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EVOLVING CURVES WITH
BOUNDARY CONDITIONS

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EVOLVING CURVES WITH BOUNDARY CONDITIONS

YOSHIKAZU GIGA

Abstract. This paper reviews the theory of generalized solutions by the level set method for the curve shortening equation in a domain subject to boundary conditions. An example of fattening is provided for the curve shortening equation with right angle boundary condition. The example provides infinitely many reasonable solutions after onset of singularities.

1. Introduction. We are concerned with evolution of an embedded curve Γ_t whose end points may touch the boundary $\partial\Omega$ of a planer domain Ω containing Γ_t. A typical boundary condition is a prescribed contact angle condition, where the angle of Γ_t and $\partial\Omega$ is prescribed at points of touching. If the angle is the right angle, we often call the Neumann boundary condition. If the end point of Γ_t is fixed, the condition is called the Dirichlet boundary condition. We consider the curvature flow equation

$$V = K \quad \text{on} \quad \Gamma_t \tag{1}$$

subject to the boundary condition on $\partial\Omega$; here V and K denote normal velocity and curvature of Γ_t, respectively. Boundary value problems often appear as a singular limit of the Allen-Cahn type equations with suitable boundary conditions at least formally. For example if the zero Neumann condition is imposed, we get (1) with the Neumann condition $[RSK]$ as a singular limit. If suitably scaled Robin type condition is imposed, we get (1) with a prescribed contact angle as a singular limit $[OS]$. However, if a non-zero boundary condition is imposed, the singular limit is not the Dirichlet problem but a prescribed contact angle problem $[ORS]$. So far these is no good interpretation of the Dirichlet problem for (1) as a singular limit of the Allen-Cahn equation.

In this paper we mainly consider the Neumann boundary value problem for (1). In other words Γ_t is required to intersect perpendicularly if Γ_t touches $\partial\Omega$. If the domain Ω is convex, as in the case of $\Omega = \mathbf{R}^2$ the whole evolution is easy to understand. For given smooth curve γ_0 the (local) solution Γ_t stays smooth and embedded. It either converges to the geodesic as time t tends to infinity or becomes convex in a finite time. In the latter case

Γ_t shrinks to either at an interior or a boundary point of Ω in a finite time. As discused in $[RSK]$ this is obtained by adapting results of Grayson $[Gr1]$ and Gage and Hamilton $[GH]$ where Ω is assumed to be $\mathbf{R}^2$. However, if Ω is non-convex, at some time the curve Γ_t may touch the boundary $\partial\Omega$ at some non-end points. To track the evolution we need to extend the solution after onset of singularities.

If Γ_t is a closed hypersurface moved by its mean curvature in $\mathbf{R}^2$, singularities may develop for $n \geqslant 2[Gr2]$. Generalized solution based on the level set method tracks the whole evolution after singularities develop $[CGG]$,$[ES]$. Moreover, the singular limit of the Allen-Cahn equation yields a varifold type weak solution (Brakke's solution $[B]$) and its support is contained in generalized solution by the level set method $[I]$,$[So]$. Relation of the Allen-Cahn equation and generalized solution is first established in $[ESS]$. Although generalized solution is always a closed set, it may develop the interior even if Γ_0 is smooth. This is recently proved by Angenent, Ilmanen and Velazquez $[AIV]$ at least for $4 \leqslant n \leqslant 7$. Such phenomena is called fattening. If the generalized solution fattens at some time, then Brakke's solution is not unique so this phenomena relates to nonuniqueness of weak solutions. There are several criteria of non fattening. For example, if Γ_0 is axisymmetric, fattening does not occur $[SS]$,$[AAG]$. See also $[BSS]$ for more general evolution equations. The level set method works for very general evolution equation $[GGG]$,$[GG]$ including

$$V = K + 1. \tag{2}$$

As proved in $[BP]$ there may occur fattening for (2) when $\Omega = \mathbf{R}^2$.

