SCHRÖDINGER EQUATIONS WITH NONLINEARITY OF INTEGRAL TYPE

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SCHRÖDINGER EQUATIONS WITH NONLINEARITY OF INTEGRAL TYPE

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Abstract. We consider the Cauchy problem for the nonlinear Schrödinger equation with interaction described by the integral of the intensity with respect to one direction in two space dimensions. Concerning the problem with finite initial time, we prove the global well-posedness in the largest space $L^2(\mathbb{R}^2)$. Concerning the problem with infinite initial time, we prove the existence of modified wave operators on a dense set of small and sufficiently regular asymptotic states.

§1 **Introduction.** In this paper we study the nonlinear Schrödinger equation

$$i\partial_t u + \frac{1}{2}\Delta u = f(u) \tag{1.1}$$

where u is a complex valued function of time and space variables denoted respectively by $t \in \mathbb{R}$ and $(x, y) \in \mathbb{R}^2$, $\partial_t = \partial/\partial t$, Δ is the Laplacian in space $\mathbb{R}^2$, and $f(u)$ is the nonlinear interaction given by

$$(f(u))(t, x, y) = \lambda(\int_{-\infty}^{x} |u|^2(t, x', y)dx')u(t, x, y) \tag{1.2}$$

with $\lambda \in \mathbb{R}$. The equation (1.1) with integral type nonlinearity (1.2) appears as a model of the propagation of laser beams under the influence of a steady transverse wind along the x-axis [1,3,18] and as a special case of the Davey-Stewartson system where the velocity potential is independent of the y-variable [2,5-7,12,16].

In spite of a large amount of literature on the nonlinear Schrödinger equations, there is only a few result on the equation (1.1) with a special nonlinearity (1.2) [1,3]. In [1] Baillon, Cazenave and Figueira proved the global existence and uniqueness of solutions of the Cauchy problem for (1.1) in the usual Sobolev spaces $H^m(\mathbb{R}^2)$ with $m \geq 1$. In [3] Cazenave proved that for any initial data $\phi \in L^2(\mathbb{R}^2) \cap L^2(\mathbb{R}_y; L^1(\mathbb{R}_x))$ (1.1) has a unique global solution $u \in C(\mathbb{R}; L^2(\mathbb{R}^2)) \cap L^1_{loc}(\mathbb{R}; L^\infty(\mathbb{R}_y; L^2(\mathbb{R}_x)))$. A somewhat remarkable fact is that a smoothing effect takes place only on the y-variable when measured by the space integrability properties. In view of these two results, we have the questions whether the Cauchy problem for (1.1) is solvable with initial data $\phi \in L^2(\mathbb{R}^2)$ and, if that is the case, whether the corresponding solution u exhibits a smoothing effect anisotropic in space directions. Our first goal is to find the function space largest possible where the Cauchy problem for (1.1) is globally well-posed for the data in $L^2(\mathbb{R}^2)$. Our second goal is to find the asymptotic form of global solutions of (1.1). To our knowledge there is no result available so far on that problem except the observation in [12] on a close relation between (1.2) and the cubic nonlinearity $\lambda|u|^2 u$ in one space dimension as regards the range of nonlinear interaction. In fact, it is shown in [12] that both nonlinear terms have the same decay rate of order $O(|t|^{-1})$ in the corresponding L^2 space as $t \to \pm\infty$. This implies in particular that the nonlinearity (1.2) falls beyond the scope of the usual framework of scattering and requires a special treatment within the theory of long range scattering [8,9,11,13,17,20]. The question thus arises what modification ensures the right comparison dynamics for the large time behavior of solutions of (1.1).

In order to state our result precisely, we introduce some notations.

