On Commutativity of Diagrams of Type II_1 Factors

Jerzy Wierzbicki

Series ♯264. September 1994

HOKKAIDO UNIVERSITY

PREPRINT SERIES IN MATHEMATICS

♯ 239 Y. Giga, N. Mizoguchi, On time periodic solutions of the Dirichlet problem for degenerate parabolic equations of nondivergence type, 23 pages. 1994.

♯ 240 C. Dohmen, Existence of Fast Decaying Solutions to a Haraux-Weissler Equation With a Prescribed Number of Zeroes, 12 pages. 1994.

♯ 241 K. Sugano, Note on H-separable Frobenius extensions, 8 pages. 1994.

♯ 242 J. Zhai, Some Estimates For The Blowing up Solutions of Semilinear Heat Equations, 11 pages. 1994.

♯ 243 N. Hayashi, K. Kato and T. Ozawa, Dilation Method and Smoothing Effect of the Schrödinger Evolution Group, 10 pages. 1994.

♯ 244 D. Lehmann, T. Suwa, Residues of holomorphic vector fields relative to singular invariant subvarieties, 26 pages. 1994.

♯ 245 H. Kubo, Slowly decaying solutions for semilinear wave equations in odd space dimensions, 30 pages. 1994.

♯ 246 T. Nakazi, M. Yamada, (A_2)-Conditions and Carleson Inequalities, 27 pages. 1994.

♯ 247 N. Hayashi, K. Kato and T. Ozawa, Dilation Method and smoothing Effect of Solutions to the Benjamin-ono Equation, 17 pages. 1994.

♯ 248 H. Kikuchi, Sheaf cohomology theory for measurable spaces, 12 pages. 1994.

♯ 249 A. Inoue, Tauberian theorems for Fourier cosine transforms, 9 pages. 1994.

♯ 250 S. Izumiya, G. T. Kossioris, Singularities for viscosity solutions of Hamilton-Jacobi equations, 23 pages. 1994.

♯ 251 H. Kubo, K. Kubota, Asymptotic behaviors of radially symmetric solutions of $\Box u = |u|^p$ for super critical values p in odd space dimensions, 51 pages. 1994.

♯ 252 T. Mikami, Large Deviations and Central Limit Theorems for Eyraud-Farlie-Gumbel-Morgenstern Processes, 9 pages. 1994.

♯ 253 T. Nishimori, Some remarks in a qualitative theory of similarity pseudogroups, 19 pages. 1994.

♯ 254 T. Suwa, Residues of complex analytic foliations relative to singular invariant subvarieties, 15 pages. 1994.

♯ 255 T. Tsujishita, On Triple Mutual Information, 7 pages. 1994.

♯ 256 T. Tsujishita, Construction of Universal Modal World based on Hyperset Theory, 15 pages. 1994.

♯ 257 A. Arai, Trace Formulas, a Golden-Thompson Inequality and Classical Limit in Boson Fock Space, 35 pages. 1994.

♯ 258 Y-G. Chen, Y. Giga, T. Hitaka and M. Honma, A Stable Difference Scheme for Computing Motion of Level Surfaces by the Mean Curvature, 18 pages. 1994.

♯ 259 K. Iwata, J. Schäfer, Markov property and cokernels of local operators, 7 pages. 1994.

♯ 260 T. Mikami, Copula fields and its applications, 14 pages. 1994.

♯ 261 A. Inoue, An Abel-Tauber theorem for Fourier sine transforms, 6 pages. 1994.

♯ 262 N. Kawazumi, Homology of hyperelliptic mapping class groups for surfaces, 13 pages. 1994.

♯ 263 Y. Giga, M. E. Gurtin, A comparison theorem for crystalline evolution in the plane, 14 pages. 1994.

♯ 264 J. Wierzbicki, On Commutativity of Diagrams of Type II_1 Factors, 26 pages. 1994.

On Commutativity of Diagrams of Type II_1 Factors

By

Jerzy Wierzbicki*

Abstract

We show that a diagram $\begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array}$ of type II_1 factors, with the relative commutant $S' \cap K = \mathbb{C}$, must be a commuting square, if we only impose some conditions on the inclusion $S \subset Q$.

§1. Introduction

The object of our study are diagrams of von Neumann algebras $\begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array}$. This kind of situation was first considered by S.Popa ([P1],[P2],[P3]) and appeared later in other papers, for example [GHJ],[P4],[P5],[SW],[K],[S],[WW]. When the von Neumann algebra K is equipped with a finite tracial weight τ and E_Q^K, E_R^K and E_S^K are τ - preserving conditional expectations of K onto Q, R and S, then a special situation may occur:

$$E_Q^K E_R^K = E_R^K E_Q^K = E_S^K.$$

Then the diagram $\begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array}$ is called a commuting square. At present, this concept plays important role in the theory of subfactors. In the special case, when $S = \mathbb{C}$, the subalgebras Q and R were called by S.Popa orthogonal ([P1]). This case, in the classical probability theory, corresponds to the condition of independence of two σ - fields.

1991 *Mathematics Subject Classification.* Primary 46L37.

*Department of Mathematics, Faculty of Science, Hokkaido University, Sapporo 060,Japan.

Everywhere in this paper, the von Neumann algebras, which form such a diagram, are type II_1 factors. This situation was considered by T.Sano and Y.Watatani in [SW], where the authors introduced the notion of angles between two subfactors Q and R. Our paper is, in a sense, an extension of the study presented in [SW]. It may seem a little surprising that for certain inclusions, say $S \subset Q$, a diagram $\begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$, with (as always in this work) $[K : S] < \infty$ and $S' \cap K = \mathbb{C}$, must be a commuting square. We will show several results of this kind. The notion of commuting square of type II_1 factors is strictly connected to the concept of so called co-commuting square of type II_1 factors, which will be described in the next section. There are plenty of diagrams $\begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$ which are completely irregular in the sense that $Q \vee R = K$, $Q \cap R = S$ and they are neither commuting nor co-commuting squares. We can construct them, like in [SW], by "tensoring" degenerated (in the sense of S.Popa's [P5]) diagrams. However, when the index $[Q : S]$ is "small" or the second relative commutant of the inclusion "$S \subset Q$" is only 2 - dimensional, then the situation is different. For example, we can prove such property:

Let $\mathcal{D} = \begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$ *be a diagram of type* II_1 *factors with* $S' \cap K = \mathbb{C}$ *and* $[K : S] < \infty$. *If* $[Q : S] = 4\cos^2 \frac{\pi}{n}$ *for a prime number n, then the diagram* $\mathcal{D}$ *is a commuting square.*

The crucial observation is that Ocneanu's convolution $E_Q^K * E_R^K$ is a scalar multiple of a projection. Then, we can use results from [SW] or [P3] to show that the "right angle" between two subfactors is not so special in certain situations.

T.Sano and Y.Watatani ([SW]) showed that commutativity of a diagram is equivalent to co-commutativity of some other related diagram. Therefore, sufficient conditions for a diagram to be, for example, a co-commuting square can immediately be reformulated in terms of a commuting square. We will try then not to repeat ourselves and we will keep rather to the co-commuting square terminology.

§2. Fourier Transform and convolution.

We recall here the Ocneanu's Fourier transform and convolution in the third relative commutant. We prove also a few properties which will be useful later. We believe that most of them are known to other persons working in this area. It is convenient to keep in mind a lemma which is an immediate consequence of [P2, Lemma 1.2.2.] and [PP1, Corollary 4.5]. If τ is a faithful, normal and finite trace on a von Neumann algebra B and A is its von Neumann subalgebra $K \subset L$ then by ${}^{\tau}E_A^B$ we denote τ preserving conditional expectation of B onto A. When there is no risk of confusion we will write it simply E_A^B or just E_A.

Lemma 2.1. *Let $K \subset A \subset B \subset L$ be all type II_1 factors and let the inclusion $K \subset L$ be extremal with $[L : K] < \infty$. If τ and τ' are the traces on L and on K', then for any $x \in K' \cap L$,*

$$ {}^{\tau}E_A^B(x) = {}^{\tau}E_{K'\cap A}^{K'\cap B}(x) = {}^{\tau'}E_{K'\cap A}^{K'\cap B}(x) \ \text{and} \ {}^{\tau'}E_{B'}^{A'}(x) = {}^{\tau'}E_{A'\cap L}^{B'\cap L}(x) = {}^{\tau}E_{A'\cap L}^{B'\cap L}(x). $$

We will often use the lemma without referring to it.

