, more geometric and elementary way to analyze the cohomology of the group Δ_g.

We fix our notations. Throughout this paper let $g \geq 2$ and Σ_g an oriented 2-dimensional C^∞ manifold of genus g. Usually the group of path-components $\pi_0\mathrm{Diff}^+(\Sigma_g)$ is denoted by Γ_g (or $\mathcal{M}_g$) and called the mapping class group of genus g, where $\mathrm{Diff}^+(\Sigma_g)$ denotes the topological groups consisting of all orientation preserving diffeomorphisms of the C^∞ manifold Σ_g endowed with the C^∞ topology. Let a point $p \in \Sigma_g$ be fixed. Similarly we denote

$$\Gamma_g^1 := \pi_0\mathrm{Diff}^+(\Sigma_g, p) \quad \text{and} \quad \Gamma_{g,1} := \pi_0\mathrm{Diff}^+(\Sigma_g, T_p\Sigma_g),$$

where

$$\mathrm{Diff}^+(\Sigma_g, p) := \{f \in \mathrm{Diff}^+(\Sigma_g); f(p) = p\}$$
$$\mathrm{Diff}^+(\Sigma_g, T_p\Sigma_g) := \{f \in \mathrm{Diff}^+(\Sigma_g, p); (df)_p = 1_{T_p\Sigma_g}\}.$$

1991 Mathematical Subject Classification. Primary 57M20. Secondary 14H15, 14H45, 20F36, 20F38, 20J05, 57N05.

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

Then we have exact sequences

$$0 \to \pi_1(\Sigma_g) \to \Gamma_g^1 \to \Gamma_g \to 1,$$
$$0 \to \mathbb{Z} \to \Gamma_{g,1} \to \Gamma_g^1 \to 1 \quad \text{and}$$
$$0 \to \pi_1(T^\times \Sigma_g) \to \Gamma_{g,1} \to \Gamma_g \to 1,$$

where we denote by $T^\times \Sigma_g$ the fiber bundle over Σ_g obtained by deleting the zero section from the tangent bundle $T\Sigma_g$.

Let $\iota \in \Gamma_g$ be the mapping class of a hyperelliptic involution. Our main object to study in this paper is the centralizer Δ_g of ι in the group Γ_g:

$$\Delta_g := \{\phi \in \Gamma_g; \phi\iota\phi^{-1} = \iota\},$$

which we call *the hyperelliptic mapping class group of genus g*.

Let k be a field. We denote its characteristic by $\mathrm{ch}\, k$. Easily one can deduce

$$H^*(\Delta_g; H^1(\Sigma_g; k)) = 0, \quad \text{if } \mathrm{ch}\, k \neq 2. \tag{0.1}$$

Introduce the fiber product $\Delta_{g,1} := \Delta_g \times_{\Gamma_g} \Gamma_{g,1}$. By (0.1), if $\mathrm{ch}\, k \not| (2-2g)$, the extension

$$0 \to \pi_1(T^\times \Sigma_g) \to \Delta_{g,1} \xrightarrow{\pi} \Delta_g \to 1$$

induces a Gysin exact sequence

$$\cdots \to H^{q+3}(\Delta_{g,1}) \xrightarrow{\pi_!} H^q(\Delta_g) \xrightarrow{\cup\epsilon} H^{q+4}(\Delta_g) \xrightarrow{\pi^*} H^{q+4}(\Delta_{g,1}) \to \cdots \tag{0.2}$$

with coefficients in the field k, where ϵ is given by

$$\epsilon := e_2 - (2-2g)^{-1}{e_1}^2 \in H^4(\Delta_g; k)$$

(Proposition 1.8). Here $e_n \in H^{2n}(\Gamma_g; \mathbb{Z})$ is the n-th Morita-Mumford class [10] [11].

Let $H_{g,1}$ denote the space of holomorphic isomorphism classes (i.e., the moduli) of triples (C, p, v), where C is a hyperelliptic curve of genus g, $p \in C$, and $v \in T^\times C := T_pC - \{0\}$. In §2 we prove that $H_{g,1}$ is an Eilenberg-MacLane space of type $(\Delta_{g,1}, 1)$, and give its description with the (braid) configuration spaces of the complex line $\mathbb{C}$. The description induces a cohomology exact sequence

$$\cdots \to H^{q-2}(B_{2g+1}) \to H^q(\Delta_{g,1}) \to H^q(B_{2g+2}) \to H^{q-1}(B_{2g+1}) \to \cdots, \tag{2.9}$$

with arbitrary (trivial) coefficients. Here B_n denotes the Artin braid group of n-strands.

The group structure of the integral cohomology of the Artin braid group B_n has been already determined completely by Vainshtein [12] based on works of Arnol'd [1] and Fuks [7]. Especially $H^*(B_n; \mathbb{Z})$ has no p-torsion provided that $p > [n/2]$. In §3, using these results and the two exact sequences stated above, we compute the cohomology $H^*(\Delta_g; k)$ in the case when $\mathrm{ch}\, k \geq g+1$. The groups $H^*(\Delta_g : k)$ with *any*trivial field coefficients have already been determined by Boedigheimer-Cohen-Peim [4]. They utilize sophisticated techniques coming from algebraic topology, while our computation is more geometric and elementary.

The author would like to express his gratitude to Prof. Y.Matsumoto, Prof. S.Morita and Dr. Y.Kasahara for helpful discussions.

1. Symmetric Mapping Class Groups.

The purpose of this section is to prove the Gysin sequence (0.2) under some slightly general situations.

Let G be a finite subgroup of the mapping class group Γ_g. The group G acts on the surface Σ_g because of the Nielsen realization [9]. Let $\mathcal{S}_g$ be a subgroup of Γ_g including G as a normal subgroup. We impose the following condition on the finite group G;

(1.1) The quotient orbifold Σ_g/G is the sphere S^2 with some elliptic points.