Notation. $L^p_x L^q_y = L^p(\mathbb{R}_x; L^q(\mathbb{R}_y))$, $L^q_y L^p_x = L^q(\mathbb{R}_y; L^p(\mathbb{R}_x))$ with norms

$$\|u\|_{L^p_x L^q_y} = (\int_{-\infty}^{\infty} \|u(x, \cdot)\|^p_{L^q(\mathbb{R}_y)} dx)^{1/p}, \quad \|u\|_{L^q_y L^p_x} = (\int_{-\infty}^{\infty} \|u(\cdot, y)\|^q_{L^p(\mathbb{R}_x)} dy)^{1/q}.$$

$U_0(t) = \exp(i(t/2)\Delta)$ denotes the free propagator. $\mathcal{F}$ denotes the Fourier transform given by

$$(\mathcal{F}u)(\xi,\eta) = \hat{u}(\xi,\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp(-i\xi x - i\eta y)u(x,y)dxdy.$$

The basic existence and uniqueness result is the following.

THEOREM 1. *For any $t_0 \in \mathbb{R}$ and $\phi \in L_x^2L_y^2$ (1.1) with (1.2) has a unique solution $u \in C(\mathbb{R}; L_x^2L_y^2) \cap L_{loc}^8(\mathbb{R}; L_y^4L_x^2)$ with $u(t_0) = \phi$. That solution u satisfies $u \in L_{loc}^q(\mathbb{R}; L_y^rL_x^2)$ for any q and r with $0 \le 2/q = 1/2 - 1/r \le 1/2$. Moreover, $\|u(t)\|_{L_x^2L_y^2} = \|\phi\|_{L_x^2L_y^2}$ for any $t \in \mathbb{R}$.*

Remark 1. Since $(q,r) = (4,\infty)$ is admissible, the solution of (1.1) with L^2 data still exhibits a smoothing effect similar to that of [3] without the assumption $\phi \in L_y^2L_x^1$.

Remark 2. By Theorem 1, the nonlinear propagator $U(t) : u(0) \to u(t)$ is well-defined, where u is the solution of the Cauchy problem for (1.1) with (1.2) with data $u(0)$ in $L_x^2L_y^2$ prescribed at initial time $t = 0$. The nonlinear propagator $U(t)$ forms a group under composition and is an isometry in $L_x^2L_y^2$. The map $\phi \to U(\cdot)\phi$ is well-defined from $L_x^2L_y^2$ to $C(\mathbb{R}; L_x^2L_y^2) \cap L_{loc}^8(\mathbb{R}; L_y^4L_x^2)$.

Moreover, we have :

THEOREM 2. *For any $T > 0$ and $\phi \in L_x^2L_y^2$ there exists $\epsilon > 0$ such that the map $\psi \to U(\cdot)\psi$ is Lipschitz from*

$$B(\phi;\epsilon) \equiv \{\psi \in L_x^2L_y^2; \|\psi - \phi\|_{L_x^2L_y^2} < \epsilon\}$$

to

$$L^\infty(-T,T; L_x^2L_y^2) \cap L^q(-T,T; L_y^rL_x^2)$$

for any q and r with $0 \le 2/q = 1/2 - 1/r \le 1/2$.

For the existence of modified wave operators for (1.1), following [8,9,17], we introduce the following three modified free dynamics

$$v_1^\pm(t) = U_0(t)\exp(-iS_\pm(t,-i\nabla))\phi_\pm, \tag{1.3}$$

$$v_2^\pm(t) = U_0(t)M(-t)\exp(-iS_\pm(t,-i\nabla))\phi_\pm, \tag{1.4}$$

$$v_3^{\pm}(t) = \exp(-iS_{\pm}(t, t^{-1}x, t^{-1}y))U_0(t)\phi_{\pm}, \tag{1.5}$$

where $\phi_{\pm}$ are the data prescribed at $t = +\infty$, $\nabla = (\partial_x, \partial_y)$,

$$S_{\pm}(t, x, y) = \pm\lambda \int_{-\infty}^{x} |\hat{\phi}_{\pm}|^2(x', y)dx' \log|t|, \tag{1.6}$$

$$M(t) = \exp(i(x^2 + y^2)/(2t)), \tag{1.7}$$

and $\exp(-iS_{\pm}(t, -i\nabla))$ are realized by the Fourier multipliers.

