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ABSTRACT

We give a sufficient condition for the uniqueness of solutions to Cauchy problem for a class of integro-partial differential operators related to the stochastic analysis, by way of the large deviations technique.

1. Introduction.

Fix $T > 0$ and $u_0(\cdot) \in C(R^n, R)$, and consider the following PDE (see [13]); for $t \in (0,T)$ and $x \in R^n$,

$$\begin{aligned} \partial u(t,x)/\partial t = & \sum_{i,j=1}^{n} a^{ij}(t,x)\partial^2 u(t,x)/\partial x_i \partial x_j \\ & + \sum_{i=1}^{n} b^i(t,x)\partial u(t,x)/\partial x_i + f(t,x,u(t,x)), \\ u(0,x) = & u_0(x). \end{aligned} \quad (1.1).$$

As a uniqueness theorem for (1.1), the following can be proved from [7], p. 140, Corollary 4.2.

Theorem 1.1. *Suppose that $(a^{ij}(t,x))_{i,j=1}^n$ is bounded and nonnegative definite, and that $f(t,x,u)$ is differentiable in u, and that the following holds;*

$$\sup\{(\partial f(t,x,u)/\partial u)/(1+|x|^2); t \in [0,T], x \in R^n, u \in R\} < \infty, \quad (1.2).$$

$$\sup\{\limsup_{|x|\to\infty} |b^i(t,x)|/(1+|x|); 0 \le t \le T, i = 1, \cdots, n\} < \infty. \quad (1.3).$$

Then any solutions $u_1(t,x)$ and $u_2(t,x) \in C^{1,2}((0,T) \times R^n, R) \cap C([0,T] \times R^n, R)$ to (1.1) coincide with each other if the following holds;

$$\limsup_{x\to\infty} |x|^{-2} \log\{\sup_{0\le t\le T} |u_1(t,x) - u_2(t,x)| + 1\} < +\infty. \quad (1.4).$$

As an example of nonunique solutions to (1.1), the following is known (see also [12]).

Theorem 1.2. *([1], Theorem). Suppose that* $n = 1$ *in (1.1) and that* $a^{11}(t,x) \equiv 1, b^1(t,x) \equiv f(t,x,u) \equiv 0$. *Then there exists a solution* $u(t,x) \in C^{1,2}((0,\infty) \times R, R) \cap C([0,\infty) \times R, R)$ *of (1.1) for which* $u(0,x) \equiv 0$, *and which is not identically zero for* $t > 0$, *and for which for any* $\varepsilon > 0$, *there exists a constant* $C = C(\varepsilon) > 0$ *such that*

$$|u(t,x)| \le C(\varepsilon)\exp(\varepsilon/t) \quad \text{on } (0,\infty) \times R. \tag{1.5}$$

In this paper, we generalize Theorem 1.1 by considering a class of integro-differential equations. In a special case our equation becomes (1.1), and our result gives a probabilistic proof to Theorem 1.1.

In section 2 we state our results. In section 3 we prove them.

2. Main results.

Let us introduce our integro-differential equation.

Fix $T > 0$ and $u_0(\cdot) \in C(R^n, R)$, and consider the following integro-differential equation; for $t \in (0,T)$ and $x \in R^n$,

$$\begin{aligned}\partial u(t,x)/\partial t = &\sum_{i,j=1}^{n} a^{ij}(t,x)\partial^2 u(t,x)/\partial x_i \partial x_j \\ &+ \sum_{i=1}^{n} b^i(t,x)\partial u(t,x)/\partial x_i + \int_{y \ne x, y \in R^n} [u(t,y) - u(t,x) \\ &- \sum_{i=1}^{n}(y_i - x_i)\partial u(t,x)/\partial x_i]\nu_{t,x}(dy) + f(t,x,u(t,x)),\\ u(0,x) = u_0(x). &\end{aligned} \tag{2.1}$$

In this section we assume the following.