Let $S \subset K$ be an irreducible inclusion of type II_1 factors with $[K : S] = \gamma^{-1} < \infty$. Consider corresponding Jones' tower

$$ S \subset K \subset^{e_S} L \subset^{e_K} L_1. $$

Note that by [PP2] the inclusion $S \subset L_1$ is extremal, and so Lemma 2.1 can be applied.

Definition 2.2. The mapping $\mathcal{F} : K' \cap L_1 \to S' \cap L$ given by

$$ \mathcal{F}(x) = \gamma^{-\frac{3}{2}} E_L(x e_S e_K) $$

will be called the Fourier transform of $K' \cap L_1$ into $S' \cap L$. By the inverse Fourier transform we mean the mapping $\mathcal{F}^* : S' \cap L \to K' \cap L_1$ defined by

$$ \mathcal{F}^*(y) = \gamma^{-\frac{3}{2}} E_{K'\cap L_1}(y e_K e_S). $$

We show that the name "inverse Fourier transform" is justified. The algebra $S'\cap L_1$ has canonical Hilbert space structure determined by the trace $\tau_{L_1} = \tau_{S'\cap L_1}$. Let us consider $S'\cap L$ and $K'\cap L$ as its Hilbert subspaces.

Proposition 2.3. *The mappings $\mathcal{F}$ and $\mathcal{F}^*$ are Hilbert space isomorphisms between $S'\cap L$ and $K'\cap L_1$. Moreover,*

$$\mathcal{F}\mathcal{F}^* = Id_{|S'\cap L} \quad \text{and} \quad \mathcal{F}^*\mathcal{F} = Id_{|K'\cap L_1}.$$

In [W] was shown a property of the Pimsner-Popa basis, which is very useful in proof of the above proposition.

Lemma 2.4. *Let $N\subset M$ be a finite index inclusion of type II_1 factors. If $\{m_i\}_i$ is a Pimsner-Popa basis of N' over M', then the trace preserving conditional expectation E_N^M of M onto N is expressed as follows:*

$$E_N^M(x) = [M:N]^{-1}\sum_i m_i x m_i^*.$$

Proof of Proposition 2.3. We show first that $\mathcal{F}\mathcal{F}^* = Id_{|S'\cap L}$. Let $\{m_i\}_{i=0}^n$ be Pimsner-Popa basis of K over S such that $m_0 = 1$. The preceding lemma gives: $E_{K'}^{S'}(y) = \gamma\sum_i m_i y m_i^*$ for trace preserving conditional expectation $E_{K'}^{S'}$. From Lemma 2.1,

$$^{\tau'}E_{K'\cap L_1}^{S'\cap L_1}(ye_Ke_S) = E_{K'}^{S'}(ye_Ke_S) = \gamma\sum_i m_i y e_K e_S m_i^*,$$

for $y\in S'\cap L$. Hence

$$\gamma^3\mathcal{F}\mathcal{F}^*(y) = \gamma E_L(\sum_i m_i y e_K e_S m_i^* e_S e_K) =$$

$$= \gamma E_L(\sum_i m_i y e_K E_S(m_i^* m_0) e_S e_K) = \gamma E_L(y e_K e_S e_K) = \gamma^2 E_L(y e_K) = \gamma^3 y.$$

We show now that $\mathcal{F}^*$ is a contraction. For $y\in S'\cap L$

$$\|\mathcal{F}(y)\|_2^2 = \gamma^{-3}\tau(E_{K'\cap L_1}(ye_Ke_S)E_{K'\cap L_1}(ye_Ke_S)^*) =$$

$$\gamma^{-1}\sum_{i,j}\tau(m_i y e_K e_S m_i^* m_j e_S e_K y^* m_j^*) =$$

$$= \gamma^{-1}\sum_{i<n}\tau(m_i y e_K e_S e_K y^* m_i^*) + \gamma^{-1}\tau(m_n y e_K p e_S e_K y^* m_n^*),$$

where $p = E_S(m_n^* m_n)$ is a projection, cf.[PP1]. Since $[p, e_K] = 0$,

$$\|\mathcal{F}(y)\|_2^2 = \sum_{i<n}\tau(m_i y e_K y^* m_i^*) + \tau(m_n y e_K p e_K y^* m_n^*) \leq \sum_{i\leq n}\tau(m_i y e_K y^* m_i^*) =$$

$$= \sum_{i\leq n}\tau(E_L(m_i y e_K y^* m_i^*)) = \gamma\tau(\sum_{i\leq n} m_i y y^* m_i^*) = \gamma\tau(y y^* \sum_{i\leq n} m_i^* m_i) =$$

$$\gamma\|y\|_2^2\tau(\sum_{i\leq n} m_i m_i^*) = \|y\|_2^2.$$

Similarly, using a Pimsner-Popa basis of L' over L_1' we obtain that $\mathcal{F}$ is a contraction too. So $\mathcal{F}$ and $\mathcal{F}^*$ are isometries and the standard polarization argument completes the proof.

Q.E.D.

We will always use notation $\langle K, S, e\rangle$, for an algebraic basic construction ([PP2]) of a finite index inclusion $S \subset K$ of type II_1 factors, where e is the corresponding Jones projection, i.e. $exe = E_S^K(x)e$, for $x \in K$ and $\langle K, S, e\rangle$ is generated, as a von Neumann algebra, by K and e. If $\langle K, S, e\rangle$ is represented on $L^2(K,\tau)$ and e is just an extension of E_S^K, then, as in [J], we will write simply $\langle K, e\rangle$. Let us consider an intermediate subfactor Q between S and K, with indices $[Q : S] = \alpha^{-1}$ and $[K : Q] = \lambda^{-1}$. By [J] we know that $\alpha\lambda = \gamma$. Let $M = \langle K, e_Q\rangle$ and $M_1 = \langle L, e_M\rangle$ be Jones' basic constructions. We have

$$S \subset Q \subset K \subset^{e_Q} M \subset L \subset^{e_M} M_1 \subset L_1,$$

where, as before, $L = \langle K, e_S\rangle$ and $L_1 = \langle L, e_K\rangle$. Obviously, $e_Q \in S' \cap L$ and $e_M \in K' \cap L_1$. Also $e_Q e_S = e_S$ and $e_M e_K = e_K$.

Proposition 2.5.

$$\mathcal{F}(\alpha^{-\frac{1}{2}} e_M) = \lambda^{-\frac{1}{2}} e_Q \textit{ and } \mathcal{F}^*(\lambda^{-\frac{1}{2}} e_Q) = \alpha^{-\frac{1}{2}} e_M.$$

Proof. We will show only the second equality. First, we remark that $e_Q e_K e_Q = \lambda e_M e_Q$. Indeed, let us represent all these operators on $L^2(L,\tau)$ and let the vector sign denote the canonical embedding of L in $L^2(L,\tau)$. Then for any $\overrightarrow{ae_Sb} \in L^2(L,\tau)$, $a,b \in K$ we have:

$$e_Q e_K e_Q \overrightarrow{ae_Sb} = \overrightarrow{e_Q E_K(e_Q a e_S b)} = \overrightarrow{e_Q E_K(e_Q a e_Q e_S b)} =$$

$$= \overrightarrow{e_Q E_Q(a) E_K(e_S) b} = \gamma \overrightarrow{e_Q E_Q(a) b} =$$

$$= \lambda \overrightarrow{E_Q(a) \alpha e_Q b} = \lambda \overrightarrow{E_M(E_Q(a) e_S b)} = \lambda e_M e_Q \overrightarrow{ae_Sb}.$$

In the last line we used the property $E_M(e_S) = \alpha e_Q$, cf. [SW, Lemma 7.1].

Now,

$$\mathcal{F}^*(e_Q) = \gamma^{-\frac{3}{2}} E_{K'\cap L_1}(e_Q e_K e_S) = \gamma^{-\frac{3}{2}} E_{K'\cap L_1}(\lambda e_M e_S) =$$

$$\gamma^{-\frac{3}{2}} \lambda e_M E_{K'\cap L_1}(e_S) = \sqrt{\frac{\lambda}{\alpha}} e_M,$$

where we used [PP1, Corollary 4.5].