The group $\langle\iota\rangle$ of order 2 generated by the hyperelliptic involution ι satisfies the condition, and we may take the hyperelliptic mapping class group Δ_g as $\mathcal{S}_g$.

Let k be a field with $\mathrm{ch}\, k \not| \sharp G$. From the condition (1.1) follows

$$H^*(G; H^1(\Sigma_g; k)) = 0.$$

Hence all the $E_2^{p,q}$ terms of the Lyndon-Hochschild-Serre (LHS) spectral sequence of the extension

$$0 \to G \to \mathcal{S}_g \to \mathcal{S}_g/G \to 1$$

with coefficients in $H^1(\Sigma_g; k)$ vanish, and so we obtain

$$H^*(\mathcal{S}_g; H^1(\Sigma_g; k)) = 0, \quad \text{if } \mathrm{ch}\, k \not| \sharp G. \tag{1.2}$$

Especially we have

$$H^*(\Delta_g; H^1(\Sigma_g; k)) = 0, \quad \text{if } \mathrm{ch}\, k \neq 2. \tag{1.3}$$

Let $\mathcal{S}_g^1$ and $\mathcal{S}_{g,1}$ denote the fiber products $\mathcal{S}_g^1 := \mathcal{S}_g \times_{\Gamma_g} \Gamma_g^1$ and $\mathcal{S}_{g,1} := \mathcal{S}_g \times_{\Gamma_g} \Gamma_{g,1}$ respectively. The group extension

$$0 \to \mathbb{Z} \to \mathcal{S}_{g,1} \to \mathcal{S}_g^1 \to 1 \tag{1.4}$$

defines the Euler class $e \in H^2(\mathcal{S}_g^1; \mathbb{Z})$. We consider the extension

$$0 \to \pi_1(\Sigma_g) \to \mathcal{S}_g^1 \to \mathcal{S}_g \to 1. \tag{1.5}$$

Let $\int_{\text{fiber}} : H^q(\mathcal{S}_g^1) \to H^{q-2}(\mathcal{S}_g)$ denote the Gysin map induced by the extension (1.5). Substituting (1.2) into a lemma of Morita ([10] Proposition 3.1), we have

Proposition 1.6. *Let k be a field with $\mathrm{ch}\, k \not| (2-2g)\sharp G$. Then the LHS spectral sequence of the extension (1.5) degenerates itself into a decomposition*

$$H^q(\mathcal{S}_g^1; k) = H^q(\mathcal{S}_g; k) \oplus e \cup H^{q-2}(\mathcal{S}_g; k).$$

Here it should be remarked that the bundle $T^\times \Sigma_g$ is aspherical (i.e., an Eilenberg-MacLane space of type $(\pi, 1)$). We have an extension

$$0 \to \pi_1(T^\times \Sigma_g) \to \mathcal{S}_{g,1} \xrightarrow{\pi} \mathcal{S}_g \to 1. \tag{1.7}$$

Proposition 1.8. *Let k be a field with $\operatorname{ch} k \not| (2-2g)\sharp G$, and introduce a 4-dimensional class $\epsilon \in H^4(\mathcal{S}_g; k)$ defined by*

$$\epsilon := e_2 - (2-2g)^{-1}{e_1}^2.$$

Here $e_n := \int_{\text{fiber}} e^{n+1} \in H^{2n}(\Gamma_g; \mathbb{Z})$ is the n-th Morita-Mumford class[10][11]. Then we have a Gysin exact sequence

$$\cdots \to H^{q+3}(\mathcal{S}_{g,1}) \xrightarrow{\pi_!} H^q(\mathcal{S}_g) \xrightarrow{\cup\epsilon} H^{q+4}(\mathcal{S}_g) \xrightarrow{\pi^*} H^{q+4}(\mathcal{S}_{g,1}) \to \cdots$$

with coefficients in k.

When $G = \langle \iota \rangle$ and $\mathcal{S}_g = \Delta_g$, the Gysin sequence (1.8) is nothing but the sequence (0.2) stated in Introduction.

Proof. Since $\operatorname{ch} k \not| (2-2g)$, we have $k[\mathcal{S}_g]$-isomorphisms

$$H^1(T^\times \Sigma_g) = H^2(T^\times \Sigma_g) = H^1(\Sigma_g),$$

which implies the LHS spectral sequence of the extension (1.7) is given by

$$E_2^{p,q} = \begin{cases} H^q(\mathcal{S}_g), & \text{if } q = 0, 3, \\ 0, & \text{otherwise.} \end{cases}$$

Hence it suffices to show that we may exchange the image $d_3(1) \in H^4(\mathcal{S}_g)$ of $1 \in E_2^{0,3} = H^0(\mathcal{S}_g)$ under the transgression d_3 for our class ϵ. From the Gysin sequence induced by the extension (1.4)

$$H^2(\mathcal{S}_g^1) \xrightarrow{\cup e} H^4(\mathcal{S}_g^1) \to H^4(\mathcal{S}_{g,1}) \quad \text{(exact)}$$

and the decomposition (1.6), the image of $d_3(1)$ in $H^4(\mathcal{S}_g^1)$ is given by

$$d_3(1) = (ae + w)e \in H^4(\mathcal{S}_g^1)$$

for some $w \in H^2(\mathcal{S}_g)$ and $a \in k$. Applying the Gysin map $\int_{\text{fiber}}$ to the above, we have

$$0 = \int_{\text{fiber}} d_3(1) = \int_{\text{fiber}} (ae^2 + we) = ae_1 + (2-2g)w,$$

which means $w = (a/(2g-2))e_1$, i.e., $d_3(1) = a(e^2 - (2-2g)^{-1}e_1 e)$. Hence we obtain

$$\begin{aligned} d_3(1) &= (1/(2-2g)) \int_{\text{fiber}} d_3(1)e = (a/(2-2g)) \int_{\text{fiber}} (e^3 - (2-2g)^{-1}e_1 e^2) \\ &= (a/(2-2g))\epsilon. \end{aligned}$$