(A.1). $a(t,x) = (a^{ij}(t,x))_{i,j=1}^n$ is a symmetric, nonnegative definite $n \times n$-matrix. $\nu_{t,x}(dy)$ is a nonnegative Borel measure on $R^n \backslash \{x\}$. $b(t,x) = (b^i(t,x))_{i=1}^n$ is globally Lipschitz continuous in x, uniformly in t. For any $s \ge 0$, there exist strong Markov processes $(X(t), P_{s,x})_{s \le t, x \in R^n}$ whose infinitesimal generator L_t is given by the following; for any infinitely differentiable function g; $R^n \mapsto R$ with a compact support, $t \ge s$, and $x \in R^n$,

$$\begin{aligned}L_t g(x) = &\sum_{i=1}^{n} b^i(t,x)\partial g(x)/\partial x_i + \sum_{i,j=1}^{n} a^{ij}(t,x)\partial^2 g(x)/\partial x_i \partial x_j \\ &+ \int_{y \ne x, y \in R^n} [g(y) - g(x) - \sum_{i=1}^{n}(y_i - x_i)\partial g(x)/\partial x_i]\nu_{t,x}(dy)\end{aligned} \tag{2.2}$$

(see [9] and [14]).

(A.2). For any $r > 0$

$$H(r) \equiv \sup\{< a(t,x)z, z > + \int_{y \neq x, y \in R^n} [\exp(< z, y-x >) - 1 \quad (2.3).$$
$$- < z, y-x >]\nu_{t,x}(dy); t \in [0,T], x \in R^n, |z| \leq r\} < \infty,$$

where $< \cdot, \cdot >$ denotes the inner product in R^n.
(A.3.1). $f(t,x,u)$ has a continuous $\partial f(t,x,u)/\partial u$, and

$$C(f) \equiv \sup\{(\partial f(t,x,u)/\partial u)/(1+|x|); t \in [0,T], x \in R^n, u \in R\} < \infty. \quad (2.4).$$

(A.3.2). $f(t,x,u)$ has a continuous $\partial f(t,x,u)/\partial u$, and

$$\tilde{C}(f) \equiv \sup\{(\partial f(t,x,u)/\partial u)/(1+|x|^2); t \in [0,T], x \in R^n, u \in R\} < \infty. \quad (2.5).$$

Remark 2.1. From (2.3), for $t \geq s$, and $x \in R^n$,

$$X(t) = x + \int_s^t b(\alpha, X(\alpha))ds + M^{[c]}(t) + M^{[d]}(t) \quad P_{s,x} - a.s., \qquad (2.6).$$

where $M^{[c]}(t)$ and $M^{[d]}(t)$ denote square integrable, continuous and purely discontinuous martingales, respectively (see [10]).

Before we state our main result, let us give some notation. Put

$$C_1 = 2\sup\{\limsup_{h \to o} |b(t,x+h) - b(t,x)|/|h|; t \geq 0, x \in R^n\}, (2.7).$$
$$C_2 = n\exp(C_1 T/2), \qquad (2.8).$$
$$C(f;r) \equiv \sup\{\partial f(t,x,u)/\partial u; t \in [0,T], |x| \leq r, u \in R\}, \qquad (2.9).$$
$$L(r) \equiv \sup\{zr - H(z); z \geq 0\}, \qquad (2.10).$$

for $r > 0$.

The following is our main result.

Theorem 2.1. *Suppose that (A.1)-(A.2) hold, and that $f(t,x,u)$ has a continuous $\partial f(t,x,u)/\partial u$. Then any solutions $u_1(t\ ,x)$ and $u_2(t,x) \in C^{1,2}((0,T) \times R^n, R) \cap C([0,T] \times R^n, R)$ to (2.1) coincide with each other if*

$$\limsup_{x \to \infty} (TL(|x|/(C_2 T)))^{-1} \qquad (2.11).$$
$$\times \{\log\{\sup_{0 \leq t \leq T} |u_1(t,x) - u_2(t,x)| + 1\} C(f;|x|)T\} < 1.$$

Remark 2.2. For $r > 0$ and $z > 0$,

$$L(r)/r \geq z - H(z)/r. \tag{2.12}$$

Let $r \to \infty$ and then $z \to \infty$, we get the following;

$$\lim_{r \to \infty} L(r)/r = \infty. \tag{2.13}$$

As an easy consequence of Theorem 2.1 and Remark 2.2, we get the following corollary whoes proof is omitted.

Corollary 2.2. *Suppose that (A.1)-(A.3.1) hold. Then any solutions* $u_1(t, x)$ *and* $u_2(t,x) \in C^{1,2}((0,T) \times R^n, R) \cap C([0,T] \times R^n, R)$ *to (2.1) coincide with each other if*

$$\limsup_{x \to \infty} |x|^{-1} \log\{ \sup_{0 \leq t \leq T} |u_1(t,x) - u_2(t,x)| + 1\} < +\infty. \tag{2.14}$$

In the same way as in the proof of Theorem 2.1, we can show that the following is true.