Let us go back to the boundary value problem for (1). To track the evolution after singularities so far only a level set method is established. If the boundary condition is the Neumann condition, a level set method is established by $[Sa]$ for convex Ω and $[GS]$ for general C^2 bounded domain. Although applicable equations may be a little bit restrictive for general Ω, they include (1) and (2) as an examples. For general prescribed contact angle problem, Sato recently extended the level set method at least for half space Ω when the angle is constant (independent of points of Ω.) Even for the Neumann problem varifold type solution is never constructed. It is not clear how relates the singular limit of the Allen-Cahn equation to generalized solutions when Ω is not convex. For convex case there is a rigorous proof in $[KKR]$ but their method does not work for general domain.

In this paper we give an example of fattening for (1) with the Neumann condition. One similar aspect of the mean curvature flow equation is that the equation is independent of the choice of orientations of Γ_t. The equation (2)depends on the orientation. The example presented here may produce infinitely many "weak" solutions although we do not clarify the meaning. Such example is not known for other problems. Compared with the example for the mean curvature flow problem, our example is far simpler but the singularity we observe is cusp type while theirs is cone type. (Note that there is no fattening unless Γ_t develops singularities.)

For the Dirichlet problem of (1) a level set method is adapted by $[SZ]$ provided that Ω containing Γ_t is convex and that the end points of Γ_t lie on $\partial\Omega$. The results includes the case Ω in $\mathbf{R}^n$ if $\partial\Omega$ is mean-convex and (1) is replaced by the mean curvature flow equations. However, it is not clear whether generalized solution depends on the choice of Ω.

Another interesting boundary value problem is studied by $[AG]$, where the end points moves by its contact angle.

We remark that the prescribed contact angle problem is a special example of three - phase problem where one phase is always standing. A three-phase problem is an interesting problem but its global evolution is so far not clear except special cases [T],[SR].

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2. Generalized interface evolution. We consider the level set equation of (1) subject to the Neumann boundary condition.

$$u_t - |\nabla u| \operatorname{div} (\nabla u/|\nabla u|) = 0 \quad \text{in } (0,\infty) \times \Omega \tag{3}$$

$$\partial u/\partial \nu = 0 \qquad \text{on } (0,\infty) \times \partial\Omega, \tag{4}$$

where ν denotes the outer unit normal of $\partial\Omega$. Each level set of u satisfying (3) moves by (1). The condition (4) gives the Neumann condition for each level set of u. We recall a definition of a generalized solution (1) subject to the Neumann boundary condition.

2.1 Definition. Let Ω be a bounded domain with C^2 boundary and D_0 be an open set in Ω. An open set D in $[0,\infty) \times \Omega$ is a generalized subsolution of (1) subject to the Neumann condition with initial data D_0 if there is continuous (viscosity) subsolution of (3),(4) such that

$$D_0 = \{x \in \Omega; u(0,x) > 0\}, D = \{(t,x) \in [0,\infty) \times \Omega; u(t,x) > 0\}.$$

If u is a continuous supersolution, D is called a generalized supersolution. If u is a solution, D is called a generalized solution. If D_0^+ and D_0^- are two disjoint open set in Ω and Γ_0 is the complement of the union of D_0^+ and D_0^-, then the complement of the union of D^+ and D^- is called the generalized interface evolution Γ with initial data Γ_0. Here $D^\pm$ is a generalized solution with initial data $D_0^\pm$, respectively. For W in $[0,\infty) \times \Omega$ its cross section at time t denotes $W(t)$*i.e.*

$$W(t) = \{x \in \Omega; (t,x) \in W\}.$$

2.2. Comparison Theorem. *Assume that D_0^i are two open sets in Ω such that D_0^2 contains $D_0^1 (i = 1,2)$. Let D^2 and D^1 be generalized super- and subsolutions with initial data D_0^2 and D_0^1 respectively. Then D^1 is contained in D^2.*

2.3. Unique Existence Theorem. *For each D_0 there is a unique generalized solution D with $D(0) = D_0$. As a result for each closed set Γ_0 separating Ω into two pieces, there is a unique generalized interface evolution with $\Gamma(0) = \Gamma_0$.*

These results are quite standard. Although we use the notion of sub and super solutions, results in 2.2 and 2.3 follows from the papers [GS],[Sa]. Indeed Theorem 2.3 is exactly the main theorem of [GS]. The comparison theorem follows from the comparison theorem for solutions of (3),(4) if $u^1 \leqslant u^2$ holds initially for functions u^i representing D^i[GS]. Since there is a continuous nondecreasing $\theta : \mathbf{R} \to \mathbf{R}$ with $\theta(0) = 0, \theta(\sigma) > 0$ for $\sigma > 0$ such that $u^1 \leqslant \theta(u^2)$ at $t = 0$[CGG, §5] and since $\theta(u^2)$ solves (3),(4) [GS],[Sa], we may always assume $u^1 \leqslant u^2$ initially. So Theorem 2.2 is proved.