THEOREM 3. *Let $\phi_+ \in \mathcal{F}(H^2(\mathbb{R}^2))$ with $\|\hat{\phi}_+\|_{L_y^\infty L_x^2}$ sufficiently small. Then (1.1) with (1.2) has a unique solution $u \in C(\mathbb{R}; L_x^2 L_y^2) \cap L_{loc}^4(\mathbb{R}; L_y^\infty L_x^2)$ such that for any θ with $1/4 < \theta < 1$ and any $j = 1, 2, 3$*

$$\sup_{t \geq 1} t^{\theta} \|u(t) - v_j^+(t)\|_{L_x^2 L_y^2} < \infty. \tag{1.8}$$

That solution u satisfies $u \in L_{loc}^q(\mathbb{R}; L_y^r L_x^2)$ and

$$\sup_{t \geq 1} t^{\theta} \|u - v_j^+\|_{L^q(t,\infty; L_y^r L_x^2)} < \infty \tag{1.9}$$

for any q, r, θ with $0 \leq 2/q = 1/2 - 1/r \leq 1/2$, $1/4 < \theta < 1$ and $j = 1, 2, 3$. A similar result holds for negative times.

Remark 3. By Theorem 3, the modified wave operators $\Omega_{\pm} : \phi_{\pm} \to u(0)$ are well-defined on the set

$$X_\rho = \{\psi \in H^2 \cap \mathcal{F}(H^2); \|\hat{\psi}\|_{L_y^\infty L_x^2} < \rho\}.$$

The free propagator $U_0(t)$ leaves X_ρ invariant and therefore Theorem 3 applies to any state on the flow $\{U_0(t)\psi\}_{-\infty}^{\infty}$ provided that $\psi \in X_\rho$ with ρ small enough.

We now state the intertwining property of the modified wave operators $\Omega_{\pm}$.

THEOREM 4. *Let ρ satisfy the smallness condition of Theorem 3. Then for any $\psi \in X_\rho$ and any $t \in \mathbb{R}$*

$$U(t)\Omega_{\pm}\psi = \Omega_{\pm}U_0(t)\psi. \tag{1.10}$$

This paper is organized as follows. In section 2 we prepare some basic estimates of the free propagator $U_0(t)$ and the nonlinear term $f(u)$ in the anisotropic space $L^r_y L^2_x$. In section 3 we prove Theorems 1 and 2. In section 4 we prove Theorems 3 and 4.

We conclude this section by giving some additional notations freely used in this paper. For any r with $1 \le r \le \infty$ we denote by r' the exponent dual to r defined by $1/r + 1/r' = 1$. For any $s \in \mathbb{R}$ we denote by $H^s = H^s(\mathbb{R}^2)$ the usual Sobolev space of order s. For any interval $I \subset \mathbb{R}$, possibly unbounded, we denote by $\bar{I}$ the closure of I in $\mathbb{R} \cup \{-\infty, +\infty\}$ equipped with the natural topology. For any interval $I \subset \mathbb{R}$ and any Banach space we denote by $C(I; X)$ the space of strongly continuous functions u from I to X and by $L^q(I; X)$ (resp. $L^q_{loc}(I; X)$) the space of measurable functions u from I to X such that $\|u(\cdot)\|_X \in L^q(I)$ (resp. $L^q_{loc}(I)$). Different positive constants might be denoted by the same letter C.

§2. Preliminary Estimates.

In this section we collect some basic estimates of the free propagator $U_0(t)$ and the nonlinear term $f(u)$ in the anisotropic space.