Q.E.D.

Definition 2.6. For $x, y \in S' \cap L$ we set

$$x * y = \mathcal{F}(\mathcal{F}^*(x)\mathcal{F}^*(y)).$$

Similarly, for $x, y \in K' \cap L_1$

$$x \hat{*} y = \mathcal{F}^*(\mathcal{F}(x)\mathcal{F}(y)).$$

Following [O], we call the operation "$*$" or "$\hat{*}$" convolution. The convolution "$\hat{*}$" in $K' \cap L_1$ should not be confused with the convolution "$*$", which we define by building up another basic construction, say $L_2 = \langle L_1, e_L \rangle$, and putting $x * y = \mathcal{F}_1(\mathcal{F}_1^*(x)\mathcal{F}_1^*(y))$, where $\mathcal{F}_1$ and $\mathcal{F}_1^*$ are the shifted Fourier and inverse Fourier transforms: $\mathcal{F}_1^* : K' \cap L_1 \to L' \cap L_2$, $\mathcal{F}_1^*(x) = \gamma^{-\frac{3}{2}} E_{L'}(x e_L e_K)$ and $\mathcal{F}_1 : L' \cap L_2 \to K' \cap L_1$, $\mathcal{F}_1(y) = \gamma^{-\frac{3}{2}} E_{L_1}(y e_K e_L)$. Similarly, we may define "$\hat{*}$" in $S' \cap K$.

Proposition 2.7.

$$\textit{For } x \in S' \cap L, \ \sqrt{\frac{\alpha}{\lambda}} e_Q * x = E_M(x).$$

$$\textit{For } y \in K' \cap L_1, \ \sqrt{\frac{\lambda}{\alpha}} e_M \hat{*} y = E_{M'}(y) = E_{M' \cap L_1}(y).$$

Proof. Let $\tilde{x} = \mathcal{F}^*(x) \in K' \cap L_1$. By [PP1, Lemma 1.2] and Proposition 2.5,

$$E_L(e_M \tilde{x} e_S e_K) = \gamma^{-1} E_L(e_M E_L(\tilde{x} e_S e_K) e_K) = \gamma^{-1} E_L(e_M E_L(\tilde{x} e_S e_K) e_M e_K) =$$

$$= \gamma^{-1} E_L(E_M(\tilde{x} e_S e_K) e_K) = E_M(\tilde{x} e_S e_K) = \gamma^{\frac{3}{2}} E_M(\mathcal{F}(\tilde{x})) = \gamma^{\frac{3}{2}} E_M(x).$$

Since $S' = \langle K', L', e_S \rangle$ and $Q' = \langle K', M', e_Q \rangle$, the proof of the other equality is almost the same.

Q.E.D.

Lemma 2.8. *Let $S \subset Q \subset K$ be type II_1 factors, $[K : S] < \infty$, $M = \langle K, Q, e_Q \rangle$ and $S' \cap Q = \mathbb{C}$. Then*

(1) *e_Q is minimal projection in $S' \cap M$.*

(2) *e_Q is central in $S' \cap M$, iff $S' \cap K = \mathbb{C}$.*

Proof. (1) Suppose $e \leq e_Q$ for some non-zero projection $e \in P(S' \cap M)$. Exactly as in [P3, Proposition 7.3] we obtain a projection $f \in S' \cap K$ such that $E_K(e) = \lambda f$ ($\lambda = [K : Q]^{-1}$). Now,

$$e = e e_Q = \lambda^{-1} E_K(e e_Q) e_Q = \lambda^{-1} E_K(e) e_Q = f e_Q.$$

Hence $[f, e_Q] = 0$; therefore $f \in S' \cap Q = \mathbb{C}$, which implies $e = e_Q$.

(2) "$\Rightarrow$" Fix $q_1, q_2 \in P(S' \cap K)$ and a number $\theta \in (0, 2\pi]$ such that $q_1 + q_2 = 1$ and $|\tau(q_1) + \exp(i\theta)\tau(q_2)| \neq 1$. $u = q_1 + \exp(i\theta) q_2$ is a unitary in $S' \cap K$ and since e_Q is central, $e_Q = e_Q u e_Q u^* e_Q$. Hence $|\tau(u)| = 1$, which gives contradiction.

"$\Leftarrow$"Let $c = C(e_Q, S' \cap M)$ be the central support of e_Q in $S' \cap M$. For any projection $f \in (S' \cap M)_c$ satisfying $\tau(f) = \lambda$, there is a unitary $u \in S' \cap M$ such that $f = ue_Qu^*$. Hence, by [PP1, Lemma 1.2] and Lemma 2.1

$$ue_Q = \lambda^{-1}E_K(ue_Q)e_Q = \lambda^{-1}\tau(ue_Q)e_Q.$$

Thus $f = \lambda^{-2}|\tau(ue_Q)|^2 e_Q$. This implies $c = e_Q$.

Q.E.D.

From now on, we consider two intermediate subfactors Q, R between S and K: $\begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array}$. We always consider non-degenerated diagrams, in the sense that $Q \not\subset R$ and $R \not\subset Q$, but we do not assume, like in [SW], that always $K = Q \vee R$ and $S = Q \cap R$. Throughout the rest of the paper we keep to the notation preceding Proposition 2.5 and we add the following one: $N = \langle K, e_R \rangle$ and $N_1 = \langle L, e_N \rangle$ will be the Jones basic constructions. For brevity we assume that $\eta = [K : R]^{-1}$ and $\beta = [R : S]^{-1}$. The inclusions

$$\begin{array}{ccc} N & \subset & L \\ \cup & & \cup \\ K & \subset & M \end{array} \quad \text{and} \quad \begin{array}{ccc} M_1 & \subset & L_1 \\ \cup & & \cup \\ L & \subset & N_1 \end{array}$$

hold true. J_K (or J_L) will be modular conjugation in $L^2(K, \tau)$ (or $L^2(L, \tau)$). We always assume that the inclusion $S \subset K$ is irreducible.

Theorem 2.9. *There exists a projection p in $R' \cap M$ such that:*

(1)

$$e_Q * e_R = \tau(e_Q e_R)\gamma^{-\frac{1}{2}} p = \sqrt{\frac{\lambda}{\alpha}} E_M(e_R) = \sqrt{\frac{\eta}{\beta}} E_{R'}(e_Q),$$

(2)

$$e_R * e_Q = J_K e_Q * e_R J_K = \tau(e_Q e_R)\gamma^{-\frac{1}{2}} J_K p J_K = \sqrt{\frac{\lambda}{\alpha}} E_{Q'}(e_R) = \sqrt{\frac{\eta}{\beta}} E_N(e_Q),$$

(3) *p is minimal and central projection in $R' \cap M$ and $p \geq e_R \vee e_Q$,*

(4) *$\hat{p} = J_K p J_K$ is minimal and central projection in $Q' \cap N$ and $\hat{p} \geq e_R \vee e_Q$.*

Proof. Let $p = \bigwedge\{f | f \geq e_Q, f \in P(R' \cap M)\}$. Suppose p_1, $0 \neq p_1 \leq p$, is another projection in $R' \cap M$. By Lemma 2.8 $e_Q p_1 = 0$ or $e_Q p_1 = e_Q$. This implies $e_Q \leq p - p_1 \leq p$ or $e_Q \leq p_1 \leq p$. So the minimality of p gives us $p_1 = 0$ or $p_1 = p$. Therefore p is minimal in $R' \cap M$.

Let $E_{R'}(e_Q) = \sum_{1 \leq i \leq n} \alpha_i p_i$, $\alpha_i > 0$ be the spectral decomposition. Then, by $e_Q \leq p$, it follows that $\sum \alpha_i p_i \leq p$ and, in consequence $\sum p_i \leq p$. Therefore, by minimality of p, we see that $n = 1$ and $E_{R'}(e_Q) = \theta p$, for some scalar θ. From this and Lemma 2.8 we see that p is also central in $R' \cap M$.