2.4 Monotone Convergence Theorem. *Let D and $\{D^j; j = 1, 2, ...\}$ be generalized solutions with initial data $D(0) = D_0$, $D^j(0) = D_0^j$. If $D_0^j (j = 1, 2, ...)$ is a nondecreasing sequence of open sets in Ω and $\cup_{j>0} D_0^j = D_0$ (in short $D_0^j \uparrow D_0$), then $D^j \uparrow D$.*

This type of result was first explicitly stated in [*AAG*, §3.4]. The proof is essentially the same so is omitted.

Here is an explanation of fattening based on Theorem 2.4. Assume that Γ_0 is smooth and the complement of Γ_0 consists of two open sets $D_0^{\pm}$ in Ω. Let $D_0^{+j} \uparrow D_0^+$ and $D_0^{-j} \uparrow D_0^-$. Let D^{+j} and D^{-j} denote a generalized solution with initial data D_0^{+j} and D_0^{-j} respectively. By Theorem 2.4 $D^{+j} \uparrow D^+$ and $D^{-j} \uparrow D^-$. Let Γ be the generalized interface evolution with $\Gamma(0) = \Gamma_0$. If $\Gamma(t)$ has an interior point at some time $t = s$, then there arises discrepancy between approximation from D^- side and D^+ side in the sense that the "limit" of $\partial(D^{\pm j}(s))$ does not agree even if Γ^j is smooth for $[0, s]$.

3. Fattening for curve shortning equation in a non-convex domain. We consider a generalized interface evolution Γ of (1) with the Neumann condition with $\Gamma(0) = \Gamma_0$, where Γ_0 is a smooth curve. As noted in §1 if Ω is convex, Γ does not fatten. If Ω is non-convex, the fattening phenomena may occur. We shall show this fact by constructing an explicit example.

3.1.Theorem on existence of fattening. *There are a non-convex domain Ω and smooth initial curve $\Gamma(0)$ such that the generalized interface evolution $\Gamma(t)$ in Ω has an interior point at some $t = s_0 > 0$. Moreover, $\Gamma(0)$ can be taken so that following properties hold. The complement of $\Gamma(0)$ consists of two disjoint open set $D_0^{\pm}$. Let $D_{\pm}$ be the generalized solution with $D_{\pm}(0) = D_0^{\pm}$. The time s_0 can be taken so that generalized evolution started with the boundary of $D_-(t)$ always develop interior instaneously provided $t > s_0$ is sufficiently close to s_0. Here $D_-(t)$ is one of evolutions which is connected and has a smooth boundary.*

Remark. The second part heuristically says that solution may bifurcate infinitely many times after $t > s_0$. This has a sharp contrast to other fattening examples.

We shall explicitly construct such a domain Ω and a curve $\Gamma(0)$. For this purpose we consider an evolution of curve $C(t)$ given by a graph of a function $y = \varphi(x, t)$ defined in an interval $[-R, R]$. The curve $C(t)$ moves by its curvature in $U = I \times \mathbf{R}$, $I = (-R, R)$ if and only if φ solves

$$\varphi_t/(1+\varphi_t^2)^{1/2} = \varphi_{xx}/(1+\varphi_x^2)^{3/2} \quad \text{for } |x| < R, t > 0.$$

The left hand side is the upward normal velocity of $C(t)$ while the right hand side is the upward curvature of $C(t)$. The Neumann condition for $C(t)$ is just a simple Neumann boundary condition for φ :

$$\varphi_x(\pm R, t) = 0.$$

In other words if the initial curve $C(0)$ is given as a graph of a function $y = g(x)$, its evolution $C(t)$ in U is given as a graph of solution for the initial-boundary problem of

$$(5) \qquad \begin{aligned} &\varphi_t = \varphi_{xx}/(1+\varphi_x^2) && \text{for } |x| < R, \quad t > 0 \\ &\varphi_x(\pm R, t) = 0 && \text{for } t > 0 \\ &\varphi(x, 0) = g(x) && \text{for } |x| < R. \end{aligned}$$