Now, if $a \neq 0$, we may exchange $d_3(1)$ for ϵ to obtain the Gysin sequence (1.8). Hence it suffices to show that $\epsilon = 0$ under the assumption $a = 0$. Suppose $a = 0$, which implies $d_3(1) = 0$ and

$$0 \to H^4(\mathcal{S}_g) \to H^4(\mathcal{S}_{g,1}) \quad \text{(exact)}. \tag{1.9}$$

If e^2 is given by $e^2 = u + ve$, $u \in H^4(\mathcal{S}_g)$, $v \in H^2(\mathcal{S}_g)$ from the decomposition (1.6), then $u = e^2 - ve$ vanishes in $H^4(\mathcal{S}_{g,1})$ and so in $H^4(\mathcal{S}_g)$ from (1.9). Hence

$$v = (2-2g)^{-1} \int_{\text{fiber}} ve = (2-2g)^{-1} \int_{\text{fiber}} e^2 = (2-2g)^{-1} e_1, \quad \text{and}$$

$$\epsilon = e_2 - ve_1 = \int_{\text{fiber}} e^3 - ve^2 = 0.$$

This completes the proof of Proposition (1.8) □

2. Description of Moduli.

For $g \geq 2$ let $H_{g,1}$ denote the space of holomorphic isomorphism classes (i.e., the moduli) of triples (C,p,v), where C is a hyperelliptic curve of genus g, $p \in C$ and $v \in T_pC - \{0\}$. In this section we prove that the space $H_{g,1}$ is an Eilenberg-MacLane space of type $(\Delta_{g,1}, 1)$ and give an description of $H_{g,1}$ with the (braid) configuration spaces of the complex line $\mathbb{C}$ to obtain a cohomology exact seqence (2.9).

It should be remarked that the holomorphic automorphism group of each triple $(C,p,v) \in H_{g,1}$ is trivial. In fact, since $g \geq 2$, any holomorphic automorphism of C is an isometry under the hyperbolic metric, and any isometry fixing the tangent space of one point coincides with the identity on the path-component containing the point. This fact implies the existence of the universal family over the moduli $H_{g,1}$. The universal Riemann surface is given as the moduli of quadruples (C,p,v,p_1), where $(C,p,v) \in H_{g,1}$ and $p_1 \in C$, and its relative tangent bundle over $H_{g,1}$ is given as the moduli $E_{g,1}$ of quintuples (C,p,v,p_1,v_1), where $(C,p,v) \in H_{g,1}$, $p_1 \in C$ and $v_1 \in T_{p_1}C$. The diagonal map $(C,p,v) \in H_{g,1} \mapsto (C,p,v,p,v) \in E_{g,1}$ is well-defined and gives the tautological section of $E_{g,1} \to H_{g,1}$. Thus the holonomy homomorphism h associated to the universal family $E_{g,1}$ has its own values in the fiber product $\Delta_{g,1}$:

$$h : \pi_1(H_{g,1}) \to \Delta_{g,1} = \Delta_g \times_{\Gamma_g} \Gamma_{g,1}.$$

Proposition 2.1. *The moduli $H_{g,1}$ is an aspherical space (i.e., an Eilenberg-MacLane space of type $(\pi,1)$), and the holonomy homomorphism $h : \pi_1(H_{g,1}) \to \Delta_{g,1}$ is an isomorphism.*

Proof. For a hyperelliptic curve C of genus g we consider a bijection θ of the set $\{1,2,\dots,2g+2\}$ onto the set of all Weierstrass points of C. In this paper we call it a level (structure) over the hyperelliptic curve C. We denote by $\tilde{H}_{g,1}$ the space of holomorphic isomorphism classes (i.e., the moduli) of quadruples (C,p,v,θ), where $(C,p,v) \in H_{g,1}$ and θ is a level over C. Since the automorphism group of each triple $(C,p,v) \in H_{g,1}$ is trivial, the map $\varpi_1 : \tilde{H}_{g,1} \to H_{g,1}$ given by forgetting the level structure forms a principal $\mathfrak{S}_{2g+2}$ bundle. Here $\mathfrak{S}_{2g+2}$ is the $2g+2$-th symmetric group. When we denote by $\tilde{H}_g$ the moduli of levelled hyperelliptic curves (C,θ), the natural map $\varpi_2 : \tilde{H}_{g,1} \to \tilde{H}_g$, $(C,p,v,\theta) \mapsto (C,\theta)$ is a C^∞ fiber bundle whose fiber is diffeomorphic to $T^\times\Sigma_g/\langle\iota\rangle$, because the automorphism group of each levelled hyperelliptic curve (C,θ) is just $\langle\iota\rangle$.

The well-known isomorphism

$$\tilde{H}_g = \{(z_4,\dots,z_{2g+2}) \in (\mathbb{P}^1 - \{0,1,\infty\})^{2g-1}; z_i \neq z_j \quad (i \neq j)\}$$

implies that $\tilde{H}_g$ is aspherical. The space $T^\times\Sigma_g/\langle\iota\rangle$ is also aspherical. Therefore the space $\tilde{H}_{g,1}$ and hence the space $H_{g,1} = \tilde{H}_{g,1}/\mathfrak{S}_{2g+2}$ are aspherical.