Identically we obtain $E_{Q'}(e_R) = \kappa g$ for a minimal and central projection g in $Q' \cap N$ and a scalar κ. It is easy to check that

$$\forall x \in S' \cap L, \quad E_{N \cap Q'}(x) = J_K E_{R' \cap M}(J_K x J_K) J_K.$$

Using this we see that for $\hat{g} = J_K g J_K$,

$$\kappa \hat{g} = J_K E_{Q' \cap N}(e_R) J_K = E_{R' \cap M}(J_K e_R J_K) = E_M(e_R) = \sqrt{\frac{\alpha}{\lambda}} e_Q * e_R.$$

Also, $\hat{g}$ is minimal and central projection in $R' \cap M$. Since $e_R \leq g$, we have $e_R \leq \hat{g}$. Hence

$$0 \neq e_S \leq e_Q \wedge e_R \leq p \wedge \hat{g},$$

which implies $p = \hat{g}$. Computation of coefficients θ and κ ends the proof. For example, let us multiply the equality $\kappa p = E_M(e_R)$ by e_Q and take the trace: $\kappa e_Q = E_M(e_R e_Q)$, and so $\kappa = \lambda^{-1} \tau(e_R e_Q)$.

Q.E.D.

Let us write $[x]$ for the range projection of an operator x. Similarly as above, we obtain the following corollary.

Corollary 2.10. *$[e_M \hat{*} e_N]$ is a minimal and central projection in $M' \cap N_1$, $[e_M \hat{*} e_N] \geq e_M \vee e_N$ and*

$$e_M \hat{*} e_N = J_L e_N \hat{*} e_M J_L = \tau(e_M e_N) \gamma^{-\frac{1}{2}} [e_M \hat{*} e_N] = \sqrt{\frac{\alpha}{\lambda}} E_{M'}(e_N) = \sqrt{\frac{\beta}{\eta}} E_{N_1}(e_M).$$

Remark. From the above theorem and corollary, it is easy to see the following property of the "shifted" convolution "$*$" in $K' \cap L_1$: $e_N * e_M = e_M \hat{*} e_N$. Then it is natural to ask, wether this relation holds true for arbitrary operators in $K' \cap L_1$. Right now, it is not clear for us.

For later convenience, we carry out some computations. From [J] we know, that $\frac{\lambda}{\beta} = \frac{\eta}{\alpha}$. We denote this quotient by δ.

Proposition 2.11.

$$\frac{\tau([e_Q * e_R])}{\tau([e_M \hat{*} e_N]} = \frac{\tau(e_Q e_R)}{\tau(e_M e_N)} = \delta.$$

Proof. From Propositions 2.5 and 2.3 and Theorem 2.9 we have:

$$\tau(e_M e_N) = \|e_M e_N\|_2^2 = \|\mathcal{F}(e_M e_N)\|_2^2 = \frac{\alpha\beta}{\lambda\eta}\|e_Q * e_R\|_2^2 =$$

$$= \frac{\alpha\beta}{\lambda\eta}\tau(e_Q e_R)^2 \gamma^{-1} \frac{\lambda\eta}{\tau(e_Q e_R)} = \delta^{-1}\tau(e_Q e_R).$$

Q.E.D.

T.Sano and Y.Watatani, cf. [SW], introduced notion of angles between subfactors Q and R. The diagram $\begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$, with $K = Q \vee R$ and $\text{Op-ang}_K(Q,R) = \{\frac{\pi}{2}\}$ will be called co-commuting square. (Cf. [K],[WW],[Wi].) The following characterization can be found in [SW].

Proposition 2.12. *The following conditions are equivalent:*

(1) *Diagram* $\begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$ *of finite factors is a co-commuting square.*

(2) *Diagram of their commutants* $\begin{matrix} R' & \subset & S' \\ \cup & & \cup \\ K' & \subset & Q' \end{matrix}$ *forms a commuting square.*

(3) *Diagram* $\begin{matrix} M & \subset & L \\ \cup & & \cup \\ K & \subset & N \end{matrix}$ *is a commuting square.*

(4) *Diagram* $\begin{matrix} M_1 & \subset & L_1 \\ \cup & & \cup \\ L & \subset & N_1 \end{matrix}$ *is a co-commuting square.*

(5) $E_M(e_R) \in \mathbb{C}$.

As an application of Theorem 2.9 we give a few other conditions on commutativity. (Remember that $K' \cap L = \mathbb{C}$ and $[L : K] = [K : S] = \gamma^{-1} < \infty$.)

Corollary 2.13. *The following conditions are equivalent:*

(1) $\begin{array}{ccc} N & \subset & L \\ \cup & & \cup \\ K & \subset & M \end{array}$ *is a co-commuting square.*

(2) $N' \cap M_1 = \mathbb{C}$.

(3) $\tau(e_N e_M) = \alpha\beta$.

(4) $e_N \hat{*} e_M$ *is a scalar.*

(5) $e_N \hat{*} e_M = e_M \hat{*} e_N$ *and* $N \vee M = L$.

Proof. "(2) $\Rightarrow$ (1)" follows from Proposition 2.12.

"(1) $\Rightarrow$ (2)" If (1) is satisfied then, from Proposition 2.12, $E_{M_1}(e_N) \in \mathbb{C}$. Hence and by Theorem 2.9 the identity is minimal in $N' \cap M_1$.

"(1) $\Leftrightarrow$ (3)" Since, by computation in 2.11, (3) is equivalent to $e_Q e_R = e_S$, this is immediate consequence of Proposition 2.12.

"(2) $\Leftrightarrow$ (4)" is obvious because, by Theorem 2.9, $e_M \hat{*} e_N$ is a scalar multiple of a minimal projection in $M' \cap N_1$.

"(1) $\Leftrightarrow$ (5)" is direct consequence of Proposition 2.5.

Q.E.D.

By the remark following Corollary 2.10, we can replace the symbol "$\hat{*}$" in (4) and (5) of the above corollary, by "$*$".

Example. Let μ be an outer action of a finite group G on type II_1 factor S. If A and B are subgroups of G with trivial intersection $A \cap B = \{e\}$, then we can consider diagram of crossed products $\begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array}$, where $Q = S \rtimes_\mu A$, $R = S \rtimes_\mu B$ and $K = S \rtimes_\mu G$. The diagram of corresponding basic constructions, cf. [SW, Lemma 5.1], can be identified with the following one $\begin{array}{ccc} N & \subset & L \\ \cup & & \cup \\ K & \subset & M \end{array}$, where

$M = (S \otimes L^\infty(G/A)) \rtimes_\mu G$, $N = (S \otimes L^\infty(G/B)) \rtimes_\mu G$, $L = (S \otimes L^\infty(G)) \rtimes_\mu G$. $L^\infty(G/A)$ ($L^\infty(G/B)$) contains functions constant on left cosets of A (of B) and the action μ is extended in obvious way, cf. [SW]. If χ_D denotes the characteristic function of a subset $D \subset G$, then

$$e_Q \approx 1 \otimes \chi_A, \quad e_R \approx 1 \otimes \chi_B, \quad \text{and} \quad e_S \approx 1 \otimes \chi_{\{e\}}.$$

One can easily verify that $R' \cap M$ is isomorphic to the subspace of $L^\infty(G)$ which is composed of functions constant on double cosets BgA, $g \in G$. Similarly $Q' \cap N \approx L^\infty(A \setminus G/B)$ and we have

$$[e_Q * e_R] \approx \chi_{BA} \quad \text{and} \quad [e_R * e_Q] \approx \chi_{AB}.$$

Also, since $\begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array}$ is a commuting square, $[e_N \hat{*} e_M] = [e_M \hat{*} e_N] = 1$.

§3. Angles for subfactors with "small" second relative commutant.

The following lemma may be considered as a generalization of Lemma 5.3 in [SW].

Lemma 3.1. *If the diagram* $\begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array}$ *is not a co-commuting square and* $\dim(Q' \cap M) = 2$, *then the only non-zero element of spectrum* $Sp(e_M e_N e_M - e_K)$ *is* $\frac{1}{1-\lambda}(\frac{\tau(e_N e_M)}{\alpha} - \lambda)$.