3.2. Remark on solvability. Although the first equation of (5) nonlinear, it is (quasilinear) strictly parabolic since the coefficient in front of φ_{xx} is bounded away from zero and bounded provided that φ_x is bounded. Solvability for smooth initial data is well-known including the mean curvature flow equation for a graph of a function in a multiple dimensional domain instead of I. Local-in -time unique existence of smooth solutions follows from the linear theory and a standard iteration procedure ($[LSU]$ and $[Fr]$). Global-in-time existence (of smooth solutions) follows from apriori bounds for $\varphi_x[LSU]$. Necessary a priori bounded was proved by Ecker and Huisken $[EH]$ including multi-dimensional case. They also proved that φ tends to constant as t tends to infinity. Large time behavior of solution with given boundary derivative $\varphi_x(\pm R, t)$ is studied by Altschuler and Wu $[AW1]$ for one-dimensional problem. An extension for multi-dimensinal case is found in $[AW2]$. (If the boundary condition is the Dirichlet type, for example $\varphi = 0$, and I is replaced by a multi-dimensional non-(mean)-convex domain, smooth global-in-time solutions may not exit. Thus, for global results the Neumann boundary condition is essential for a multi-dimensional domain; see Oliker and Ural'ceva $[OU]$ for existence of global weak solutions for Dirichlet problem.)

Here we only need the existence of a unique local-in-time smooth solution φ of (5) for smooth g satisfying the boundary condition $g_x(\pm R) = 0$. (In this case we even reduce the problem for periodic boundary value problem with period 4R by reflecting φ with respect to $x = R$ so we need not worry about the boundary). We consider a special initial data having only one inflection point.

3.3. Choice of initial data. Let g be a smooth function on $[-R, R]$. Our assumptions are

(i) $g_x(\pm R) = 0$

(ii) $g_{xx} < 0$ for $x < 0$ and $g_{xx} > 0$ for $x > 0$ so that $x = 0$ is the only inflection point *i.e.*, $g_{xx}(0) = 0$ and that $g' < 0$ on $I = (-R, R)$ by (i).

(iii) $g_{xxx}(0) > 0, g_{xxxx}(0) < 0$ (and $g(0) = 0$ for simplicity.)

Such a g is of course exists and its typical graph is as in figure 1. The condition (iii)(and(ii)) guarantees that $g(x) < -g(-x)$ for small $x > 0$. In particular g is not odd so that $C(0)$ has no symmetry. Let φ be a solution of (5) with g satisfying (i)-(iii). Let $x(t)$ be an inflection point of $\varphi(x, t)$.

3.4. Lemma on motion of inflection point $x(t)$. *Let g be a smooth function on $[-R, R]$ satisfying (i)-(iii). Then $x(t)$ is uniquely determined and a smooth function on a short time interval $[0, \eta]$ with positive time derivatives, i.e.$x'(t) > 0$.*

Proof. Since $g_{xxx}(0) \neq 0$, there is a unique solution $x(t)$ near zero if

$$\varphi_{xx}(x(t)) = 0 \quad \text{for small } t$$

by the implicit function theorem. Since g has only one inflection points, $x(t)$ is unique in $[-R, R]$ for small t. By the implicit function theorem $x(t)$ is smooth for small t. Differentiating $\varphi_{xx}(x(t), t) = 0$ in time yields

$$\varphi_{xxx}(x(t), t)x'(t) + \varphi_{xxt}(x(t), t) = 0$$

Differentiating $(5)_1$ in x twice yields

$$\begin{aligned}\varphi_{xxt} &= (\varphi_{xx}/(1+\varphi_x^2))_{xx} \\ &= \varphi_{xx} \cdot (1/(1+\varphi_x^2))_{xx} + 2\varphi_{xxx} \cdot (1/(1+\varphi_x^2))_x + \varphi_{xxxx}/(1+\varphi_x^2).\end{aligned}$$