LEMMA 2.1. *U_0 satisfies the following estimates : (1) Let r and δ satisfy $2 \le r \le \infty, \delta = 1/2 - 1/r$. Then for $t \neq 0$*

$$\|U_0(t)\phi\|_{L^r_y L^2_x} \le (2\pi|t|)^{-\delta} \|\phi\|_{L^{r'}_y L^2_x}. \tag{2.1}$$

(2) For any (q, r) with $0 \le 2/q = 1/2 - 1/r \le 1/2$,

$$\|U_0(\cdot)\phi\|_{L^q(\mathbb{R}; L^r_y L^2_x)} \le C\|\phi\|_{L^2_y L^2_x}. \tag{2.2}$$

(3) For any (q_1, r_1) and (q_2, r_2) with $0 \le 2/q_j = 1/2 - 1/r_j \le 1/2$, $j = 1, 2$, for any interval $I \subset \mathbb{R}$ which may be unbounded, and for any $s \in \bar{I}$ the operator G_s defined by

$$(G_s u)(t) = \int_s^t U_0(t - \tau) u(\tau) d\tau \tag{2.3}$$

satisfies the estimate

$$\|G_s u\|_{L^{q_1}(I; L^{r_1}_y L^2_x)} \le C\|u\|_{L^{q'_2}(I; L^{r'_2}_y L^2_x)} \tag{2.4}$$

where C is independent of I and s.

Proof. By the decomposition

$$U_0(t) = [1 \otimes \exp(i(t/2)\partial_y^2)][\exp(i(t/2)\partial_x^2) \otimes 1],$$

unitarity in L_x^2 of $\exp(i(t/2)\partial_x^2)$, the representation of $\exp(i(t/2)\partial_y^2)$ by the integral kernel $(2\pi it)^{-1/2}\exp(i(y-y')^2/(2t))$, Minkowski's integral inequality, and the Riesz-Thorin interpolation theorem, we obtain

$$
\begin{aligned}
&\|U_0(t)\phi\|_{L_y^r L_x^2} \\
&= \Big(\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty}|(2\pi it)^{-1/2}\int_{-\infty}^{\infty}\exp(i(y-y')^2/(2t))\phi(x,y')dy'|^2dx\Big)^{r/2}dy\Big)^{1/r} \\
&\le \Big(\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty}|(2\pi it)^{-1/2}\int_{-\infty}^{\infty}\exp(i(y-y')^2/(2t))\phi(x,y')dy'|^r dy\Big)^{2/r}dx\Big)^{1/2} \\
&\le (2\pi|t|)^{-\delta}\Big(\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty}|\phi(x,y)|^{r'}dy\Big)^{2/r'}dx\Big)^{1/2} \\
&\le (2\pi|t|)^{-\delta}\Big(\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty}|\phi(x,y)|^2dx\Big)^{r'/2}dy\Big)^{2/r'} = (2\pi|t|)^{-\delta}\|\phi\|_{L_y^{r'}L_x^2}.
\end{aligned}
$$

This proves (2.1). Given (2.1), the rest of the lemma is proved in the standard way as in [4,10,14,15,21] if we replace complex valued functions by L_x^2 valued functions.

QED

LEMMA 2.2. *Let $r_j, 0 \le j \le 3$, satisfy $1 \le r_j \le \infty$ and $1/r_0 = 1/r_1 + 1/r_2 + 1/r_3$. Then*

$$
\|\psi_1\int_{-\infty}^{x}(\psi_2\psi_3)(x',y)dx'\|_{L_y^{r_0}L_x^2} \le \prod_{j=1}^{3}\|\psi_j\|_{L_y^{r_j}L_x^2}. \tag{2.5}
$$

Proof. The result follows from the Schwarz and Hölder inequalities in x and y variables, respectively, since

$$
\|\psi_1\int_{-\infty}^{x}(\psi_2\psi_3)(x',y)dx'\|_{L_x^2} \le \|\psi_1\int_{-\infty}^{\infty}|\psi_2||\psi_3|dx'\|_{L_x^2} \le \prod_{j=1}^{3}\|\psi_j\|_{L_x^2}.
$$