Proof. Note that our assumption $\dim(Q' \cap M) = 2$ implies $Q \vee R = N \cap M = K$. Let us denote $E_0 = E_M E_N E_M$ and $E = E_0 - E_K$. It is sufficient to show (see [SW, Corollary 3.1]) that $E^2 = \frac{1}{1-\eta}(\frac{\tau(e_N e_M)}{\beta} - \eta)E$. Since E has the bimodule property, by [PP1, Lemma 1.2], we need only to make sure that $E^2(e_S) = \frac{1}{1-\lambda}(\frac{\tau(e_N e_M)}{\alpha} - \lambda)E(e_S)$. By [SW, Lemma 7.1] and Theorem 2.9, for the projection $\hat{p} = [e_R * e_Q] \in Q' \cap N$ we have

$$E_0(e_S) = E_M E_N(\alpha e_Q) = \frac{\tau(e_Q e_R)}{\delta} E_M(\hat{p}).$$

Since $\dim(Q' \cap M) = 2$,

$$E_M(\hat{p}) = \frac{\tau(\hat{p}e_Q)}{\tau(e_Q)}e_Q + \frac{\tau(\hat{p}(1-e_Q))}{\tau(1-e_Q)}(1-e_Q) = e_Q + \frac{\tau(\hat{p})-\lambda}{1-\lambda}(1-e_Q).$$

Therefore, setting

$$\theta = \tau(e_Qe_R)/\eta \quad \text{and} \quad \kappa = \frac{\lambda}{\eta}\frac{\eta - \tau(e_Qe_R)}{1-\lambda}, \quad \text{we get}$$

$$E_0(e_S) = \alpha E_0(e_Q) = \alpha\theta e_Q + \alpha\kappa(1-e_Q).$$

Hence,

$$E_0^2(e_S) = E_0(\alpha\theta e_Q + \alpha\kappa(1-e_Q)) = (\theta-\kappa)E_0(e_S) + \alpha\kappa \quad \text{which gives}$$

$$E^2(e_S) = (E_0 - E_K)(E_0 - E_K)(e_S) = (E_0^2 - E_K)(e_S) = (\theta-\kappa)E_0(e_S) + \alpha\kappa - \gamma$$

$$= (\theta-\kappa)E(e_S) + (\theta-\kappa)\gamma + \alpha\kappa - \gamma = (\theta-\kappa)E(e_S)$$

and

$$\theta - \kappa = \frac{1}{1-\lambda}(\frac{\tau(e_Qe_R)}{\eta} - \lambda) = \frac{1}{1-\lambda}(\frac{\tau(e_Ne_M)}{\alpha} - \lambda).$$

Q.E.D.

Theorem 3.2. *Let* $\mathcal{D} = \begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$ *be a diagram of type type* II_1 *factors* ($[K:S] < \infty$, $S' \cap K = \mathbb{C}$) *and let* $[K:R] < [K:Q]$. *If* $\dim Q' \cap M = 2$ ($M = \langle K, Q, e_Q \rangle$), *then* $\mathcal{D}$ *is a co-commuting square.*

Proof. Suppose that $\mathcal{D}$ is not a co-commuting square. From the preceding lemma, the spectrum $Sp(e_Me_Ne_M - e_K) - \{0\} = \{\sigma\}$, where $\sigma = \frac{1}{1-\lambda}(\frac{\tau(e_Ne_M)}{\alpha} - \lambda)$. Since

$$Sp(e_Ne_Me_N - e_K) - \{0\} = Sp(e_Ne_M - e_K)(e_Ne_M - e_K)^* - \{0\}$$

$$= Sp(e_Ne_M - e_K)^*(e_Ne_M - e_K) - \{0\} = \{\sigma\},$$

it follows that

$$e_Me_Ne_M = e_K + \sigma g, \quad e_Ne_Me_N = e_K + \sigma f,$$

for some projections $g, f \in S' \cap L$. Then, by Proposition 2.12, $\tau(e_M e_N) \neq \gamma$ $(= [K : S]^{-1})$, which implies $\tau(g) = (\tau(e_M e_N) - \gamma)\frac{1}{\sigma} = \alpha - \gamma$, but

$$f \leq e_N - e_K \text{ and } \tau(f) = \tau(g),$$

and so $\alpha - \gamma \leq \beta - \gamma$, which gives contradiction with $\lambda < \eta$.

Q.E.D.

Corollary 3.3. *If* $\mathcal{D} = \begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array}$, $\dim Q' \cap M = \dim R' \cap M = 2$ *and* $[K : Q] \neq [K : R]$, *then* $\mathcal{D}$ *is a co-commuting square.*

§4. Angles for subfactors with "small" indices.

We use here results of S.Popa, cf. [P3], to consider another sufficient conditions which lead to the same effect. Let us outline some of those results. We assume the following notations: $P_n(x)$ will be Jones' polynomials,

$$P_{-1} \equiv P_0 \equiv 1 \quad P_{n+1}(x) = P_n(x) - xP_{n-1},$$

if $x > \frac{1}{4}$ then

$$\Lambda_0(x) = \{\frac{P_{k-1}(x)}{P_{k-2}(x)} \mid 2 \leq k \leq n-2 \} = \{x\frac{P_{k-1}(x)}{P_k(x)} \mid 0 \leq k \leq n-4 \} \text{ and } \Lambda_1 = \emptyset,$$

if $x \leq \frac{1}{4}$ then

$$\Lambda_0(x) = \{x\frac{P_{k-1}(x)}{P_k(x)} \mid 0 \leq k \} \cup \{\frac{P_{k+1}(x)}{P_k(x)} \mid 0 \leq k \}$$

$$\text{and} \quad \Lambda_1 = [\frac{1 - \sqrt{1-4x}}{2}, \frac{1 + \sqrt{1-4x}}{2}],$$

$$\Lambda(x) = \Lambda_0(x) \cup \Lambda_1(x) \quad \text{and} \quad \tilde{\Lambda}(x) = \Lambda(x)) \cap (x + \Lambda(x)).$$

If $B \subset A$ is a finite index inclusion of type II_1 factors, then

$$P(A, B) = \{e \in P(A) | E_B(e) \text{ is a scalar } \}$$

and as in [P3] we denote: $\Lambda(A,B) = \{\tau(e) | e \in P(A,B)\}$.

Theorem 4.1. *For a finite index inclusion of type II_1 factors $B \subset A$, $\Lambda(A,B) \subset \Lambda([A:B]^{-1})$. Moreover, if $[A:B] \leq 4$, then $\Lambda(A,B) = \Lambda([A:B]^{-1})$.*

Next proposition is a slight modification of [P3, Proposition 4.5]. We will set

$$\mathcal{L}^s(B \subset A) = \{F |\ F \text{ is factor, } B \subset F \subset A \text{ and } [A:F]^{-1} = s\}.$$

Proposition 4.2. *Let $D \subset B \subset A$ be a triple of type II_1 factors with $[A:D] < \infty$ and $D' \cap B = \mathbb{C}$. If $f \in P(D' \cap A)$ and $\tau(f) \in \Lambda_0(t)$ ($t = [A:B]^{-1}$), then there exists $F \in \mathcal{L}^s(D \subset B)$, $s = \frac{t^{k+1}}{P_k(t)^2}$, where $k \geq 0$ is determined by $\tau(f) = t\frac{P_{k-1}(t)}{P_k(t)}$ or by $\tau(f) = \frac{P_{k+1}(t)}{P_k(t)}$.*

Proof. We may assume that $\tau(f) = t\frac{P_{k-1}(t)}{P_k(t)}$, for some $k \geq 0$. Since evidently $f \in P(A,B)$, exactly as in [P3, Proposition 4.5] we obtain a subfactor $F \subset B$ with $[B:F]^{-1} = s$. It is easy to verify that the additional assumption $f \in D'$ yields $D \subset F$.

Q.E.D.

Let us now come back to diagrams of type II$_1$ factors. We keep to all the notations from the preceding sections as well as to the assumptions $[K:S] < \infty$ and $S' \cap K = \mathbb{C}$.