Theorem 2.3. *Suppose that (A.1)-(A.2) and (A.3.2) hold, and that* $\nu_{t,x}(dy) \equiv 0$. *Then any solutions* $u_1(t,x)$ *and* $u_2(t,x) \in C^{1,2}((0,T) \times R^n, R) \cap C([0,T] \times R^n, R)$ *to (2.1) coincide with each other if*

$$\limsup_{x \to \infty} |x|^{-2} \log\{ \sup_{0 \leq t \leq T} |u_1(t,x) - u_2(t,x)| + 1\} < +\infty. \tag{2.15}$$

Remark 2.3. The condition (2.5) and (2.15) are equivalent to (1.2) and (1.4), respectively.

We close this section by the following example.

Example 2.1. Suppose that $n = 1$, that $a(t,x) \equiv 0$, and that $\nu_{t,x}(dy) \equiv \delta_{1+x}(dy)$. Then for $z > 0$,

$$H(z) = \exp(z) - 1 - z, \tag{2.16}$$

and for $r > 0$,

$$L(r) = (1 + r)\log(1 + r) - r. \tag{2.17}$$

$$\lim_{r \to \infty} C_2 T L(r/(C_2 T))/L(r) = 1. \tag{2.18}$$

3. Proof of results.

In this section we prove Theorems in section 2.

Before we prove them, let us give the technical result whose proof will be given later.

For $r > 0$, $t \geq s \geq 0$, and $x \in R^n$, put

$$\tau_r \equiv \inf\{t > 0; |X(t)| > r\}, \tag{3.1}$$

$$C_3 = (16H(1)n^2)^{-1}. \tag{3.2}$$

Then the following result can be proved by way of the idea of Freidlin-Wentzell theory (see [3], [5] and also [2] for other approaches).

Lemma 3.1. *(I). Suppose that (A.1)-(A.2) hold. Then for any $x \in R^n$,*

$$\lim_{r\to\infty} (TL(r/(C_2T)))^{-1} \log P_{0,x}(\tau_r \leq T) \leq -1. \tag{3.3}$$

(II). Suppose that (A.1)-(A.2) hold, and that $\nu_{t,x}(dy) \equiv 0$. Then for any $t > s \geq 0$ and $x \in R^n$,

$$\limsup_{r\to\infty} r^{-2} \log P_{s,x}(\tau_r \leq t) \leq -C_3 \exp(-C_1(t-s))/(t-s) \tag{3.4}$$

(see (2.7)-(2.8) for notation).

Let us first prove Theorem 2.1 from Lemma 3.1, (I).

Proof of Theorem 2.1. For $0 \leq t \leq T$, $x \in R^n$, and $u, v \in R$, put

$$F(t,x:u,v) \equiv \begin{cases} \{f(t,x,u) - f(t,x,v)\}/(u-v) & \text{if } u \neq v, \\ \partial f(t,x,u)/\partial u & \text{if } u = v \end{cases} \tag{3.5}$$

which is continuous from the assumption, and put

$$w(t,x) \equiv u_1(t,x) - u_2(t,x). \tag{3.6}$$

Then $w(t,x)$ satisfies the following integro-differential equation; for $t \in (0,T)$ and $x \in R^n$,

$$\begin{aligned} \partial w(t,x)/\partial t = &\sum_{i,j=1}^{n} a^{ij}(t,x)\partial^2 w(t,x)/\partial x_i \partial x_j + \sum_{i=1}^{n} b^i(t,x)\partial w(t,x)/\partial x_i \\ &+ \int_{y\neq x, y\in R^n} [w(t,y) - w(t,x) \\ &\qquad - \sum_{i=1}^{n}(y_i - x_i)\partial w(t,x)/\partial x_i]\nu_{t,x}(dy) \\ &+ F(t,x;u_1(t,x),u_2(t,x))w(t,x), \\ w(0,x) = 0. & \end{aligned} \tag{3.7}$$