In view of a theorem of Birman-Hilden [3] we have an extension

$$(2.2) \qquad 0 \to \langle\iota\rangle \to \Delta_g \to \Gamma_0^{2g+2} \to 1,$$

where $\Gamma_0^n = \pi_0\mathrm{Diff}^+(S^2, \{n\text{-point set}\})$. It should be noted that Γ_0^n admits mapping classes permuting the n reference points. Let $\tilde{\Delta}_g$ and $\tilde{\Gamma}_0^{2g+2}$ denote the kernels of the natural surjections $\Delta_g \to \mathfrak{S}_{2g+2}$ and $\Gamma_0^{2g+2} \to \mathfrak{S}_{2g+2}$, respectively. As is well-known, $\pi_1(\tilde{H}_g) = \tilde{\Gamma}_0^{2g+2}$. From the comparison of the restriction of (2.2) to the subgroup $\tilde{\Gamma}_0^{2g+2}$ with the homotopy exact sequence associated to the fiber bundle $\varpi_2 : \tilde{H}_{g,1} \to \tilde{H}_g$ follows a natural extension

$$0 \to \pi_1(T^\times \Sigma_g) \to \pi_1(\tilde{H}_{g,1}) \to \tilde{\Delta}_g \to 1,$$

which implies a natural extension

$$0 \to \pi_1(T^\times \Sigma_g) \to \pi_1(H_{g,1}) \to \Delta_g \to 1.$$

Thus the holonomy homomorphism h is an isomorphism. □

The configuration space $F_n\mathbb{C}$ of ordered n points on the complex line $\mathbb{C}$ is defined by

$$F_n\mathbb{C} := \{(a_1, \ldots, a_n) \in \mathbb{C}^n; \quad a_i \neq a_j \quad \text{if } i \neq j\},$$

on which the n-th symmetric group $\mathfrak{S}_n$ acts freely by permuting of the components z_i's. We denote by $B_n\mathbb{C}$ the quotient space $F_n\mathbb{C}/\mathfrak{S}_n$. By definition the Artin braid group B_n is equal to the fundamental group $\pi_1(B_n\mathbb{C})$. We introduce a complex manifold X_n, $n \geq 2$, defined by

$$X_n := \left\{ f \in \mathbb{C}[z]; \begin{array}{l} f = z^n + s_2 z^{n-2} + \cdots + s_{n-1}z + s_n \\ f(z) \text{ has no multiple roots.} \end{array} \right\},$$

which is an open set of $\mathbb{C}^{n-1}$. Clearly we have a natural isomorphism

$$\begin{aligned} &B_n\mathbb{C} \cong X_n \times \mathbb{C} \\ &(a_1, \ldots, a_n) \mapsto (\prod\nolimits_{i=1}^n (z - a_i + \frac{1}{n}\sum\nolimits_{i=1}^n a_i), \quad \frac{1}{n}\sum\nolimits_{i=1}^n a_i), \end{aligned}$$

so that we have

$$X_n \simeq K(B_n, 1). \tag{2.3}$$

Now we introduce a subset of the moduli $H_{g,1}$ by

$$W_g := \{(C, p, v); p \text{ is a Weierstrass point.}\} \subset H_{g,1}.$$

We shall give natural holomorphic isomorphisms

$$X_{2g+1} \cong W_g \quad \text{and} \quad X_{2g+2} \cong H_{g,1} - W_g.$$

Let $f = \prod_{i=1}^{2g+1}(z - a_i)$ be a polynomial belonging to X_{2g+1}. We define the hyperelliptic curve $H(f)$ by glueing two affine plane curves

$$\begin{aligned} &{w_1}^2 = f(z_1) = \prod_{i=1}^{2g+1} (z_1 - a_i) \quad \text{and} \\ &w^2 = z^{2g+2} f(z^{-1}) = z \prod_{i=1}^{2g+1} (1 - a_i z) \end{aligned}$$

along $\{z_1 \neq 0\}$ and $\{z \neq 0\}$ under the relations

$$z_1 z = 1 \quad \text{and} \quad w = {z_1}^{-(g+1)} w_1.$$

We denote by ∞ the point $(z, w) = (0, 0)$ and call it *the point at infinity*. Since w is a coordinate centered at ∞, we may define

$$v(f) := \left(\frac{d}{dw}\right)_\infty \in T_\infty H(f).$$

Thus we obtain a natural holomorphic map

$$H_{2g+1} : X_{2g+1} \to M_{g,1}, \quad f \mapsto (H(f), \infty, v(f)),$$

whose image is included in W_g. When we set

$$f_{(\lambda)}(z) := \prod_{i=1}^{2g+1} (z - \lambda a_i) \in X_{2g+1} \quad (\lambda \in \mathbb{C} - \{0\}),$$

we deduce

(2.4) $$(H(f), \infty, \lambda v(f)) = (H(f_{(\lambda^2)}), \infty, v(f_{(\lambda^2)})) \in M_{g,1}$$

for any $\lambda \in \mathbb{C} - \{0\}$. A straightforward argument involved with (2.4) shows