Lemma 4.3.

$$\tau([e_Q * e_R]) \in [[\tilde{\Lambda}(\lambda) \cap \tilde{\Lambda}(\eta)] \cup \{1\}] \cap \delta[[\tilde{\Lambda}(\alpha) \cap \tilde{\Lambda}(\beta))] \cup \{1\}].$$

Proof. Suppose that $p = [e_Q * e_R] \neq 1$. Since $p \in R' \cap M$ and $R' \cap K = \mathbb{C}$, by Lemma 2.1, $p \in P(M,K)$, and so $\tau(p) \in \Lambda(M,K) \subset \Lambda(\lambda)$. The same reasoning applied to the projection $p - e_Q \in S' \cap M$ also gives $\tau(p) - \lambda \in \Lambda(\lambda)$. Hence $\tau(p) \in \tilde{\Lambda}(\lambda)$. Identically, we obtain $\tau(\hat{p}) = \tau(J_K p J_K) \in \tilde{\Lambda}(\eta)$, but $\tau(p) = \tau(\hat{p})$; therefore,

$$\tau(p) \in \tilde{\Lambda}(\lambda) \cap \tilde{\Lambda}(\eta).$$

Again, the projection $q = [e_M \hat{*} e_N] \in P(N_1, L)$ and analogically we obtain:

$$\tau(q) \in \tilde{\Lambda}(\alpha) \cap \tilde{\Lambda}(\beta)$$

or $\tau(q) = 1$. Now, by Proposition 2.11, $\tau(p) = \delta\tau(q)$, which completes the proof.

Q.E.D.

Remark. Note that the condition $\tau(p) \in \Lambda(\lambda)$, which is weaker than the above lemma, is a kind of generalization (at least in $S' \cap K = \mathbb{C}$ case) of Popa's [P3, Theorem 6.1]. Indeed, if diagram $\begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$ is a commuting square then $\tau(p) = \delta$. Hence

$$\delta = \frac{\lambda}{\beta} \in \Lambda(\lambda) \Leftrightarrow \beta \in \lambda\Lambda(\lambda)^{-1} = \Lambda(\lambda),$$

where the last equality comes directly from the definition of the set $\Lambda(\lambda)$.

Remark. Incidentally, with a little extra effort we can make the Popa's condition a bit stronger. If diagram $\begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$ is a commuting square ($S' \cap K = \mathbb{C}$) then $\beta \in \{\lambda, \frac{\lambda}{1-\lambda}\} \cup \Lambda_1(\lambda)$ and $\alpha \in \{\eta, \frac{\eta}{1-\eta}\} \cup \Lambda_1(\eta)$. In the proof we use Proposition 4.2 to obtain a factor in $\mathcal{L}^s(N \subset M_1)$ with "too big" index s^{-1}.

Example. It is easy to see that for $\lambda^{-1} < 2 + \sqrt{5}$, the set $\tilde{\Lambda}(\lambda)$ is finite. Let λ be a transcendental and η algebraic numbers, $1 < \lambda^{-1}, \eta^{-1} < 2 + \sqrt{5}$. Then, by the above lemma, the diagram $\begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$ is a co-commuting square.

Example. It is interesting to see, how some of the above results reflect in the group theory. Let μ be an outer action of a finite group G on a type II_1 factor S. Let, for example, H and F be non-trivial subgroups of G such that $H \cap F = \{1\}$. We can consider diagram $\begin{matrix} S \rtimes_\mu H & \subset & S \rtimes_\mu G \\ \cup & & \cup \\ S & \subset & S \rtimes_\mu F \end{matrix}$ and apply [NT], [SW], Proposition 4.2 and Theorem 2.9 to get a property of groups which does not seem quite trivial. If $[G : H] = 3$ and $G \neq HF$, then $|F| = 2$ and there is an intermediate subgroup D such that $F \subset D \subset G$, with $[G : D] = 3$ or $[G : D] = 4$.

$$\begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array}$$

Theorem 4.4. *Let* $\mathcal{D} = \begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array}$ *be a diagram of type* II_1 *factors with* $S' \cap K = \mathbb{C}$ *and* $[K : S] < \infty$. *If* $[K : Q] = 4\cos^2\frac{\pi}{n}$ *for a prime number* n, *then the diagram* $\mathcal{D}$ *is a co-commuting square.*

Proof. Let $n \geq 5$ be a prime number and $[K : Q] = \lambda^{-1} = 4\cos^2\frac{\pi}{n}$. By Lemma 4.3, it is sufficient to prove that the set $\tilde{\Lambda}(\lambda)$ is empty. From its definition, it is not empty, iff there exist integers k, s, such that

$$(*) \qquad n - 2 \geq s > k \geq 2 \text{ and } \frac{P_{k-1}(\lambda)}{P_{k-2}(\lambda)} = \frac{P_{s-1}(\lambda)}{P_{s-2}(\lambda)} + \lambda.$$

From [J, Lemma 4.2.4], the equation is equivalent to the following one:

$$\frac{1}{\sin\frac{2\pi}{n}} = \frac{1}{\tan\frac{k\pi}{n}} - \frac{1}{\tan\frac{s\pi}{n}}.$$

From this we see that if k, s satisfy $(*)$ then $k' = n - s, s' = n - k$ also satisfy $(*)$. Therefore, we are allowed to assume that $k + s \leq n$. Directly from the definition of Jones' polynomials

$$P_{k-1}(\lambda)P_{s-2}(\lambda) - P_{s-1}(\lambda)P_{k-2}(\lambda) = \lambda(P_{k-2}(\lambda)P_{s-3}(\lambda) - P_{s-2}(\lambda)P_{k-3}(\lambda)).$$

Then $(*)$ is equivalent to the following equation

$$(**) \qquad P_{k-2}(\lambda)P_{s-2}(\lambda) - P_{k-2}(\lambda)P_{s-3}(\lambda) + P_{s-2}(\lambda)P_{k-3}(\lambda) = 0.$$

By [J, Lemma 4.2.4], the degree of the above polynomial does not exceed $[\frac{k-1}{2}] + [\frac{s-1}{2}]$. We show that the degree of minimum polynomial of λ is $\frac{n-1}{2}$. We use the terminology and results presented in [ST]. If we denote $\zeta = \exp(\frac{2\pi i}{n})$, then $\lambda = \frac{1}{4} - \frac{1}{4}(\frac{1-\zeta}{1+\zeta})^2$ is an element of the cyclotomic field $Q(\zeta)$. By Theorem 2.5 and Lemma 3.4 in [ST], degree of the minimum polynomial of λ is $\frac{n-1}{m}$, where m is the number of solutions in l, $1 \leq l \leq n-1$ of the following equation:

$$\lambda = \frac{1}{4} - \frac{1}{4}(\frac{1-\zeta^l}{1+\zeta^l})^2.$$

It is easy matter to check that $m = 2$. Now, the inequality $[\frac{k-1}{2}] + [\frac{s-1}{2}] \geq \frac{n-1}{2}$ clearly contradicts our assumption $k + s \leq n$.

Q.E.D.

Corollary 4.5. *Let* $\mathcal{D} = \begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array}$ *be a diagram of type II_1 factors with $S' \cap K = \mathbb{C}$ and $[K : S] < \infty$. If $[Q : S] = 4\cos^2\frac{\pi}{n}$ for a prime number n, then the diagram $\mathcal{D}$ is a commuting square.*

Proof. This is an immediate consequence from Proposition 2.12 and Theorem 4.4.

Q.E.D.

Example. T.Teruya, cf. [Te], considered characteristic intermediate subfactors. If $S \subset Q \subset K$, $S' \cap K = \mathbb{C}$ is a triple of type II_1 factors then, analogically to group theory, he called Q a characteristic intermediate subfactor, if for any automorphism $\sigma \in \mathrm{Aut}(K)$ such that $\sigma(S) = S$ we have also $\sigma(Q) = Q$. Also, he showed that, if K is an extension of S by a finite group G, then this notion coincides with characteristic subgroups. By Theorem 4.4 we can easily obtain examples of characteristic intermediate subfactors. Let n and m, $n \neq m$, $n, m > 4$ be any prime numbers and let K be a hyperfinite type II_1 factor. Let $Q \subset K$, and $S \subset Q$ be II_1 subfactors with $[K : Q] = 4\cos^2\frac{\pi}{n}$ and $[Q : S] = 4\cos^2\frac{\pi}{m}$. Then, no matter how we pick up S, the subfactor Q is characteristic.