Since $x(t)$ is the inflection point of φ, one observes that

$$\varphi_{xxt}(x(t), t) = \varphi_{xxxx}(x(t), t)/(1+\varphi_x^2(x(t), t)).$$

Since $g_{xxx}(0) > 0$ and $g_{xxxx}(0) < 0$, using the above formula for $x'(t)$ we end up with

$$x'(t) = -\varphi_{xxxx}/(\varphi_{xxx} \cdot (1+\varphi_x^2)) > 0 \text{ for small } t,$$

where the right hand side is evaluated at $(x(t), t)$.□

3.5. Remark on motion of inflection point. Differentiating $(5)_1$ in x twice as above we observe that φ_{xx} solves a linear parabolic equation with variable coefficients. If we appeal to the theory of zeros of solutions of linear parabolic equations [*An*], we can conclude that $x(t)$ is a smooth curve for $t > 0$ until it possibly hits the boundary points without assuming §3.3(iii).

We next consider envelope of $y = \varphi(x, t) (0 \leqslant t \leqslant \eta)$ in an interval $[0, x(\eta)]$. Since $x(t)$ gives a smooth diffeomorphism from $[0, \eta]$ to its image, its inverse function $t(x)$ is smooth in $[0, x(\eta)]$. We set

$$\psi(x) = \varphi(x, t(x))$$

so that ψ is smooth on $[0, x(\eta)]$.

3.6. Lemma on envelope of solution curves. *Let φ be the solution of (3.1) with initial data g in § 3.3. Then*

$$\varphi(x, t) \leqslant \psi(x) \qquad \text{for} \quad 0 \leqslant x \leqslant x(\eta),\ 0 \leqslant t \leqslant \eta.$$

The equality holds if and only if $x(t) = x$. If $x = x(t)$, then

$$\varphi_x(x, t) = \psi_x(x).$$

Moreover $\psi_{xx}(x) > 0$ for $0 < x < x(\eta)$. Here $x(t)$ is the inflection point of $\varphi(x, t)$.

Proof. Since $x'(t) > 0$, there is a unique s, $0 \leqslant s \leqslant \eta$ such that $x = x(s)$. If $t < s$, then

$$\varphi_t(x(s), t) = \varphi_{xx}(x(s), t)/(1+\varphi_x^2(x(s), t)) > 0$$

since $x(t) < x(s)$. Similarly, if $t > s$, then $\varphi_t(x(s),t) < 0$ and if $t = s$ then $\varphi_t(x(t),t) = 0$. Thus $\varphi(x(s),t)$ takes its unique maximum $\psi(x)$ at $t = s$ as a function of t. If $t = s$ so that $x = x(t)$, then

$$\begin{aligned}\psi_x(x) &= \varphi_x(x,t(x)) + \varphi_t(x,t(x))t_x(x)\\ &= \varphi_x(x,t) + \varphi_t(x(t),t)t_x(x).\end{aligned}$$

Since $\varphi_t(x(t),t) = 0$, we end up with $\psi_x(x) = \varphi_x(x,t)$.

It remains to prove $\psi_{xx} > 0$. Similarly to above, differentiating ψ_x in x and noting $\varphi_{xx}(x,t(x)) = 0$ yields

$$\begin{aligned}\psi_{xx}(x) &= 2\varphi_{xt}t_x + \varphi_{tt}t_x^2 = (2\varphi_{xxx} + t_x\varphi_{xxt})t_x\mu,\ \mu = (1+\varphi_x^2)^{-1}\\ &= \varphi_{xxx}t_x\mu \text{ by the formula for } x'(t);\end{aligned}$$

here partial derivative of φ is evaluated at $(x,t(x))$. Since $\varphi_{xxx} > 0$ by the choice of g and η we obtain $\psi_{xx} > 0$.□

3.7. Motion in a bounded domain. Clearly, for each x_0, $0 < x_0 < x(\eta)$, there is a smooth function h on $[0,x(\eta)]$ such that $\psi \leqslant h$ on $(0,x(\eta))$ and that $\psi = h$ only on $[x_0,x(\eta)]$. The graph of h touchs that of ψ from above and the leftest touching point denotes $P = (x_0,\psi(x_0))$. We extend h outside $[0,x(\eta)]$ so that

$$\varphi(x,t) < h(x) \text{ for } 0 \leqslant t \leqslant \eta, \text{ and for } x,\ -R \leqslant x \leqslant 0 \text{ or } x(\eta) < x \leqslant R.$$