Lemma 2.5. *The holomorphic map $H_{2g+1} : X_{2g+1} \to W_g$ is an isomorphism:*

$$H_{2g+1} : X_{2g+1} \cong W_g.$$

Next let $f = \prod_{i=1}^{2g+2}(z - a_i)$ be a polynomial belonging to X_{2g+2} in turn. We define the elliptic curve $H(f)$ by glueing two affine plane curves

$$w_1{}^2 = f(z_1) = \prod_{i=1}^{2g+2} (z_1 - a_i) \quad \text{and}$$

$$w^2 = z^{2g+2} f(z^{-1}) = \prod_{i=1}^{2g+2} (1 - a_i z)$$

along $\{z_1 \neq 0\}$ and $\{z \neq 0\}$ under the relations

$$z_1 z = 1 \quad \text{and} \quad w = {z_1}^{-(g+1)} w_1.$$

We denote by ∞_f the point $(z, w) = (0, 1)$ and call it *the point at infinity associated to the polynomial* f. Since z is a coordinate centered at ∞_f, we define

$$u(f) := \left(\frac{d}{dz}\right)_{\infty_f} \in T_{\infty_f} H(f).$$

Here it should be remarked that $(dw/dz)_{\infty_f} = 0$. Thus we obtain a natural holomorphic map

$$H_{2g+2} : X_{2g+2} \to M_{g,1}, \quad f \mapsto (H(f), \infty_f, u(f)),$$

whose image is included in $H_{g,1} - W_g$. When we set

$$f_{(\lambda)}(z) := \prod_{i=1}^{2g+2} (z - \lambda a_i) \in X_{2g+2} \quad (\lambda \in \mathbb{C} - \{0\}),$$

we deduce

(2.6) $$(H(f), \infty, \lambda u(f)) = (H(f_{(\lambda)}), \infty_{f_{(\lambda)}}, u(f_{(\lambda)})) \in M_{g,1}$$

for any $\lambda \in \mathbb{C} - \{0\}$. A straightforward argument involved with (2.6) shows

Lemma 2.7. *The holomorphic map $H_{2g+2} : X_{2g+2} \to H_{g,1} - W_g$ is an isomorphism:*

$$H_{2g+2} : X_{2g+2} \cong H_{g,1} - W_g.$$

It is necessary to describe the tubular neighbourhood of X_{2g+1} embedded in $H_{g,1}$ through the map H_{2g+1}. We remark

$$\frac{d}{dz} z^{2g+2} f(z^{-1}) = 1 \quad \text{at } z = 0 \text{ for any } f \in X_{2g+1}.$$

Hence there exist two open neighbourhoods $\mathcal{U}_1$ and $\mathcal{U}_2$ of $X_{2g+1} \times \{0\}$ in $X_{2g+1} \times \mathbb{C}$ such that the map

$$\begin{aligned} X_{2g+1} \times \mathbb{C} \supset \mathcal{U}_1 &\to X_{2g+1} \times \mathbb{C} \\ (f, z) &\mapsto (f, z^{2g+2} f(z^{-1})) \end{aligned}$$

is a holomorphic isomorphism of $\mathcal{U}_1$ onto $\mathcal{U}_2$. We denote the inverse of this isomorphism by

$$\mathcal{U}_2 \to \mathcal{U}_1, \quad (f, \zeta) \mapsto (f, z(f, \zeta)).$$

When the point $(f, w) \in X_{2g+1} \times \mathbb{C}$ satisfies $(f, w^2) \in \mathcal{U}_2$, the point

$$p(f, w) := (z(f, w^2), w) \in \mathbb{C}^2.$$

belongs to the curve $H(f)$ and w is a coordinate of $H(f)$ near the point $p(f, w)$. We endow each $H(f)$ with the hyperbolic metric. Consider the open neighbourhood $\mathcal{U}$ of $X_{2g+1} \times \{0\}$ in $X_{2g+1} \times \mathbb{C}$ defined by

$$\mathcal{U} := \left\{ (f, w) \in X_{2g+1} \times \mathbb{C}; \begin{array}{l} (f, w^2) \in \mathcal{U}_2 \\ \infty, \text{ i.e., } (z, w) = (0, 0) \text{ is the unique} \\ \text{closest Weierstrass point of } p(z, w) \\ \text{with respect to the hyperbolic metric.} \end{array} \right\}.$$

The map

$$H_{\mathcal{U}} : \mathcal{U} \to H_{g,1}, \quad (f, w) \mapsto (H(f), p(f, w), \left(\frac{d}{dw}\right)_{p(f,w)})$$

is an open embedding. In fact, it suffices to show the injectivity of the map $H_{\mathcal{U}}$. Suppose

$$(H(f), p(f, w), \left(\frac{d}{dw}\right)_{p(f,w)}) \overset{\varphi}{\cong} (H(h), p(h, x), \left(\frac{d}{dw}\right)_{p(h,x)})$$

for $(f, w), (h, x) \in \mathcal{U}$. The holomorphic isomorphism φ preserves the hyperbolic metric and the Weierstrass points, and so maps the infinity ∞ to the infinity ∞. This implies that there exists an affine transformation $z \mapsto \lambda z + \nu$, $\mu, \nu \in \mathbb{C}$, $\mu \neq 0$, mapping the roots of f onto those of h. From the definition of X_{2g+1} follows $\nu = 0$, namely, $h = f_{(\mu)}$. If $(z', w') = \varphi(z, w)$ $((z, w) \in H(f))$, we have

$$z' = \mu^{-1}z \quad \text{and} \quad (w')^2 = (z')^{2g+2} f_{(\mu)}((z')^{-1}) = \mu^{-1}w^2.$$

Hence we obtain $w' = \mu_1{}^{-1}w$ for some $\mu_1 = \pm\sqrt{\mu}$, and $x = \mu_1{}^{-1}w$ from $p(h, x) = \varphi(p(f, w))$. Furthermore

$$\left(\frac{d}{dw}\right)_{p(h,x)} = \varphi(\left(\frac{d}{dw}\right)_{p(f,w)}) = \mu_1{}^{-1}\left(\frac{d}{dw}\right)_{p(h,x)},$$

which shows $\mu_1 = 1$. Consequently we obtain $x = w$ and $h = f_{(\mu_1{}^2)} = f$, as was to be shown.