For $t \in [0,T]$ and $x \in R^n$, applying the Ito formula for $w(t-s, X(s)) \times \exp(\int_0^s F(t-\alpha, X(\alpha); u_1(\alpha, X(\alpha)), u_2(\alpha, X(\alpha)))d\alpha)(0 \leq s \leq t)$, we get the following;

$$\begin{aligned} w(t,x) = {} & E_{0,x}[w(0,X(t))\exp(\int_0^t F(t-\alpha, X(\alpha) \\ & ; u_1(\alpha, X(\alpha)), u_2(\alpha, X(\alpha)))d\alpha); t < \tau_r] \\ & + E_{0,x}[w(t-\tau_r, X(\tau_r))\exp(\int_0^{\tau_r} F(t-\alpha, X(\alpha) \\ & ; u_1(\alpha, X(\alpha)), u_2(\alpha, X(\alpha)))d\alpha); t \geq \tau_r] \\ = {} & E_{0,x}[w(t-\tau_r, X(\tau_r))\exp(\int_0^{\tau_r} F(t-\alpha, X(\alpha) \\ & ; u_1(\alpha, X(\alpha)), u_2(\alpha, X(\alpha)))d\alpha); t \geq \tau_r] \end{aligned} \tag{3.8}$$

from (3.7).

Form Lemma 3.1, (I) and (2.11), we get, for $x \in R^n$,

$$\begin{aligned} & \sup_{0 \leq t \leq T} |w(t,x)| \\ \leq {} & \sup_{0 \leq t \leq T, |y|=r} |w(t,y)| \exp(TC(f;r))P_{0,x}(T \geq \tau_r) \\ \to {} & 0 \quad (\text{as } r \to \infty\ ). \end{aligned} \tag{3.9}$$

Q.E.D.

Next we prove Theorem 2.3 from Lemma 3.1, (II).

Proof of Theorem 2.3. Denote by C_4 the quantity on the right hand side of (2.15). Take $t_1 \in [0,T]$ sufficiently small so that

$$C_4 + t_1\tilde{C}(f) < C_3 \exp(-C_1 t_1)/t_1. \tag{3.10}$$

(see (2.7), (2.8) and (3.2)).

In the same way as in (3.9), we get, for $x \in R^n$,

$$\begin{aligned} & \sup_{0 \leq t \leq t_1} |w(t,x)| \\ \leq {} & \sup_{0 \leq t \leq t_1, |y|=r} |w(t,y)| \exp(t_1\tilde{C}(f)(1+r^2))P_{0,x}(t_1 \geq \tau_r) \\ \to {} & 0 \quad (\text{as } r \to \infty \text{ from Lemma 3.1, (II) and (2.15)}). \end{aligned} \tag{3.11}$$

For $t_1 \leq t \leq \min(2t_1, T)$ and $x \in R^n$, consider $w(t-t_1, x)$ and $P_{t_1,x}(t \geq \tau_r)$ $(r > 0)$. Then in the same way as above, we can show that $w(t-t_1, x)$ $(t_1 \leq t \leq \min(2t_1, T)$, $x \in R^n)$ is identically zero. Inductively we can show that Theorem 2.3 is true.

Q.E.D.

Before we prove Lemma 3.1, let us prove the following lemma.

Lemma 3.2. *For any $t \geq s \geq 0$ and $x \in R^n$, the following holds;*

$$\sup_{s\leq\gamma\leq t}|X(\gamma)| \leq (|x| + (t-s)\sup_{s\leq\gamma\leq t}|b(\gamma,o)| + \sup_{s\leq\gamma\leq t}|M^{[c]}(\gamma) + M^{[d]}(\gamma)|)\exp(C_1(t-s)/2) \quad P_{s,x}-a.s. \tag{3.12}$$

(see (2.7) for natation).
Proof. For $\gamma \geq s \geq 0$ and $x \in R^n$

$$\begin{aligned} X(\gamma) &= x + \int_s^\gamma b(\alpha, X(\alpha))d\alpha + M^{[c]}(\gamma) + M^{[d]}(\gamma) \\ &= x + \int_s^\gamma b(\alpha, o)d\alpha + M^{[c]}(\gamma) + M^{[d]}(\gamma) \\ &\quad + \int_s^\gamma [b(\alpha, X(\alpha)) - b(\alpha, o)]d\alpha, \end{aligned} \tag{3.13}$$

$P_{s,x} - a.s.$. By Gronwall's inequality, from (3.13), the proof is over.
Q.E.D.