Then the domain (see fig.2)

$$\Omega = \{(x,y); |x| < R,\ k(x) < y < h(x)\}$$

always contains $C(t)$,the graph of $y = \varphi(x,t)$, for $0 \leqslant t \leqslant \eta$ until the time t_0 when $C(t_0)$ first touchs $\partial\Omega$ at P, provided that a function k is sufficiently negative, say $k < \inf_I g$. We also observe that $C(t)$ touches $\partial\Omega$ for $t_0 \leqslant t \leqslant \eta$ but $C(t)$ is always contained in $\bar{\Omega}$ for $t \leqslant \eta$. By modifying h and k near $x = \pm R$, we may assume that Ω is a bounded domain with smooth boundary. This domain is certainly not convex and it has a bump around P, since ψ is convex.

3.8. Proposition on evolution in the bumped domain. *For given $C(0)$ let x_0 and Ω be taken as in § 3.7.*

(i) *Ω contains $C(t)$ for $0 \leqslant t < \eta$ except $t = t_0$ such that x_0 is the inflection point of $y = \varphi(x,t_0)$i.e.$x(t_0) = x_0$.*
(ii) *$C(t_0)$ is contained in Ω except the point $P = (x_0,\psi(x_0))$ on the boundary $\partial\Omega$.*
(iii) *A generalized interface evolution Γ in Ω with initial data $\Gamma(0) = C(0)$ agrees with C in Ω up to $t = t_0$; i.e. $\Gamma(t) = C(t)$ for $0 \leqslant t \leqslant t_0$.*

The first two properties easily follow from definitions of $\Omega, \psi, C(t)$. For $0 \leqslant t < t_0$ $C(t)$ does not touch the boundary. Since $C(t)$ is a smooth evolution in a convex domain U

it should agree with generalized solution in U and for $t < t_0$ it agrees with $\Gamma(t)$ in Ω. Note that $\Gamma(t)$ of generalized interface evolution is upper semicontinuous and left lower semicontinuous as a set valued function of t. Since the evolution of $C(t)$ is smooth up to $t = t_0$, the set $\Gamma(t_0)$ agrees with a $C(t_0)$ in Ω.□

We divide Ω into three disjoint sets $\Gamma(t_0)$,

$$D_{+0} = \{(x,y) \in \Omega, y > \varphi(x,t_0), \quad |x| < R\} \quad \text{and}$$
$$D_{-0} = \{(x,y) \in \Omega, y < \varphi(x,t_0), \quad |x| < R\}.$$

We consider generalized evolutions $D_\pm(t)$ of $D_{\pm 0}$.

3.9. Lemma. *(i) For $\delta > 0$ the set $D_+(t)$ is contained in*

$$W = \{(x,y) \in \Omega; y > \varphi(x_0,t_0), |x| < R\} \cup \{(x,y) \in \Omega; y > (-\delta + \psi'(x_0))(x - x_0) + \psi(x_0)\}$$

for all $t \geqslant 0$. (cf.fig.3).

$$\text{(ii) } D_-(t - t_0) = \{(x,y) \in \Omega; \quad y < \varphi(x,t)] \quad \text{for } t_0 \leqslant t \leqslant \eta.$$

Proof. (i) Since W is a standing supersolution and $W \supset D_{+0}$, by comparison $W \supset D_+(t)$. (ii) This is by approximation. Let

$$D^j_{-0} = \{(x,y) \in \Omega; y < \varphi(x,t_0) - 1/j, |x| < R\}$$

so that $D^j_{-0} \uparrow D_{-0}$, as $j \to \infty$. Let D^j_- be a generalized solution with initial data D^j_{-0} in Ω so that

$$D^j_-(t - t_0) = \{(x,y) \in \Omega \ ; \ y < \varphi(x,t) - 1/j, \ |x| < R\}$$

since $y = \varphi(x,t) - 1/j$ never touches $\partial\Omega$. Since $D^j_- \uparrow D_-$ by monotone convergence, we see