Thus the normal bundle of the embedding $H_{2g+1} : X_{2g+1} \to H_{g,1}$ is complex analytically trivial. From the Thom isomorphisms

$$\begin{aligned} H^{q+2}(H_{g,1}, H_{g,1} - W_g) &\cong H^q(X_{2g+1}) \\ H_{q+2}(H_{g,1}, H_{g,1} - W_g) &\cong H_q(X_{2g+1}), \end{aligned} \tag{2.8}$$

follows a cohomology exact sequence

$$\cdots \to H^{q-2}(X_{2g+1}) \to H^q(H_{g,1}) \to H^q(X_{2g+2}) \to H^{q-1}(X_{2g+1}) \to \cdots$$

with arbitrary (trivial) coefficients. Since $H_{g,1} \simeq K(\Delta_{g,1}, 1)$ (2.1) and $X_n \simeq K(B_n, 1)$ (2.3), we obtain a cohomology exact sequence

$$\cdots \to H^{q-2}(B_{2g+1}) \to H^q(\Delta_{g,1}) \to H^q(B_{2g+2}) \to H^{q-1}(B_{2g+1}) \to \cdots \tag{2.9}$$

with arbitrary (trivial) coefficients.

We conclude this section with two remarks.

First, using this description of the moduli $H_{g,1}$, one obtains the following presentation of the group $\Delta_{g,1}$:

$$\begin{aligned} \text{generators: } & \sigma_i, \quad 1 \le i \le 2g+1, \\ \text{relations: } & \sigma_i\sigma_j = \sigma_j\sigma_i, \quad \text{if } |i-j| \ge 2, \\ & \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad \text{for } 1 \le i \le 2g, \\ & (\sigma_2\sigma_3\cdots\sigma_{2g+1})^{2g+1} = \sigma_1\sigma_2\cdots\sigma_{2g+1}\sigma_{2g+1}\cdots\sigma_2\sigma_1. \end{aligned}$$

A presentation of the group Δ_g^1 is obtained by adding a single relation:

$$(\sigma_1\sigma_2\cdots\sigma_{2g+1})^{2g+2} = 1$$

to the above.

Second the sequence (2.9) gives us some information on the cohomology $H^*(\Delta_g; \mathbb{F}_2)$. As has been proved by Fuks [7], the natural surjection $\phi_n : B_n \to \mathfrak{S}_n$ of the Artin braid group B_n to the n-th symmetric group $\mathfrak{S}_n$ induces a surjection

$$\phi_n{}^* : H^*(\mathfrak{S}_n; \mathbb{F}_2) \to H^*(B_n; \mathbb{F}_2).$$

Clearly the surjection $\phi_{2g+2} : B_{2g+2} \to \mathfrak{S}_{2g+2}$ passes through $\Delta_{g,1}$. Hence $H^*(\Delta_{g,1}; \mathbb{F}_2) \to H^*(B_{2g+2}; \mathbb{F}_2)$ is surjective and the sequence (2.9) decomposes itself into the short exact sequences

$$0 \to H^{q-2}(B_{2g+1}; \mathbb{F}_2) \to H^q(\Delta_{g,1}; \mathbb{F}_2) \to H^q(B_{2g+2}; \mathbb{F}_2) \to 0. \tag{2.10}$$

Since the $\mathbb{F}_2$-Betti numbers of B_n have been determined by Fuks [7], one can obtain those of $\Delta_{g,1}$.

3. Computations.

Now we shall give computations of the cohomology of the hyperelliptic mapping class group Δ_g based on the results in the preceding sections.

We begin by the first and the second cohomology. The sequence (0.2) implies

$$H^q(\Delta_{g,1}; k) = H^q(\Delta_g; k), \quad \text{if } q = 1, 2 \text{ and } \operatorname{ch} k \not| (2-2g).$$

In view of the presentation of the group Δ_g given by Birman-Hilden [3] Theorem 8, p.110, we have

$$H_1(\Delta_g; \mathbb{Z}) = \begin{cases} \mathbb{Z}/2(2g+1), & \text{if } g \text{ is even,} \\ \mathbb{Z}/4(2g+1), & \text{if } g \text{ is odd.} \end{cases} \tag{3.1}$$

Hence

$$H^1(\Delta_{g,1}; k) = H^1(\Delta_g; k) = \begin{cases} 0, & \text{if } \operatorname{ch} k \not| 2(g-1)(2g+1), \\ k, & \text{if } \operatorname{ch} k \mid (2g+1) \text{ and } \operatorname{ch} k \not| 2(g-1). \end{cases} \tag{3.2}$$

From a result of Arnol'd [1] we have $H_1(B_n; \mathbb{Z}) = \mathbb{Z}$ and $H^2(B_{2g+2}; k) = 0$, if $\operatorname{ch} k \neq 2$. Substituting them into (2.9), we obtain

$$0 \to H^1(\Delta_{g,1}) \to k \to k \to H^2(\Delta_{g,1}) \to 0 \quad \text{(exact)},$$

and so

$$H^2(\Delta_{g,1}; k) = H^2(\Delta_g; k) \cong H^1(\Delta_{g,1}; k) \quad \text{if } \operatorname{ch} k \not| 2(g-1). \tag{3.3}$$

Next we study the third cohomology. From Arnol'd [1] we have

$$H^2(B_{2g+2}; k) = H^3(B_{2g+2}; k) = 0, \quad \text{if } \operatorname{ch} k \neq 2.$$

This implies that the map in (2.9)

$$H^1(B_{2g+1})(\cong k) \to H^3(\Delta_{g,1})$$

is an isomorphism for $\operatorname{ch} k \neq 2$. Consider the Gysin map induced by the extension (1.7)

$$\pi_! : H^3(\Delta_{g,1}; k) \to H^0(\Delta_g; k),$$

or equivalently, the map

$$i^* : H^3(H_{g,1}; k) \to H^3(T^\times H(f); k) = k$$

induced by the natural map

$$i : T^\times H(f) \to H_{g,1}, \quad (p, v) \mapsto (H(f), p, v),$$

where $f \in X_{2g+2}$ and $T^\times H(f)$ is the bundle obtained by deleting the zero section from the tangent bundle of the Riemann surface $H(f)$.