QED

LEMMA 2.3. *There exists a constant C such that*

$$
\|\psi_1\int_{-\infty}^{x}(\psi_2\psi_3)(x',y)dx'\|_{L_x^2L_y^2} \tag{2.6}
$$

$$\leq \begin{cases} C\|\psi_1\|_{L_x^2L_y^2} \prod_{j=2}^3 \|\psi_j\|_{L_x^2L_y^2}^{1/2} \|\partial_y\psi_j\|_{L_x^2L_y^2}^{1/2}, \\ C\|\psi_3\|_{L_x^2L_y^2} \prod_{j=1}^2 \|\psi_j\|_{L_x^2L_y^2}^{1/2} \|\partial_y\psi_j\|_{L_x^2L_y^2}^{1/2}. \end{cases}$$

Proof. By Lemma 2.2, the left hand side of the last inequality is bounded by

$$\|\psi_1\|_{L_x^2L_y^2}\|\psi_2\|_{L_y^\infty L_x^2}\|\psi_3\|_{L_y^\infty L_x^2}$$

or

$$\|\psi_3\|_{L_x^2L_y^2}\|\psi_1\|_{L_y^\infty L_x^2}\|\psi_2\|_{L_y^\infty L_x^2}.$$

The result therefore follows from the inequality of Gagliardo and Nirenberg, since

$$\|\psi\|_{L_y^\infty L_x^2} \leq \|\psi\|_{L_x^2 L_y^\infty}$$

$$\leq C\|\|\psi\|_{L_y^2}^{1/2}\|\partial_y\psi\|_{L_y^2}^{1/2}\|_{L_x^2} \leq C\|\psi\|_{L_x^2L_y^2}^{1/2}\|\partial_y\psi\|_{L_x^2L_y^2}^{1/2}.$$

QED

§3. Proofs of Theorems 1 and 2. For $T > 0$ let $I = [t_0 - T, t_0 + T]$ and let $X(I)$ be the Banach space defined by $X(I) = C(I; L_x^2L_y^2) \cap L^8(I; L_y^4L_x^2)$ with norm

$$|||u||| = \|u\|_{L^\infty(I;L_x^2L_y^2)} + \|u\|_{L^8(I;L_y^4L_x^2)}.$$

For $\phi \in L_x^2L_y^2$ and $u \in X(I)$, we define

$$(\Phi(u))(t) = U(t-t_0)\phi - i(G_{t_0}(f(u)))(t). \tag{3.1}$$

By Lemmas 2.1 and 2.2, we have for $u, v \in X(I)$

$$|||\Phi(u)||| \leq C\|\phi\|_{L_x^2L_y^2} + C\|f(u)\|_{L^{8/7}(I;L_y^{4/3}L_x^2)} \tag{3.2}$$

$$\leq C\|\phi\|_{L_x^2L_y^2} + CT^{1/2}|||u|||^3,$$

$$|||\Phi(u) - \Phi(v)||| \leq CT^{1/2}(|||u|||^2 + |||v|||^2)|||u-v|||. \tag{3.3}$$

This shows that for any $\phi \in L_x^2L_y^2$ there exists $T > 0$ such that (3.1) has a unique fixed point $u \in X(I)$. The rest of the theorems are proved in the standard way as [4,15,19] if we replace complex valued functions by L_x^2 valued functions.

QED

§4. Proofs of Theorems 3 and 4. For definiteness we consider the case $t > 0$ only and the superscript and subscript $+$ and the variable t will often be omitted when this causes no confusion. Let $\phi \in \mathcal{F}(H^2)$ and define

$$S(t,x,y) = \lambda \int_{-\infty}^{x} |\hat{\phi}|^2(x',y)dx' \log t \tag{4.1}$$

for $t \geq 1$. By a direct calculation, the Sobolev imbedding theorem, and Lemma 2.3,

$$\|\Delta(e^{-iS}\hat{\phi})\|_{L_x^2 L_y^2} \leq C(1+(\log t)^2)(\|\hat{\phi}\|_{H^2} + \|\hat{\phi}\|_{H^2}^5). \tag{4.2}$$