Indeed, from Proposition 4.2 and [J], we get $S' \cap K = \mathbb{C}$. If for some $\sigma \in \mathrm{Aut}(K, S)$, $Q \neq \sigma(Q) = R$ then, by Theorem 4.4, the diagram $\begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array}$ is commuting and co-commuting square, which, by [SW, Theorem 7.1], contradicts the assumption $n \neq m$.

Example. Let μ be an outer action of the symmetric group S_3 on a type II_1 factor S. Consider the following diagram:

$$\begin{array}{ccc} S \rtimes_\mu \langle(1,2)\rangle & \subset & S \rtimes_\mu S_3 \\ \cup & & \cup \\ S & \subset & S \rtimes_\mu \langle(2,3)\rangle \end{array}.$$

Clearly, it is not co-commuting square, and so for $n = 6$ the statement of Theorem 4.4 is not satisfied.

The prime numbers n are not the only ones for which the set $\tilde{\Lambda}(\lambda)$, $\lambda^{-1} = 4\cos^2\frac{\pi}{n}$ is empty. We can easily check that for $n = 4$ or $n = 9$ it is empty too. With a little digital help, we conjecture that it is empty for all odd numbers (≥ 5). For n even $\tilde{\Lambda}(\lambda)$ is not empty, because it contains $\{1-\lambda, 2\lambda\}$. We were only able to prove the following

Proposition 4.6. *Let $[K : Q] = \lambda^{-1} = 4\cos^2\frac{\pi}{n}$ and let the diagram* $\mathcal{D} = \begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array}$ *be not co-commuting square. Then n is not a prime number and the following statements hold true*

(1) *There exists a factor*

$$P \in \bigcup_{0 \leq k \leq n-4} \mathcal{L}^{s_k}(R \subset K), \text{ where } \quad s_k = \frac{\lambda^{k+1}}{P_k(\lambda)^2}.$$

(2) *If $2\lambda^{-1} \neq [K : R] = \eta^{-1} < (\lambda^{-1} - 1)^2$, then n must be even, the inclusion "$R \subset K$" is isomorphic to "$Q \subset K$" and*

$$\cos(\text{Op-ang}_K(Q, R)) = \{(\frac{\lambda}{1-\lambda})^2\}.$$

(3) *If $n = 6$, then the principal graph of the inclusion "$Q \subset K$" is A_5.*

Proof. (1) If the diagram $\begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array}$ is not a co-commuting square, then the projection $p = [e_Q * e_R] \in P(M, K)$ so, by Proposition 4.2, we obtain a factor $P \in \mathcal{L}^{s_k}(R \subset K)$, for some integer $k \in [0, n-4]$.

(2) Since $(0, (\lambda^{-1} - 1)^2) \cap \{s_k^{-1} \mid 0 \leq k \leq n-4\} = \{\lambda^{-1}\}$, the only possibility for the factor P in (1) is $[K : P] = \lambda^{-1}$. Since $[P : R] = \frac{\lambda}{\eta} < \lambda(\lambda^{-1} - 1)^2 < \frac{9}{4}$, by [J], $[P : R] = 1$ or $[P : R] = 2$, with the second possibility eliminated by the assumption $\frac{\lambda}{\eta} \neq 2$. From the construction of the factor P (see [P3, Proposition

4.5]), in the case of $k = 0$ or $k = n - 4$, we see that it is a downward basic construction for the pair $K \subset M$. Thus $M = \langle K, R, \tilde{e}_R \rangle$ where $\tilde{e}_R$ is minimal and central projection in $R' \cap M$. By Theorem 2.9 and Lemma 4.3, $p\tilde{e}_R = 0$; therefore, $R' \cap M = \mathbb{C}p \oplus \tilde{e}_R$. Otherwise, there would be another projection, say $f \in R' \cap M$, $f + p + \tilde{e}_R = 1$, $\tau(f) \geq \lambda$, which gives contradiction with $\lambda > 1/4$. By [PP1, Lemma 1.8], there is a unitary $u \in K$ such that $\tilde{e}_R = ue_Qu^*$ and $R = uQu^*$, which gives desired isomorphism.

Since $e_Q + \tilde{e}_R$ is a projection and $E_K(e_Q + \tilde{e}_R) = 2\lambda < 1$, by Theorem 4.1, there exists an integer k, $2 \leq k \leq n - 3$ such that

$$\frac{P_{k-1}(\lambda)}{P_{k-2}(\lambda)} = 2\lambda, \quad \text{hence} \quad P_{k-1}(\lambda) - \lambda P_{k-2}(\lambda) = \lambda P_{k-2}, \text{ and so}$$

$$P_k(\lambda) = -P_k(\lambda) + P_{k-1}(\lambda) \text{ and consequently } \frac{P_k(\lambda)}{P_{k-1}(\lambda)} = \frac{1}{2}.$$

Therefore n can not be odd. Value of the angle is computed in Lemma 3.1.

(3) If the principal graph of "$Q \subset K$" is D_4, then $Q' \cap M = \mathbb{C}e_Q \oplus \mathbb{C}f \oplus \mathbb{C}g$ for some projections f and g such that $\tau(f) = \tau(g) = \lambda (= \frac{1}{3})$. If f_1 is a projection in $S' \cap M$, then, since $E_K(f_1)$ is a scalar, $\tau(f_1) \geq \lambda$. Therefore, if $S' \cap M \neq Q' \cap M$ then $S' \cap M = \mathbb{C}e_Q \oplus M_2(\mathbb{C})$. Since f and g are Jones projections $(E_K(f) = \lambda, E_K(g) = \lambda)$ they are central (see [PP1] or Lemma 2.8). Thus we have $S' \cap M = Q' \cap M$ and, in consequence,

$$R' \cap M = (R' \cap M) \cap (Q' \cap M) = R' \cap Q' \cap M = K' \cap M = \mathbb{C}.$$

Then, by Corollary 2.13, $\mathcal{D}$ must be a co-commuting square.

Q.E.D.

Almost identically as in (3) above we can prove a little more.

Corollary 4.7. *If* $\mathcal{D} = \begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$ *is a diagram of type II_1 factors, $Q \vee R = K$ ($[K : S] < \infty$, $S' \cap K = \mathbb{C}$) and K is a crossed product of Q by an action μ of a finite group G, $K = Q \rtimes_\mu G$, then $\mathcal{D}$ is a co-commuting square.*

Example. Let $S \subset Q \subset K$ be a triple of type II_1 factors with $S' \cap K = \mathbb{C}$. Let Q be a crossed product of S by a finite abelian group and let K be a crossed product of Q by any finite group. From the above corollary and by [Wi, Theorem 6.], we see that if there is another intermediate subfactor R between S and K such that $R \cap Q = S$ and $R \vee Q = K$, then the inclusion "$S \subset K$" has finite depth.

Corollary 4.8. *If* $\mathcal{D} = \begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$ *is like above,* $[K:S] \neq \frac{P_k(\lambda)^2}{\lambda^{k+1}}$ *for* $0 < k < n-4$, $[K:R] \neq [K:Q] < 2+\sqrt{5}$, *$Q$ and R are the only intermediate subfactors between S and K, then $\mathcal{D}$ is co-commuting square.*

Proof. If $p = [e_Q * e_R]$ is like in Theorem 2.9, then $p \in P(M,K)$ and $p - e_Q \in P(M,K)$ and hence $\tau(e_Q)$ is an element of the algebraic difference $\Lambda(\lambda) - \Lambda(\lambda)$. Since $\lambda^{-1} < 2+\sqrt{5}$ and (in case $\lambda^{-1} \geq 4$) $\Lambda_1(\lambda) - \Lambda_1(\lambda) = [-\sqrt{1-4\lambda}, \sqrt{1-4\lambda}]$, we can write a little more:

$$\lambda \in (\Lambda_0(\lambda) - \Lambda_0(\lambda)) \cup (\Lambda_0(\lambda) - \Lambda_1(\lambda)) \cup (\Lambda_1(\lambda) - \Lambda_0(\lambda)).$$

In any case $\tau(p) \in \Lambda_0(\lambda)$ or $\tau(p - e_Q) \in \Lambda_0(\lambda)$. If $p \neq 1$, then by Proposition 4.2, we obtain an intermediate subfactor F between S and K. By our assumption, F can not be S. If $F = R$ then, identically as in Proposition 4.6(2), R is downward basic construction for pair $K \subset M$, which gives contradiction with $[K:R] \neq [K:Q]$. If $F = Q$ then $\tau(p - e_Q) = \lambda$, which implies non-singularity of "$Q \subset K$". This means that either there is an intermediate (of index 2) between Q and K or K is an extension of Q by cyclic group of 3 elements. First possibility is eliminated by our assumptions and in the second case we may apply Corollary 4.7.