Proposition 3.4. *Let k be a field with $\operatorname{ch} k \neq 2$. The Gysin map induced by the extension (1.7)*

$$\pi_! : H^3(\Delta_{g,1}; k)(\cong k) \to H^0(\Delta_g; k)(\cong k)$$

is an isomorphism if and only if $\operatorname{ch} k \not| \, g(g+1)(2g+1)$. Especially, if $\operatorname{ch} k \not| \, 2(g-1)$, the necessary and sufficient condition for the class ϵ to be non-zero is $\operatorname{ch} k \mid g(g+1)(2g+1)$.

Proof. Let A denote the set of all Weierstrass points on $H(f)$ and F_a the fiber of $T^\times H(f)$ over the point $a \in A$. Set $U := T^\times H(f) - \bigcup_{a \in A} F_a$. We have the commutative diagram

$$\begin{array}{ccccc} H^1(B_{2g+1}) & \cong & H^3(H_{g,1}, W_g) & \xrightarrow{\cong} & H^3(H_{g,1}) \\ \downarrow & & \downarrow & & \downarrow \\ H^1(\bigcup_{a \in A} F_a) & \cong & H^3(T^\times H(f), U) & \to & H^3(T^\times H(f)), \end{array}$$

since the map i is transversal to the submanifold $W_g(\cong X_{2g+1}) \subset H_{g,1}$. Here all the vertical arrows are induced by the map i and the 2 left isomorphisms are the Thom isomorphisms (2.8). From (2.4) the image of the generator of $H_1(F_a; \mathbb{Z})$ under i_* is represented by the loop in X_{2g+1}, $f_{(\exp 4\pi\sqrt{-1}t)}$, $0 \leq t \leq 1$. The loop induces a braid in B_{2g+1}:

$$(\sigma_1 \sigma_2 \cdots \sigma_{2g})^{2(2g+1)}.$$

Therefore $i^* : H^*(H_{g,1}) \cong k \to H^3(T^\times H(f)) \cong k$ is equivalent to the multiplication by $8g(g+1)(2g+1)$, which completes the proof. □

When k is a field with $\operatorname{ch} k \not| \, 2(g-1)g(g+1)(2g+1)$, we may choose an element $\xi \in H^3(\Delta_{g,1}; k)$ satisfying $\pi_!(\xi) = 1 \in H^0(\Delta_g; k)$. The map defined by

$$H^q(\Delta_g) \to H^{q+3}(\Delta_{g,1}), \quad u \mapsto \xi \cup \pi^* u$$

gives a right-inverse of the Gysin map $\pi_!$. Consequently

Corollary 3.5. *Let k be a field with $\operatorname{ch} k \not| \, 2(g-1)g(g+1)(2g+1)$. Then we have an isomorphism*

$$H^*(\Delta_{g,1}; k) = (k \oplus k\xi) \otimes H^*(\Delta_g; k),$$

where $\xi \in H^3(\Delta_{g,1}; k)$.

According to Vainshtein [12], $H^q(B_n; k)$ vanishes if $q \geq 2$ and $\operatorname{ch} k > [n/2]$.

Corollary 3.6. *If k is a field with $\operatorname{ch} k = 0$ or $(\geq g+2$ and $\neq 2g+1)$, then we have*

$$H^*(\Delta_g; k) = k \quad (\text{in } \dim 0).$$

Proof. From (3.2) and (3.3) follows $H^1(\Delta_{g,1}) = H^2(\Delta_{g,1}) = 0$. As was already shown, $H^3(\Delta_{g,1}) = k$. Substituting the results of Vainshtein [12] stated above into (2.9), we obtain $H^q(\Delta_{g,1}) = 0$ for $q \geq 4$. The corollary follows from the previous one. □

Remark. This result and Theorems 3.8, 3.12 are included in those of Boedigheimer-Cohen-Peim [4]. In the case $\operatorname{ch} k = 0$, this result follows from Arnol'd [1] by a consideration involved with differential forms (cf. [8]).

For the rest of this paper we study two easy cases.

First we consider the case when $p = 2g+1$ is a prime number and when k is a field with $\operatorname{ch} k = p = 2g+1$. (3.1) induces a surjection

$$\chi : \Delta_g \to \mathbb{Z}/2g+1 = \mathbb{Z}/p. \tag{3.7}$$

Theorem 3.8 (F.R.Cohen [2][5]). *Suppose $p = 2g+1$ is a prime number and let k be a field with $\operatorname{ch} k = p$. Then the surjection χ (3.7) induces an isomorphism*

$$\chi^* : H^*(\mathbb{Z}/p; k) \cong H^*(\Delta_g; k).$$

Proof. Under the representation of Birman and Hilden [3] Theorem 8, p.110, the mapping class γ defined by

$$\gamma := (\tau_1\tau_2\cdots\tau_{2g})(\tau_{2g+1}\cdots\tau_2{\tau_1}^2\tau_2\cdots\tau_{2g+1})$$

is of period $2g+1$ and satisfies $\chi(\gamma) \neq 0 \in \mathbb{Z}/p$. This implies that the surjection χ splits, and so the homomorphism $\chi^* : H^*(\mathbb{Z}/p; k) \to H^*(\Delta_g; k)$ is injective. Recall that $H^q(\mathbb{Z}/p; k) = k$ for each $q \geq 0$. Consequently it suffices to show

$$\dim H^q(\Delta_g; k) \leq 1 \tag{3.9.q}$$

for each $q \geq 1$. In view of the results of Vainshtein [12] quoted in the proof of Corollary 3.6 we have

$$H^*(B_{2g+1}; k) \cong H^*(B_{2g+2}; k) \cong H^*(S^1; k).$$

Substituting them into the sequence (2.9), we obtain $H^q(\Delta_{g,1}) = 0$ for $q \geq 4$. This implies

$$H^{q-4}(\Delta_g) \to H^q(\Delta_g) \to H^q(\Delta_{g,1}) = 0 \quad \text{(exact)}$$

for $q \geq 4$. Therefore the proof of (3.9.q) is reduced to that in the case $q \leq 3$. (3.9.1) and (3.9.2) follow from (3.2) and (3.3). (3.9.3) is already proved before Proposition 3.4. This completes the proof of the theorem. □