We define $v(t) = v_2(t)$ by

$$v(t) = U_0(t)M(-t)\exp(-iS(t,-i\nabla))\phi. \tag{4.3}$$

By the factorization

$$U_0(t) = M(t)D(t)\mathcal{F}M(t)$$

with

$$(D(t)\psi)(x) = (it)^{-1}\psi(t^{-1}x),$$

$v(t)$ is rewritten as

$$v(t) = M(t)D(t)\exp(-iS(t))\hat{\phi}, \tag{4.4}$$

where the variables (x,y) in $S(t,x,y)$ are omitted. By the representation (4.4), we have

$$f(v) = \lambda(\int_{-\infty}^{x} |D(t)\hat{\phi}|^2(x',y)dx')v$$

$$= \lambda(iD(t)(\int_{-\infty}^{x} |\hat{\phi}|^2(x',y)dx'))M(t)D(t)\exp(-iS(t))\hat{\phi}$$

$$= M(t)D(t)\exp(-iS(t))(\lambda t^{-1}f(\hat{\phi})).$$

In the same way as in [8,11,17], we obtain

$$(i\partial_t + \frac{1}{2}\Delta)v = M(t)D(t)((\partial_t S) + \frac{1}{2t^2}\Delta)\exp(-iS(t))\hat{\phi}.$$

Since $(\partial_t S)\hat{\phi} = \lambda t^{-1} f(\hat{\phi})$, we have

$$R(v) \equiv (i\partial_t + \frac{1}{2}\Delta)v - f(v) = MD(\frac{1}{2t^2}\Delta)e^{-iS}\hat{\phi}$$

and therefore, by (4.2)

$$\|R(v)\|_{L^2_x L^2_y} \le Ct^{-2}(1+(\log t)^2)(\|\hat{\phi}\|_{H^2} + \|\hat{\phi}\|^5_{H^2}). \tag{4.5}$$

We now solve the integral equation

$$\begin{aligned} u(t) &= v(t) + i\int_t^\infty U_0(t-\tau)(f(u) - (i\partial_\tau + \frac{1}{2}\Delta)v)(\tau)d\tau \\ &= v(t) + i\int_t^\infty U_0(t-\tau)(f(u) - f(v) - R(v))(\tau)d\tau. \end{aligned} \tag{4.6}$$

For that purpose we regard (4.6) as an equation of $w \equiv u - v$ and consider the map

$$(\Phi(w))(t) = i\int_t^\infty U_0(t-\tau)(f(w+v) - f(v) - R(v))(\tau)d\tau \tag{4.7}$$

on the function space

$$Y_\theta(T) = \{w \in C([T,\infty); L^2_x L^2_y) \cap L^4(T,\infty; L^\infty_y L^2_x);$$

$$|||w||| = \sup_{t \ge T} t^\theta(\|w(t)\|_{L^2_x L^2_y} + \|w\|_{L^4(t,\infty;L^\infty_y L^2_x)}) < \infty\}$$

where $1/4 < \theta < 1$ and $T > 0$ is sufficiently large. We decompose $f(w+v) - f(v)$ as

$$f(w+v) - f(v) = f(w) + Q(w) + L(w), \tag{4.8}$$

where

$$Q(w) = \lambda v\int_{-\infty}^x |w|^2 dx' + 2\lambda w\int_{-\infty}^x \mathrm{Re}(\bar{v}w)dx',$$

$$L(w) = \lambda w\int_{-\infty}^x |v|^2 dx' + 2\lambda v\int_{-\infty}^x \mathrm{Re}(\bar{v}w)dx'.$$