Q.E.D.

§5. Some consequences of the commutativity of a diagram.

We assume in this section that diagram $\begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$ is a commuting square, or equivalently, that diagram $\begin{matrix} N & \subset & L \\ \cup & & \cup \\ K & \subset & M \end{matrix}$ is a co-commuting square.

Proposition 5.1. *There exists Jones' projection f for the inclusion $M \subset L$ (i.e. $f \in L$ and $E_M(f) = [L : M]^{-1} = \alpha$) such that*

$$e_R = e_R f = [e_Q * e_R] f.$$

Proof. Let $p = [e_Q * e_R]$. Take any Jones' projection $e_0 \in L$ and the corresponding downward basic construction D so that $L = \langle M, D, e_0 \rangle$. Take such unitary $u \in M$ that $upu^* \in D$. If we set $e_1 = u^* e_0 u$ and $D_1 = u^* D u$, then, by [PP1, Corollary 1.8], $L = \langle M, D_1, e_1 \rangle$ and $p \in D_1$. Since $\tau(e_1 p) = \alpha \tau(p) = \eta = \tau(e_R)$, we can take such unitary $v \in M_p$ that $e_R = v^* e_1 p v$. It is easy to check that $f = (v^* + 1 - p) e_1 (v + 1 - p)$ will do the job.

Q.E.D.

It is natural to ask what the von Neumann algebras $A = M \cap \{e_R\}$ and $B = R \vee \{e_Q\}$ look like. Obviously, $R \subset B \subset A \subset M$ and if, in addition, the diagram $\begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array}$ is also a co-commuting square then $A = B$ is a factor and $\begin{array}{ccc} K & \subset & M \\ \cup & & \cup \\ R & \subset & A \end{array}$ is a commuting and co-commuting square too. (Cf. [K],[P5],[Wi].)

Let f be like in the above proposition and let M_0 be the corresponding downward basic construction, $M_0 = M \cap \{f\}$ and $L = \langle M, M_0, f \rangle$. For a projection e and a von Neumann algebra F we will denote

$$V(e, F) = \bigvee \{ueu^* \mid u \text{ is a unitary in } F\ \}.$$

Lemma 5.2. *Let $p = [e_Q * e_R]$. Then*

$$A_p = B_p = (M_0)_p = A e_Q A = B e_Q B = R e_Q R$$

$$\textit{and} \quad p = V(e_Q, R) = V(e_Q, B) = V(e_Q, A).$$

Proof. First, we show that $A e_Q A = R e_Q R$. Let $x \in A$. By [PP1, Lemma 1.2], $x e_Q = y e_Q$, for $y \in K$. Multiplying the equality $y e_Q e_R = e_R y e_Q$ by e_S we obtain

$ye_S = e_R y e_S = E_R(y)e_S$ which implies $y \in R$. Therefore $Ae_QA = Re_QR = Be_QB$.

Obviously $V(e_Q, R) \in R' \cap M$, hence (by Theorem 2.9) $V(e_Q, R) \geq p$. The equality follows from $ue_Qu^* \leq p$ which hold for all unitaries $u \in R$. By the first part of the proof, it follows that also $p = V(e_Q, B) = V(e_Q, A)$. Now, it is easy to see that $A_p = B_p = Re_QR$.

Proposition 5.1 implies $(M_0)_p \subset A_p$. From Theorem 2.9, A_p is a factor. Now, computation of indices completes the proof.

Q.E.D.

Remark. Let $T = Re_QR$ be the set characterized above. We remark that T is a basic construction for pair $S_p \subset R_p$ and a downward basic construction for pair $M_p \subset L_p$. We can write

$$T = \langle R_p, S_p, e_Q \rangle \quad \text{and} \quad L_p = \langle M_p, T, e_R \rangle.$$

Finally, we obtain the following decomposition for von Neumann algebras A and B.

Corollary 5.3. *Let as before $p = [e_Q * e_R]$. Then*

$$A = T \oplus M_{1-p} \quad \textit{and} \quad B = T \oplus R_{1-p}.$$

Proof. The elements $x = r + \sum_i a_i e_Q b_i$, with $r, a_i, b_i \in R$ form a dense $*$-subalgebra of B. By Theorem 2.9 we see that $(1-p)x(1-p) = (1-p)r$. Since p is central in B,

$$B = B_p \oplus B_{1-p} = T \oplus R_{1-p}.$$

If we apply this equality to the algebra $Q \vee \{e_R\}$, then, using the modular conjugation J_K, we obtain the remaining one.

Q.E.D.

Acknowledgements

I am very indebted to my teacher into the subject - Professor Y.Watatani, for his patient and attentive supervision. I would like to express my gratitude to Professors S.Koshi, A.Kishimoto and M.Hayashi for their help and taking care of me during my stay in Sapporo. My stay here is supported by Monbusho Fellowship. Finally, it is a pleasure to thank the staff at the Department of Mathematics of the Hokkaido University for their kindness and help in many situations.

References

[B] D.Bisch, A note on intermediate subfactors, to appear in *Pac. J. Math.*

[GHJ] F.Goodman, P.de la Harpe and V.F.R.Jones, *Coxeter Graphs and Towers of Algebras*, MSRI Publ.**14**, Springer-Verlag, New York, 1989.

[J1] V.F.R.Jones, Index for subfactors, *Invent. Math.* v.**72** (1983),1-25.

[K] Y.Kawahigashi, Classification of paragroup actions on subfactors, Preprint.

[NT] M.Nakamura and Z.Takeda, On the fundamental theorem of the Galois theory for finite factors, *Proc. Jap. Ac.* **36** (1960) 258-260.

[O] A.Ocneanu, *Quantum symmetry, differential geometry of finite graphs and classification of subfactors,* Tokyo University Seminary Notes **45**, (1991).

[P1] S.Popa, Orthogonal pairs of $*$-subalgebras in finite von Neumann algebras, *J. Operator Theory* **9** (1983), 253-268.

[P2] S.Popa, Maximal injective subalgebras in factors associated with free groups, *Adv. Math.* v.**50** (1983), 27-48.

[P3] S.Popa, Relative dimension, towers of projections and commuting squares of subfactors, *Pac. J. Math.* v.**137**, (1989), 181-207.

[P4] S.Popa, Classification of subfactors: the reduction to commuting square, *Invent. Math.* v.**101** (1990), 19-43.

[P5] S.Popa, Classification of amenable subfactors of type II, to appear in *Acta Math.*

[PP1] M.Pimsner and S.Popa, Entropy and index for subfactors, *Ann. Sci. Ec. Norm. Sup.* **19** (1986), 57-106.

[PP2] ————, Iterating the basic construction, *Trans. Am. Math. Soc.* **310** (1988), 127-133.

[S] T.Sano, Commuting co-commuting squares and finite dimensional Kac algebras, Preprint (1993).

[ST] I.N.Steward and D.O.Tall *Algebraic Number Theory*, Chapman and Hall, New York 1987.

[SW] T.Sano and Y.Watatani, Angles between two subfactors, to appear in *J.*

Operator Theory.

[SZ] S.Stratila and L.Zsido, *Lectures on von Neumann Algebras*, Abacus Press/ Ed. Academiei, Tunbridge Wells/ Bucuresti, 1979.

[T] M.Takesaki, *Theory of Operator Algebra I*, Springer, Berlin, 1979.

[Te] T.Teruya, Characteristic intermediate subfactors, Preprint (1994).

[W] Y.Watatani, *Index for C^*-subalgebras*, Memoirs A.M.S. No. 424 (1990).

[Wi] J.Wierzbicki, An estimate of the depth from an intermediate subfactor, to appear in Publ. RIMS, Kyoto Univ..

[WW] J.Wierzbicki and Y.Watatani, Commuting squares and relative entropy for two subfactors, Preprint (1992).