Finally we prove Lemma 3.1.
Proof of Lemma 3.1. From Lemma 3.2, for any $s \leq t$, $r > 0$ and $x(|x| < r)$,

$$\begin{aligned} &P_{s,x}(\tau_r \leq t) \\ &= P_{s,x}(\sup_{s\leq\gamma\leq t}|X(\gamma)| \geq r) \\ &\leq P_{s,x}(1 \leq (1 - \exp(C_1(t-s)/2)(|x| + (t-s)\sup_{s\leq\gamma\leq t}|b(\gamma,o)|)/r)^{-1} \\ &\qquad \times \exp(C_1(t-s)/2)/r \sup_{s\leq\gamma\leq t}|M^{[c]}(\gamma) + M^{[d]}(\gamma)|) \\ &\leq P_{s,x}(1 \leq 2^{-1}\exp(C_1(t-s)/2)/r \sup_{s\leq\gamma\leq t}|M^{[c]}(\gamma) + M^{[d]}(\gamma)|) \\ &\qquad \text{(for sufficiently large } r > 0) \\ &\leq \sum_{i=1}^n P_{s,x}(1 \leq 2^{-1}n\exp(C_1(t-s)/2)/r \sup_{s\leq\gamma\leq t}|M^{[c]}(\gamma)^i + M^{[d]}(\gamma)^i|) \\ &\leq \sum_{i=1}^n \sup_{s\leq\gamma\leq t, |y|\leq r} P_{\gamma,y}(1/2 \leq 2^{-1}n\exp(C_1(t-s)/2)/r \\ &\qquad \times |M^{[c]}(t)^i + M^{[d]}(t)^i|) \end{aligned} \tag{3.14}$$

by the strong Markov property of $(X(t), P_{s,x})_{s\leq t, x\in R^n}$ (see [5], p. 151).
Put

$$C_5 \equiv n \exp(C_1(t-s)/2). \tag{3.15}$$

Then for each $i = 1, \cdots, n$, $\gamma \in [s,t]$ and $y \in R^n$, and any $R > 0$

$$\begin{aligned} & P_{\gamma,y}(1 \le C_5/r|M^{[c]}(t)^i + M^{[d]}(t)^i|) \qquad (3.16). \\ & \le \exp(-rR)E_{\gamma,y}[\exp(RC_5(M^{[c]}(t)^i + M^{[d]}(t)^i)) \\ & \quad + \exp(-RC_5(M^{[c]}(t)^i + M^{[d]}(t)^i))] \end{aligned}$$

by the exponential Chebychev's inequality (see [5], p. 151), and

$$\begin{aligned} & E_{\gamma,y}[\exp(RC_5(M^{[c]}(t)^i + M^{[d]}(t)^i))] \qquad (3.17). \\ & = E_{\gamma,y}[\exp(RC_5(M^{[c]}(t)^i + M^{[d]}(t)^i) - \int_\gamma^t \{a^{ii}(\alpha, X(\alpha))(RC_5)^2 \\ & \quad + \int_{z \ne x, z \in R^n} [\exp(RC_5(z_i - x_i)) - 1 - RC_5(z_i - x_i)]\nu_{\alpha, X(\alpha)}(dz)\}d\alpha \\ & \quad + \int_\gamma^t \{a^{ii}(\alpha, X(\alpha))(RC_5)^2 \\ & \quad + \int_{z \ne x, z \in R^n} [\exp(RC_5(z_i - x_i)) - 1 - RC_5(z_i - x_i)]\nu_{\alpha, X(\alpha)}(dz)\}d\alpha)] \\ & \le \exp[(t-\gamma)H(RC_5)] \end{aligned}$$

(see [5], p. 148). In the same way as in (3.17),

$$E_{\gamma,y}[\exp(-RC_5(M^{[c]}(t)^i + M^{[d]}(t)^i))] \le \exp[(t-\gamma)H(RC_5)]. \qquad (3.18).$$

From (3.14)-(3.18), we get for $s \le t$ and $x(|x| < r)$,

$$\log P_{s,x}(\tau_r \le t) \le \log(2n) - (t-s)(r/(t-s)R - H(RC_5)). \qquad (3.19).$$

Taking the infimum in $R > 0$ on the left hand side of (3.19),

$$\log P_{s,x}(\tau_r \le t) \le \log(2n) - (t-s)L(r/((t-s)C_5)). \qquad (3.20).$$

From (3.20),

$$\limsup_{r \to \infty} \{TL(r/(TC_2))\}^{-1} \log P_{0,x}(\tau_r \le T) \le -1, \qquad (3.21).$$

which shows that (I) is true (see (2.8) for notation).

Let us prove (II). For any $t > s \ge 0$ and $x \in R^n$,

$$(t-s)L(r/((t-s)C_2)) = (16(t-s)H(1)n^2)^{-1}\exp(-C_1(t-s))). \quad (3.22).$$

(3.20) and (3.22) completes the proof.

Q.E.D.

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