Finally we consider the case $p = g+1$ is a prime number and k is a field with $\operatorname{ch} k = g+1$.

Proposition 3.10. *The Poincaré series of the cohomology group $H^*(\Delta_g; k)$ is given by*

$$1 + t^3 + t^{2g} + t^{2g+1}.$$

Proof. From (3.2) and (3.3) follows $H^1(\Delta_{g,1}) = H^2(\Delta_{g,1}) = 0$. It is already shown that $H^3(\Delta_{g,1}) = k$. As has been proved by Vainshtein [12],

$$H^*(B_{2g+1}; k) \cong H^*(S^1; k), \quad \text{and}$$
$$H^q(B_{2g+2}; k) = \begin{cases} k, & \text{if } q = 0, 1, 2g, 2g+1, \\ 0, & \text{otherwise.} \end{cases}$$

Substituting them into (2.9), we have an isomorphism $H^q(\Delta_{g,1}) \cong H^q(B_{2g+2})$ for $q \geq 4$. The proposition follows immediately. □

$H^1(\Delta_g) = H^2(\Delta_g) = 0$ by (3.2) and (3.3). Hence $e_1 = 0 \in H^2(\Delta_g)$ and so

$$\epsilon = e_2 \in H^4(\Delta_g; k), \quad \text{if } \operatorname{ch} k = g+1. \tag{3.11}$$

Theorem 3.12. *Suppose $p = g + 1$ is a prime number and k is a field with $\mathrm{ch}\, k = p = g + 1$. Then the Poincaré series of the cohomology group $H^*(\Delta_g; k)$ is given by*

$$(1 + t^3 + t^{2g} + t^{2g+1})(1 - t^4)^{-1}.$$

Especially the group $H^*(\Delta_g; k)$ is a free module over the polynomial algebra $k[e_2]$ (freely) generated by the second Morita-Mumford class $e_2 \in H^4(\Delta_g; k)$.

Proof. It suffices to show that the Gysin map in (0.2) $\pi_! : H^q(\Delta_{g,1}) \to H^{q-3}(\Delta_g)$ is a zero map for each $q \geq 3$.

It is already proved in Proposition 3.4 for the case $q = 3$. If $q \neq 3, 2g, 2g + 1$, then $H^q(\Delta_{g,1}) = 0$ and the Gysin map is also a zero map. Especially the Poincare series in question coincides with

$$(1 + t^3 + t^{2g} + t^{2g+1})(1 - t^4)^{-1} \equiv (1 + t^3)(1 - t^4)^{-1}$$

modulo t^{2g}. Since $g + 1 (\geq 3)$ is a prime number, g is even and $2g - 3 \equiv 1$, $2g - 2 \equiv 2 \mod 4$, which implies $H^{2g-3}(\Delta_g) = H^{2g-2}(\Delta_g) = 0$. Thus the Gysin map $\pi_! : H^q(\Delta_{g,1}) \to H^{q-3}(\Delta_g)$ is a zero map for $q = 2g, 2g + 1$, therefore, for all $q \geq 3$. This completes the proof. □

REFERENCES

[1] V.I. Arnol'd, *On some topological invariants of algebraic functions*, Transactions Moscow Math. Soc. **21** (1970), 30–52.

[2] D.J. Benson and F.R. Cohen, *Mapping class groups of low genus and their cohomology*, Mem. Amer. Math. Soc. **443** (1991).

[3] J. Birman and H. Hilden, *On the mapping class groups of closed surfaces as covering spaces*, Ann. of Math. Studies **66** (1971), no. Advances in the theory of Riemann surfaces, 81–115.

[4] C.F. Boedigheimer, F.R. Cohen and M.D. Peim, *Mapping class groups and function spaces*, preprint.

[5] F. R. Cohen, *Homology of mapping class groups for surfaces of low genus*, Contemp. Math. **58** (1987), 21–30.

[6] ———, *On the mapping class groups for punctured spheres, the hyperelliptic mapping class groups, $SO(3)$, $Spin^c(3)$*, Amer. J.of Math. **115** (1993), 389–434.

[7] D.B. Fuks, *Cohomologies of the group COS mod 2*, Functional Anal. Appl. 4 (1970), 143–151.

[8] N. Kawazumi, *On the homotopy type of the moduli space of n-point sets of $\mathbb{P}^1$*, J. Fac. Sci. Univ. Tokyo Ser. IA **37** (1990), 263–287.

[9] S.P. Kerckhoff, *The Nielsen realization problem*, Ann. of Math. **117** (1983), 235–265.

[10] S. Morita, *Characteristic classes of surface bundles*, Invent. math. 90 (1987), 551–577.

[11] D. Mumford, *Towards an enumerative geometry of the moduli space of curves*, Arithmetic and Geometry., Progr. Math. **36** (1983), 271–328.

[12] F.V. Vainshtein, *Cohomologies of braid groups*, Funct. Anal. Appl. **12** (1978), 135–137.