For the cubic part of (4.8) we use Lemma 2.2 to obtain

$$\|f(w)\|_{L^1_y L^2_x} \le C\|w\|^2_{L^2_y L^2_x}\|w\|_{L^\infty_y L^2_x}.$$

By the Hölder inequality in time, we have

$$\begin{aligned} \|f(w)\|_{L^{4/3}(t,\infty;L^1_y L^2_x)} &\le C|||w|||^2(\int_t^\infty (\tau^{-2\theta}\|w\|_{L^\infty_y L^2_x})^{4/3}d\tau)^{3/4} \\ &\le Ct^{1/4-2\theta}|||w|||^3, \end{aligned}$$

so that by Lemma 2.1

$$|||G_\infty f(w)||| \leq CT^{1/4-\theta}|||w|||^3. \tag{4.9}$$

Similarly, for $w_1, w_2 \in Y_\theta(T)$

$$|||G_\infty(f(w_1) - f(w_2))||| \leq CT^{1/4-\theta}(|||w_1|||^2 + |||w_2|||^2)|||w_1 - w_2|||. \tag{4.10}$$

For the quadratic part of (4.8), we use Lemma 2.2 to obtain

$$\|Q(w)\|_{L^2_x L^2_y} \leq C\|v\|_{L^\infty_y L^2_x}\|w\|_{L^2_y L^2_x}\|w\|_{L^\infty_y L^2_x}.$$

We note here that

$$\|v\|_{L^\infty_y L^2_x} = t^{-1/2}\|\hat{\phi}\|_{L^\infty_y L^2_x}. \tag{4.11}$$

By (4.11) and the Hölder inequality in time, we have

$$\begin{aligned}\|Q(w)\|_{L^1(t,\infty;L^2_x L^2_y)} &\leq C\|\hat{\phi}\|_{L^\infty_y L^2_x}|||w|||\int_t^\infty \tau^{-1/2-\theta}\|w\|_{L^\infty_y L^2_x}\,d\tau \\ &\leq Ct^{1/4-2\theta}\|\hat{\phi}\|_{L^\infty_y L^2_x}|||w|||^2,\end{aligned}$$

so that by Lemma 2.1

$$|||G_\infty Q(w)||| \leq CT^{1/4-\theta}\|\hat{\phi}\|_{L^\infty_y L^2_x}|||w|||^2. \tag{4.12}$$

Similarly, for $w_1, w_2 \in Y_\theta(T)$

$$|||G_\infty(Q(w_1) - Q(w_2))||| \leq CT^{1/4-\theta}\|\hat{\phi}\|_{L^\infty_y L^2_x}(|||w_1||| + |||w_2|||)|||w_1 - w_2|||. \tag{4.13}$$

For the linear part of (4.8), we use Lemma 2.2 and (4.11) to obtain

$$\begin{aligned}\|L(w)\|_{L^1(t,\infty;L^2_x L^2_y)} &\leq C\|\hat{\phi}\|_{L^\infty_y L^2_x}\int_t^\infty \tau^{-1}\|w\|_{L^2_x L^2_y}\,d\tau \\ &\leq Ct^{-\theta}\|\hat{\phi}\|_{L^\infty_y L^2_x}|||w|||,\end{aligned}$$

so that by Lemma 2.1

$$|||G_\infty L(w)||| \leq C\|\hat{\phi}\|_{L^\infty_y L^2_x}|||w|||. \tag{4.14}$$

Similarly, for $w_1, w_2 \in Y_\theta(T)$

$$|||G_\infty(L(w_1) - L(w_2))||| \leq C\|\hat{\phi}\|_{L^\infty_y L^2_x}|||w_1 - w_2|||. \tag{4.15}$$

Finally, for the remainder $R(v)$ we use Lemma 2.1 and (4.5) to obtain

$$|||G_\infty R(v)||| \le CT^{-1}(1+(\log T)^2)(\|\hat{\phi}\|_{H^2} + \|\hat{\phi}\|_{H^2}^5). \tag{4.16}$$