$$D_-(t - t_0) = \{(x,y) \in \Omega, y < \varphi(x,t)\}.$$

Proof of Theorem 3.1. Since $\varphi(x_0,t) < \varphi(x_0,t_0)$ for $t_0 < t \leqslant \eta$ there is $y_0 = y_0(t)$, $\varphi(x_0,t) < y < \varphi(x_0,y_0)$ such that a neighborhood of $Z = (x_0,y_0)$ is not contained either $D_-(t)$ or W. By Lemma 3.9 $\Gamma(t) = \Omega \setminus (D_+(t-t_0) \cup D_-(t-t_0))$ has an interior point for $\eta > t > t_0$. A similar argument yields fattening of the generalized interface evolution starting with $\partial D_-(t), t_0 < t < \eta$. The proof is now complete by taking $s_0 = t_0$.

3.10. Remark. Professor Stephan Luckhaus kindly pointed out that there is a simpler example of fattening for (1) with the Neumann condition. Let $\Gamma(0)$ be an axisymmetric closed curve which agrees with circle except a small neighborhord of one of crossing of $\Gamma(0)$ to the axis. Near this crossing $\Gamma(0)$ is assumed to be concave and elsewhere to be convex. By [$Gr1$] the solution C of (1) in $\mathbf{R}^2$ with initial data becomes convex at some time t_0. Let E be a closure of smoothly bounded domain in the plane such that $C(t)$ first touches E at time t_0. It is not difficult to show the existence of such E. If we consider (1) in $\Omega = \mathbf{R}^2 \setminus E$ with the Neumann condition, it is easy to imagine that the generalized interface evolution Γ with initial data $\Gamma(0)$ fattens after $t > t_0$. We do not discuss any detail here. The main difference between our example and this example is that the bifurcation of solution occurs infinitely many times in our example but only once ($t = t_0$) in this example.

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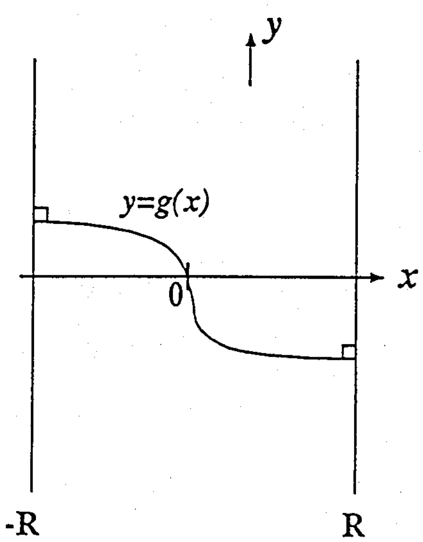

figure 1

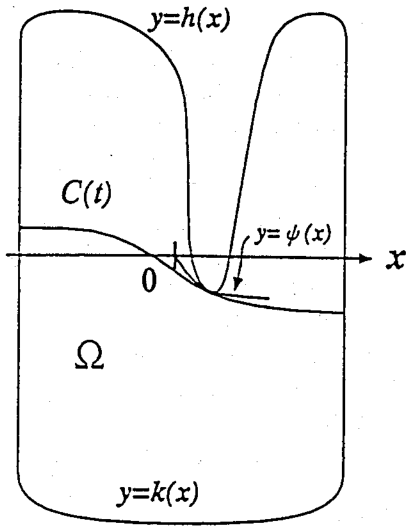

figure 2

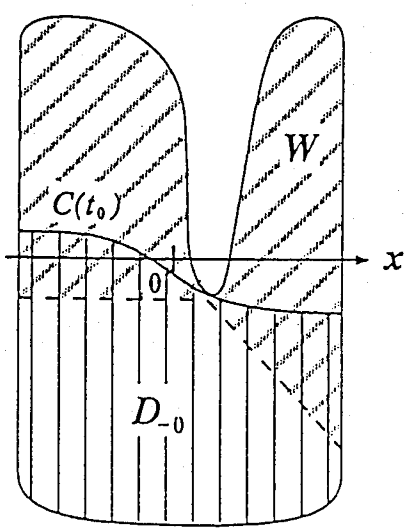

figure 3

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