Combining (4.7)-(4.9), (4.12), (4.14), (4.16), we have

$$\begin{aligned} |||\Phi(w)||| \le{} & C\|\hat{\phi}\|_{L_y^\infty L_x^2}|||w||| + CT^{1/4-\theta}(|||w||| + \|\hat{\phi}\|_{L_y^\infty L_x^2})|||w|||^2 \\ & + CT^{-1}(1+(\log T)^2)(\|\hat{\phi}\|_{H^2} + \|\hat{\phi}\|_{H^2}^5). \end{aligned} \tag{4.17}$$

Similarly, combining (4.10), (4.13), (4.15), we have for $w_1, w_2 \in Y_\theta(T)$

$$\begin{aligned} |||\Phi(w_1) - \Phi(w_2)||| \le{} & C\|\hat{\phi}\|_{L_y^\infty L_x^2}|||w_1 - w_2||| \\ & + CT^{1/4-\theta}(|||w_1|||^2 + |||w_2|||^2 + \|\hat{\phi}\|_{L_y^\infty L_x^2}^2)|||w_1 - w_2|||. \end{aligned} \tag{4.18}$$

It follows from (4.17) and (4.18) that (4.7) has a unique fixed point $w \in Y_\theta(T)$ provided that $\|\hat{\phi}\|_{L_y^\infty L_x^2}$ is sufficiently small and T is sufficiently large. By Theorem 1, the solution $u = v + w$ just defined on $[T,\infty)$ extends to the whole line $\mathbb{R}$. We now prove the uniqueness. Since we already know the uniqueness in the space $Y_\theta(T)$ with $1/4 < \theta < 1$ and T large enough, it suffices to prove that if u is a solution of (4.6) with

$$u \in C([T,\infty); L_x^2 L_y^2) \cap L_{loc}^4(T,\infty; L_y^\infty L_x^2)$$

and

$$\sup_{t\ge T} t^\theta \|u(t) - v(t)\|_{L_x^2 L_y^2} \equiv K < \infty,$$

then u satisfies

$$\sup_{t \ge T} t^\theta \|u - v\|_{L^4(t,\infty; L_y^\infty L_x^2)} < \infty.$$

Let $T \le t \le T'$. Then in the same way as above, we have

$$\begin{aligned} \|u - v\|_{L^4(t,T'; L_y^\infty L_x^2)} \le{} & CK^2 t^{1/4-\theta}\|u - v\|_{L^4(t,T';L_y^\infty L_x^2)} \\ & + CKt^{1/4-\theta}\|\hat{\phi}\|_{L_y^\infty L_x^2}\|u - v\|_{L^4(t,T';L_y^\infty L_x^2)} + CKt^{-\theta}\|\hat{\phi}\|_{L_y^\infty L_x^2} \\ & + Ct^{-1}(1+(\log t)^2)(\|\hat{\phi}\|_{H^2} + \|\hat{\phi}\|_{H^2}^5). \end{aligned} \tag{4.19}$$

This implies

$$\begin{aligned} \|u - v\|_{L^4(t,T';L_y^\infty L_x^2)} \le{} & CKt^{-\theta}\|\hat{\phi}\|_{L_y^\infty L_x^2} \\ & + Ct^{-1}(1+(\log t)^2)(\|\hat{\phi}\|_{H^2} + \|\hat{\phi}\|_{H^2}^5) \end{aligned} \tag{4.20}$$

provided that

$$CK(\|\hat{\phi}\|_{L_y^\infty L_x^2} + K)T^{1/4-\theta} \le \frac{1}{2}.$$

Since the right hand side on (4.20) is independent of T', we have $u - v \in L^4(T,\infty; L_y^\infty L_x^2)$ and we may replace the norm in $L^4(t,T'; L_y^\infty L_x^2)$ of the left hand side on (4.20) by that in $L^4(t,\infty; L_y^\infty L_x^2)$. This proves the required uniqueness.

The rest of the proof proceeds in the same way as in [8,17].

QED

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